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Irreducible triangulations are small

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ABSTRACT

A triangulation of a surface is *irreducible* if there is no edge whose contraction produces another triangulation of the surface. We prove that every irreducible triangulation of a surface with Euler genus $g \geqslant 1$ has at most $13g - 4$ vertices. The best previous bound was $171g - 72$.

1. Introduction

Irreducible triangulations are the building blocks of graphs embedded in surfaces, in the sense that every triangulation can be constructed from an irreducible triangulation by vertex splitting. Yet there are only finitely many irreducible triangulations of each surface, as proved by Barnette and Edelson [4,5]. Applications of irreducible triangulations include geometric representations [2,6], generating triangulations [16,17,20,24], diagonal flips [9,13,22], flexible triangulations [7], and an extremal problem regarding cliques in graphs on surfaces [11]. In this paper, we prove the best known upper bound on the order of an irreducible triangulation of a surface.

For background on graph theory see [10]. We consider simple, finite, undirected graphs. To *contract* an edge vw in a graph means to delete vw, identify v and w, and replace any parallel edges by a single edge. The inverse operation is called *vertex splitting*. Let G be a graph. For a vertex $v \in V(G)$, let $N_G(v) := \{w \in V(G)\colon\ vw \in E(G)\}$ and let G_v be the subgraph of G induced by $\{v\} \cup N_G(v)$. For $A \subseteq V(G)$, let $N_G(A) := \bigcup\{N_G(v)\colon\ v \in A\}$. Let $e(A)$ be the number of edges in G with both endpoints in A. For $A, B \subseteq V(G)$, let $e(A, B)$ be the number of edges in G with one endpoint in A and one endpoint in B. For $v \in V(G)$, let $e(v, B) := e(\{v\}, B)$.

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For background on graphs embedded in surfaces see [19]. Every surface is homeomorphic to $\mathbb{S}_g$, the orientable surface with g handles, or to $\mathbb{N}_h$, the non-orientable surface with h crosscaps. The *Euler genus* of $\mathbb{S}_g$ is $2g$. The *Euler genus* of $\mathbb{N}_h$ is h. The *Euler genus* of a graph G, denoted by $\mathbf{eg}(G)$, is the minimum Euler genus of a surface in which G embeds. A *triangulation* of a surface Σ is a 2-cell embedding of a graph in Σ, such that each face is bounded by three edges, and each pair of faces shares at most one edge. A triangulation G of Σ is *irreducible* if there is no edge in G whose contraction produces another triangulation of Σ. Equivalently, for $\Sigma \neq \mathbb{S}_0$, a triangulation G of Σ is irreducible if and only if every edge of G is in a triangle that forms a non-contractible cycle in Σ [19].

Recall that there are finitely many irreducible triangulations of each surface. For example, K_4 is the only irreducible triangulation of the sphere $\mathbb{S}_0$ [23], while K_6 and $K_7 - E(K_3)$ are the only irreducible triangulations of the projective plane $\mathbb{N}_1$ [3]. The complete list of irreducible triangulations has also been computed for the torus $\mathbb{S}_1$ [14], the double torus $\mathbb{S}_2$ [24], the Klein bottle $\mathbb{N}_2$ [15,26], as well as $\mathbb{N}_3$ and $\mathbb{N}_4$ [24]. Gao, Richmond and Thomassen [12] proved the first explicit upper bound on the order of an irreducible triangulation of an arbitrary surface. In particular, every irreducible triangulation of a surface with Euler genus $g \geqslant 1$ has at most $(12g+18)^4$ vertices. Nakamoto and Ota [21] improved this bound to $171g - 72$, which prior to this paper was the best known upper bound on the order of an irreducible triangulation of an arbitrary surface. In the case of orientable surfaces, Cheng et al. [8] improved this bound to $120g$. We prove:

Theorem 1. *Every irreducible triangulation of a surface with Euler genus $g \geqslant 1$ has at most $13g - 4$ vertices.*

The largest known irreducible triangulations of $\mathbb{S}_g$ and of $\mathbb{N}_h$ respectively have $\lfloor \frac{17}{2} g \rfloor$ and $\lfloor \frac{11}{2} h \rfloor$ vertices [25]. Thus the upper bound in Theorem 1 is within a factor of $\frac{26}{11}$ of optimal.

2. Background lemmas

At the heart of our proof, and that of Nakamoto and Ota [21], is the following lemma independently due to Archdeacon [1] and Miler [18]. Two graphs are *compatible* if they have at most two vertices in common.

Lemma 2. *(See [1,18].) If G and H are compatible graphs, then*

$$\mathbf{eg}(G \cup H) \geqslant \mathbf{eg}(G) + \mathbf{eg}(H).$$

Nakamoto and Ota [21] proved:

Lemma 3. *(See [21].) Let G be an irreducible triangulation of a surface with positive Euler genus. Then G has minimum degree at least 4. Moreover, for every vertex v of G, the subgraph G_v has minimum degree at least 4 and $\mathbf{eg}(G_v) \geqslant 1$.*

The following definition and lemma is implicit in [21]. An independent set S of a graph G is *ordered* if either $S = \emptyset$, or S contains a vertex v such that $S - \{v\}$ is ordered, and G_v and $\bigcup\{G_w\colon\ w \in S - \{v\}\}$ are compatible. Lemmas 2 and 3 then imply:

Lemma 4. *(See [21].) Let G be an irreducible triangulation of a surface with positive Euler genus. If S is an ordered independent set of G, then*

$$\mathbf{eg}(G) \geqslant \mathbf{eg}\left(\bigcup_{v \in S} G_v\right) \geqslant |S|.$$

3. A simple proof

In this section we give a simple proof that every irreducible triangulation of a surface with Euler genus $g \geqslant 1$ has at most $25g - 12$ vertices. The constant 25, while greater than the constant in Theorem 1, is still less than the constant in previous results. The proof follows the approach developed by Nakamoto and Ota [21] (using Lemma 4). This section also serves as a helpful introduction to the more complicated proof of Theorem 1 to come.

Let G be an irreducible triangulation of a surface with Euler genus $g \geqslant 1$. Let S be a maximal ordered independent set in G such that $\deg_G(v) \leqslant 6$ for all $v \in S$. Define

$$N := N_G(S),$$
$$A := \bigl\{v \in V(G) - (S \cup N)\colon e(v, N) \geqslant 3\bigr\},$$
$$Z := \bigl\{v \in V(G) - (S \cup N)\colon e(v, N) \leqslant 2\bigr\}.$$

Thus $\{S, N, A, Z\}$ is a partition of $V(G)$.

Suppose that $\deg_G(v) \leqslant 6$ for some vertex $v \in Z$. Since $v \notin N_G(S)$, the set $S \cup \{v\}$ is independent. Since $e(v, N) \leqslant 2$, the subgraphs G_v and $\bigcup\{G_w\colon w \in S\}$ are compatible. Since S is ordered, $S \cup \{v\}$ is ordered. Hence $S \cup \{v\}$ contradicts the maximality of S. Now assume that $\deg_G(v) \geqslant 7$ for all $v \in Z$. Thus

$$7|Z| \leqslant \sum_{v \in Z} \deg_G(v) = e(N, Z) + e(A, Z) + 2e(Z). \tag{1}$$

By Lemma 3, each vertex in A has degree at least 4, implying

$$4|A| \leqslant \sum_{v \in A} \deg_G(v) = e(N, A) + e(A, Z) + 2e(A). \tag{2}$$

By the definition of A,

$$3|A| \leqslant \sum_{v \in A} e(v, N) = e(N, A). \tag{3}$$

By Euler's formula applied to G,

$$\begin{aligned} &e(S, N) + e(N) + e(N, A) + e(N, Z) + e(A) + e(A, Z) + e(Z) \\ &\quad = \bigl|E(G)\bigr| = 3\bigl(\bigl|V(G)\bigr| + g - 2\bigr) = 3\bigl(|S| + |N| + |A| + |Z| + g - 2\bigr). \end{aligned} \tag{4}$$

Summing (1), (2), (3) and $2 \times (4)$ gives

$$|A| + |Z| + 2e(S, N) + 2e(N) + e(N, Z) \leqslant 6|S| + 6|N| + 6(g - 2).$$

Every vertex in N has a neighbour in S. Thus $e(S, N) \geqslant |N|$. By Lemma 3, $G[N]$ has minimum degree at least 3, and thus $2e(N) \geqslant 3|N|$. Since $e(N, Z) \geqslant 0$,

$$|N| + |A| + |Z| \leqslant 6|S| + 2|N| + 6(g - 2).$$

Since every vertex in S has degree at most 6, we have $|N| \leqslant 6|S|$. Thus

$$\bigl|V(G)\bigr| = |S| + |N| + |A| + |Z| \leqslant 19|S| + 6(g - 2).$$

By Lemma 4, $|S| \leqslant \mathbf{eg}(G) \leqslant g$. Therefore $|V(G)| \leqslant 25g - 12$.

4. Proof of Theorem 1

The proof of Theorem 1 builds on the proof in Section 3 by:

- introducing a more powerful approach than Lemma 4 for applying Lemma 2, thus enabling Lemma 2 to be applied to subgraphs with Euler genus possibly greater than 1 (whereas Lemma 4 applies Lemma 2 to subgraphs with Euler genus equal to 1);
- choosing an independent set S more carefully than in Section 3 so that low-degree vertices are heavily favoured in S;
- partitioning $V(G)$ into the similar sets S, N, A, Z as in Section 3, and further partitioning S and A according to the vertex degrees;
- introducing multiple partitions of N, one for each value of the degree of a vertex in S.

First we introduce a key definition. Let T be a binary tree rooted at a node r; that is, every non-leaf node of T has exactly two child nodes. Let $L(T)$ be the set of leaves of T. For each node x of T, let $T[x]$ be the subtree of T rooted at x. Suppose that each leaf $u \in L(T)$ is associated with a given subgraph $G\langle u\rangle$ of some graph G. For each non-leaf node x of T, define

$$G\langle x\rangle := \bigcup_{u\in L(T[x])} G\langle u\rangle.$$

Thus $G\langle x\rangle = G\langle a\rangle \cup G\langle b\rangle$, where a and b are the children of x. The pair $(T, \{G\langle u\rangle\colon u \in L(T)\})$ is a *tree representation* in G if $G\langle a\rangle$ and $G\langle b\rangle$ are compatible for each pair of nodes a and b with a common parent x. In this case, $\mathbf{eg}(G\langle x\rangle) \geqslant \mathbf{eg}(G\langle a\rangle) + \mathbf{eg}(G\langle b\rangle)$ by Lemma 2. This implies the following strengthening of Lemma 4.

Lemma 5. *If $(T, \{G\langle u\rangle\colon u \in L(T)\})$ is a tree representation in G, then*

$$\mathbf{eg}(G) \geqslant \sum_{u\in L(T)} \mathbf{eg}\bigl(G\langle u\rangle\bigr).$$

Let S be a set of vertices in a graph G. A tree representation $(T, \{G\langle u\rangle\colon u \in L(T)\})$ in G *respects* S if $L(T) = S$ and $G\langle u\rangle = G_u$ for each $u \in S$; henceforth denoted $(T, \{G_u\colon u \in S\})$.

Let G be an irreducible triangulation of a surface with Euler genus $g \geqslant 1$. By Lemma 3, G has minimum degree at least 4. Let S be an independent set of G such that $\deg_G(v) \leqslant 9$ for all $v \in S$. For $i \in [4, 9]$, define

$$S_i := \bigl\{v \in S\colon \deg_G(v) = i\bigr\},$$
$$\widehat{S_i} := S_4 \cup \cdots \cup S_i = \bigl\{v \in S\colon \deg_G(v) \leqslant i\bigr\}, \quad \text{and}$$
$$H_i := \bigcup_{v\in\widehat{S_i}} G_v.$$

Observe that $H_4 \subseteq H_5 \subseteq \cdots \subseteq H_9$. We say that S is *good* if there is a tree representation $(T, \{G_u\colon u \in S\})$ respecting S such that for all $i \in [4, 9]$, for every component X of H_i, there is a node $x \in V(T)$ such that

$$L\bigl(T[x]\bigr) = \widehat{S_i} \cap V(X) \quad \text{and} \quad X = G\langle x\rangle.$$

Note that these two conditions are equivalent.

For each good independent set S of G, let $\phi(S)$ be the vector $(|S_4|, |S_5|, \dots, |S_9|)$. Define $(a_4, \dots, a_9) \succ (b_4, \dots, b_9)$ if there exists $j \in [4, 8]$ such that $a_i = b_i$ for all $i \in [4, j]$, and $a_{j+1} > b_{j+1}$. Thus $\succcurlyeq$ is a linear ordering. Hence there is a good independent set S such that $\phi(S) \succcurlyeq \phi(S')$ for every other good independent set S'. Fix S throughout the remainder of the proof, and let $(T, \{G_v\colon v \in S\})$ be a tree representation respecting S.

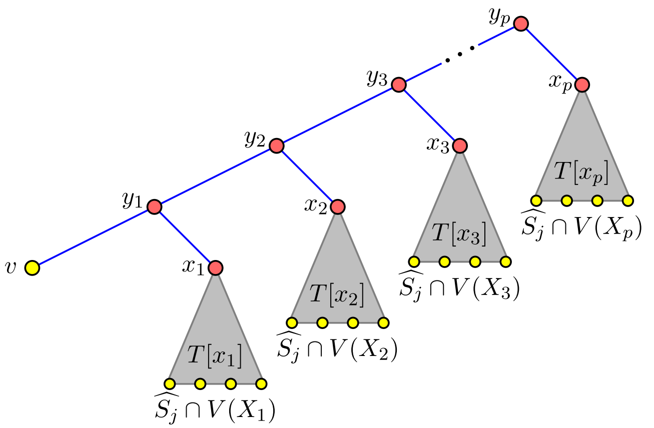

Fig. 1. Construction of T'.

Lemma 6. *Let $i \in [4, 9]$. Suppose that v is a vertex in $G - V(H_i)$, such that* $\deg_G(v) \leqslant i$. *Then v has at least three neighbours in some component of H_i.*

Proof. Suppose on the contrary that v has at most two neighbours in each component of H_i. Let $j := \deg_G(v)$. We now prove that $S' := \widehat{S_j} \cup \{v\}$ is a good independent set.

Say the components of H_j are $X_1, \dots, X_p$, where $X_1, \dots, X_q$ are the components of H_j that intersect $N_G(v)$. For $\ell \in [1, p]$, the component X_ℓ is a subgraph of some component of H_i. Thus v has at most two neighbours in X_ℓ. That is, G_v and X_ℓ are compatible. By assumption, for each $\ell \in [1, p]$, there is a node $x_\ell \in V(T)$ such that $L(T[x_\ell]) = \widehat{S_j} \cap V(X_\ell)$ and $X_\ell = G\langle x_\ell \rangle$. Thus $T[x_\ell] \cap T[x_k] = \emptyset$ for distinct $\ell, k \in [1, p]$.

Let T' be the tree obtained from the forest $\bigcup\{T[x_\ell]\colon \ell \in [1, p]\}$ by adding a path $(v, y_1, \dots, y_p)$, where each y_ℓ is adjacent to x_ℓ. Root T' at y_p, as illustrated in Fig. 1. Observe that

$$L(T') = \bigcup_{\ell \in [1,p]} L\big(T[x_\ell]\big) \cup \{v\} = \bigcup_{\ell \in [1,p]} \big(\widehat{S_j} \cap V(X_\ell)\big) \cup \{v\} = \widehat{S_j} \cup \{v\} = S'.$$

Let $G\langle u \rangle := G_u$ for each leaf $u \in L(T')$. Thus $G\langle x_\ell \rangle = X_\ell$ in T', and associated with the node y_ℓ is the subgraph $G\langle y_\ell \rangle = \bigcup\{X_k\colon k \in [1, \ell]\} \cup G_v$. The children of y_1 are x_1 and v, and for $\ell \in [2, p]$, the children of y_ℓ are x_ℓ and $y_{\ell-1}$. Since v has at most two neighbours in X_ℓ, and since $(X_1 \cup \dots \cup X_{\ell-1}) \cap X_\ell = \emptyset$, the subgraphs $G\langle y_{\ell-1} \rangle$ and $G\langle x_\ell \rangle$ are compatible.

Define $H'_4, \dots, H'_9$ with respect to S'. We must prove that for each $k \in [4, 9]$ and for each component X of H'_k, there is a node $z \in V(T')$ such that $X = G\langle z \rangle$.

First suppose that $k \in [j, 9]$. Since every vertex in S' has degree at most j, we have $\widehat{S'_k} = \widehat{S'_j}$. Thus $H'_k = H'_j$, and each component X of H'_k is a component of H'_j. Hence, either $X = X_1 \cup \dots \cup X_q \cup G_v$, or $X = X_\ell$ for some $\ell \in [q+1, p]$. In the first case, $X = G\langle y_q \rangle$. In the second case, $X = G\langle x_\ell \rangle$.

Now suppose that $k \in [4, j-1]$. Thus $S'_k = S_k$ and $H'_k = H_k$. Each component X of H'_k is a subset of X_ℓ for some $\ell \in [1, p]$, and there is a node $z \in T[x_\ell] \subseteq T'$ such that $X = G\langle z \rangle$.

This proves that $(T', \{G_u\colon u \in S'\})$ is a tree representation respecting S'. Thus S' is a good independent set. Moreover, $\phi(S') = (|S_4|, \dots, |S_{j-1}|, |S_j| + 1, 0, \dots, 0)$. Thus $\phi(S') \succ \phi(S)$. This contradiction proves that v has at least three neighbours in some component of H_i. $\square$

4.1. Properties of the neighbourhood of S

Recall that S is a good independent set such that $\phi(S) \succcurlyeq \phi(S')$ for every other good independent set S'. First note that Lemmas 3 and 5 imply:

$$g \geqslant \mathbf{eg}(G) \geqslant \sum_{u \in S} \mathbf{eg}(G_u) \geqslant |S|. \tag{5}$$

Partition $N := N_G(S)$ as follows. For $i \in [4,9]$, define

$$U_i := N_G(\widehat{S_i}),$$
$$Y_i := N - U_i,$$
$$V_i := \{v \in Y_i \colon \deg_G(v) \leqslant i\}, \quad \text{and}$$
$$W_i := \{v \in Y_i \colon \deg_G(v) \geqslant i+1\}.$$

Thus $\{U_i, Y_i\}$ and $\{U_i, V_i, W_i\}$ are partitions of N (for each $i \in [4,9]$). Also note that $U_4 \subseteq U_5 \subseteq \cdots \subseteq U_9$, and $H_i = \bigcup\{G_v \colon v \in \widehat{S_i}\}$ is a spanning subgraph of $G[\widehat{S_i} \cup U_i]$. Each vertex in N has at least one neighbour in S, and each vertex in S_i has i neighbours in N. Thus

$$|N| \leqslant e(S,N) = \sum_{i \in [4,9]} i|S_i|. \tag{6}$$

For $i \in [5,9]$, each vertex in $U_i - U_{i-1}$ has at least one neighbour in S_i, and each vertex in S_i has at most i neighbours in $U_i - U_{i-1}$. Thus

$$|U_i| \leqslant |U_{i-1}| + i|S_i|. \tag{7}$$

For $i \in [4,9]$, let c_i be the number of components of H_i. Thus

$$\sum_{j \in [4,i]} j|S_j| \geqslant \bigl|E(H_i)\bigr| \geqslant \bigl|V(H_i)\bigr| - c_i = |U_i| + |\widehat{S_i}| - c_i.$$

Hence

$$|U_i| \leqslant c_i + \sum_{j \in [4,i]} (j-1)|S_j|. \tag{8}$$

Consider a vertex $v \in U_i$ for some $i \in [4,9]$. Thus v is adjacent to some vertex $w \in \widehat{S_i}$. It follows from Lemma 3 that $G[N_G(w)]$ has minimum degree at least 3. In particular, v has at least three neighbours in $N_G(w)$, which is a subset of U_i. Thus

$$3|U_i| \leqslant \sum_{v \in U_i} e(v, U_i) = 2e(U_i). \tag{9}$$

Consider a vertex $v \in V_i$ for some $i \in [4,9]$. Thus v is in $G - V(H_i)$ and $\deg_G(v) \leqslant i$. By Lemma 6, v has at least three neighbours in H_i, implying $e(v, U_i) \geqslant 3$ since $N_G(v) \cap \widehat{S_i} = \emptyset$. Hence for $i \in [4,9]$,

$$3|V_i| \leqslant \sum_{v \in V_i} e(v, U_i) = e(U_i, V_i) \leqslant e(U_i, Y_i). \tag{10}$$

4.2. Beyond the neighbourhood of S

As in Section 3, partition $V(G) - (S \cup N)$ as

$$A := \{v \in V(G) - (S \cup N) \colon e(v,N) \geqslant 3\} \quad \text{and}$$
$$Z := \{v \in V(G) - (S \cup N) \colon e(v,N) \leqslant 2\}.$$

Thus $\{S, N, A, Z\}$ is a partition of $V(G)$. Further partition A as follows. For $i \in [4,9]$, let

$$A_i := \{v \in A \colon \deg_G(v) = i\}, \quad \text{and let}$$
$$A_{10} := \{v \in A \colon \deg_G(v) \geqslant 10\}.$$

Thus $\{A_4, \dots, A_{10}\}$ is a partition of A. For $i \in [4,9]$, let

$$\widehat{A_i} := A_4 \cup \cdots \cup A_i.$$

Consider a vertex $v \in A \cup Z$ such that $i = \deg_G(v) \in [4,9]$. By Lemma 6, v has at least three neighbours in H_i, implying $e(v,N) \geqslant 3$ since $N_G(v) \cap S = \emptyset$. Thus $v \in A$. Hence $\deg_G(v) \geqslant 10$ for every vertex $v \in Z$. Since $N_G(Z) \subseteq A \cup Z \cup N$,

$$10|Z| \leqslant \sum_{v \in Z} \deg_G(v) = 2e(Z) + e(N,Z) + e(A,Z). \tag{11}$$

Since $N_G(A) \subseteq A \cup Z \cup N$,

$$\sum_{i \in [4,10]} i|A_i| \leqslant \sum_{v \in A} \deg_G(v) = 2e(A) + e(N,A) + e(A,Z). \tag{12}$$

4.3. Global inequalities

Let $i \in [4,9]$. Consider the sum of the degrees of the vertices in Y_i. Each vertex in V_i has degree at least 4, and each vertex in W_i has degree at least $i+1$. Each neighbour of a vertex in Y_i is in $S_{i+1} \cup \cdots \cup S_9 \cup U_i \cup Y_i \cup A \cup Z$. Hence

$$\begin{aligned}
4|V_i| + (i+1)|W_i| &\leqslant \sum_{v \in Y_i} \deg_G(v) \\
&\leqslant e(Y_i, S_{i+1} \cup \cdots \cup S_9) + e(U_i, Y_i) + 2e(Y_i) + e(N,Z) + e(Y_i, A) \\
&\leqslant \sum_{j \in [i+1,9]} j|S_j| + e(U_i, Y_i) + 2e(Y_i) + e(N,Z) + e(N,A) - e(U_i, A).
\end{aligned}$$

Consider a vertex $v \in \widehat{A_i}$. Thus v is in $G - V(H_i)$ and $\deg_G(v) \leqslant i$. By Lemma 6, v has at least three neighbours in some component of H_i, implying $e(v, U_i) \geqslant 3$ since $N_G(v) \cap S = \emptyset$. Thus

$$3|\widehat{A_i}| \leqslant \sum_{v \in \widehat{A_i}} e(v, U_i) \leqslant e(U_i, A).$$

Hence

$$4|V_i| + (i+1)|W_i| + 3|\widehat{A_i}| \leqslant \sum_{j \in [i+1,9]} j|S_j| + e(U_i, Y_i) + 2e(Y_i) + e(N,Z) + e(N,A). \tag{13}$$

As proved above, each vertex in $\widehat{A_i}$ has at least three neighbours in some component of H_i. Let $X_1, \dots, X_{c_i}$ be the components of H_i. Let $\{D_1, \dots, D_{c_i}\}$ be a partition of $\widehat{A_i}$ such that for each $\ell \in [1, c_i]$, each vertex in D_ℓ has at least three neighbours in X_ℓ. Let B_ℓ be the bipartite subgraph of G with parts $(\widehat{S_i} \cap V(X_\ell)) \cup D_\ell$ and $V(X_\ell) \cap U_i$.

Let $(T, \{G_u\colon u \in S\})$ be a tree representation respecting S. For each $\ell \in [1, c_i]$, there is a node x_ℓ in T such that $G\langle x_\ell \rangle = X_\ell$. Let T' be the tree obtained from T by replacing each subtree $T[x_\ell]$ by the single node x_ℓ. Thus x_ℓ is a leaf in T'. Redefine $G\langle x_\ell \rangle := B_\ell$. Every other leaf in T' is a vertex in $S - \widehat{S_i}$. Thus $L(T') = (S - \widehat{S_i}) \cup \{x_\ell\colon \ell \in [1, c_i]\}$. For each $u \in S - \widehat{S_i}$, leave $G\langle u \rangle = G_u$ unchanged. Now $D_\ell \cap D_k = \emptyset$ for distinct $\ell, k \in [1, c_i]$, and $G_u \cap D_\ell = \emptyset$ for all $u \in S - \widehat{S_i}$. Thus $(T', \{G\langle u \rangle\colon u \in L(T')\})$ is a tree representation in G. By Lemma 5,

$$g \geqslant \mathbf{eg}(G) \geqslant \sum_{u \in L(T')} \mathbf{eg}\big(G\langle u \rangle\big) = \sum_{u \in S - \widehat{S_i}} \mathbf{eg}(G_u) + \sum_{\ell \in [1, c_i]} \mathbf{eg}(B_\ell).$$

By Lemma 3, $\mathbf{eg}(G_u) \geqslant 1$ for all $u \in S - \widehat{S_i}$. Euler's formula applied to the bipartite graph B_ℓ implies that $|E(B_\ell)| \leqslant 2(|V(B_\ell)| + \mathbf{eg}(B_\ell) - 2)$. Thus

$$g \geqslant |S - \widehat{S}_i| + \sum_{\ell \in [1,c_i]} \left(\frac{1}{2} \bigl|E(B_\ell)\bigr| - \bigl|V(B_\ell)\bigr| + 2 \right).$$

Since $B_\ell \cap B_k = \emptyset$ for distinct $\ell, k \in [1, c_i]$, and $\bigcup \{V(B_\ell)\colon\ \ell \in [1, c_i]\} = \widehat{S}_i \cup U_i \cup \widehat{A}_i$,

$$g \geqslant |S - \widehat{S}_i| - \bigl(|\widehat{S}_i| + |U_i| + |\widehat{A}_i|\bigr) + 2c_i + \frac{1}{2} \sum_{\ell \in [1,c_i]} \bigl|E(B_\ell)\bigr|.$$

Each vertex in $\widehat{A}_i$ is incident to at least three edges in some B_ℓ. Thus

$$\sum_{\ell \in [1,c_i]} \bigl|E(B_\ell)\bigr| \geqslant 3|\widehat{A}_i| + e(\widehat{S}_i, U_i) = 3|\widehat{A}_i| + \sum_{j \in [4,i]} j|S_j|.$$

Hence

$$\begin{aligned} g &\geqslant |S - \widehat{S}_i| - \bigl(|\widehat{S}_i| + |U_i| + |\widehat{A}_i|\bigr) + 2c_i + \frac{3}{2}|\widehat{A}_i| + \frac{1}{2} \sum_{j \in [4,i]} j|S_j| \\ &= |S| - |U_i| + \frac{1}{2}|\widehat{A}_i| + 2c_i + \frac{1}{2} \sum_{j \in [4,i]} (j-4)|S_j|. \end{aligned} \tag{14}$$

At this point, the reader is invited to check, using their favourite linear programming software, that inequalities (3)–(14) and the obvious equalities imply that $|V(G)| \leqslant 13g - \frac{24}{7}$. (Also note that removing any one of these inequalities leads to a worse bound.) What follows is a concise proof of this inequality, which we include for completeness.

4.4. Summing the inequalities

The notation $(x.y)$ refers to the inequality with label (x), taken with $i = y$. (For instance, (8.4) stands for inequality (8) with $i = 4$.)

Summing $4 \times$ (8.4), $8 \times$ (8.5), $4 \times$ (8.7), $4 \times$ (14.4), $4 \times$ (14.5), and $2 \times$ (14.7), and since $c_4 \geqslant 0$,

$$\begin{aligned} &4|U_5| + 2|U_7| + 5|A_4| + 3|A_5| + |A_6| + |A_7| + 10|S_8| + 10|S_9| \\ &\quad \leqslant 38|S_4| + 35|S_5| + 8|S_6| + 11|S_7| + 10g. \end{aligned} \tag{15}$$

Summing $2 \times$ (9.6), $4 \times$ (9.8), $2 \times$ (10.6), $5 \times$ (10.8), $2 \times$ (13.6), and $3 \times$ (13.8) yields

$$\begin{aligned} &6|U_6| + 12|U_8| + 14|V_6| + 27|V_8| + 14|W_6| + 27|W_8| + 15|A_4| + 15|A_5| \\ &\qquad + 15|A_6| + 9|A_7| + 9|A_8| \\ &\quad \leqslant 14|S_7| + 16|S_8| + 45|S_9| + 4e(U_6) + 8e(U_8) + 4e(U_6, Y_6) + 8e(U_8, Y_8) + 4e(Y_6) \\ &\qquad + 6e(Y_8) + 5e(N, Z) + 5e(N, A). \end{aligned}$$

Since $6e(Y_8) \leqslant 8e(Y_8)$ and $e(N) = e(U_i) + e(U_i, Y_i) + e(Y_i)$ for $i = 6$ and 8, the above inequality becomes

$$\begin{aligned} &6|U_6| + 12|U_8| + 14|V_6| + 27|V_8| + 14|W_6| + 27|W_8| + 15|A_4| + 15|A_5| + 15|A_6| \\ &\qquad + 9|A_7| + 9|A_8| \\ &\quad \leqslant 14|S_7| + 16|S_8| + 45|S_9| + 12e(N) + 5e(N, Z) + 5e(N, A). \end{aligned} \tag{16}$$

Summing $12 \times$ (7.6), $5 \times$ (7.7), and $11 \times$ (7.8) gives

$$7|U_6| + 11|U_8| \leqslant 12|U_5| + 6|U_7| + 72|S_6| + 35|S_7| + 88|S_8|. \tag{17}$$

Summing (16) and (17), and since $|N| = |U_i| + |V_i| + |W_i|$ for $i = 6$ and 8, we obtain

$$\begin{aligned}&36|N| + |V_6| + 4|V_8| + |W_6| + 4|W_8| + 15|A_4| + 15|A_5| + 15|A_6| + 9|A_7| + 9|A_8| \\ &\quad \leqslant 12|U_5| + 6|U_7| + 72|S_6| + 49|S_7| + 104|S_8| + 45|S_9| + 12e(N) + 5e(N,Z) + 5e(N,A).\end{aligned}$$

Since trivially $|V_6| + 4|V_8| + |W_6| + 4|W_8| \geqslant 0$ and $e(N,Z) \geqslant 0$, the previous inequality implies

$$\begin{aligned}&36|N| + 15|A_4| + 15|A_5| + 15|A_6| + 9|A_7| + 9|A_8| \\ &\quad \leqslant 12|U_5| + 6|U_7| + 72|S_6| + 49|S_7| + 104|S_8| + 45|S_9| + 12e(N) \\ &\qquad + 6e(N,Z) + 5e(N,A).\end{aligned} \tag{18}$$

Summing (3), $6 \times (11)$ and $6 \times (12)$ gives

$$3|A| + 6 \sum_{j \in [4,10]} j|A_j| + 60|Z| \leqslant 12e(A) + 7e(N,A) + 12e(A,Z) + 12e(Z) + 6e(N,Z). \tag{19}$$

Since $|A| = \sum_{i \in [4,10]} |A_i|$, summing $3 \times (15)$ with (18) and (19) gives

$$\begin{aligned}&57|A| + 3|A_8| + 6|A_{10}| + 36|N| + 60|Z| \\ &\quad \leqslant 114|S_4| + 105|S_5| + 96|S_6| + 82|S_7| + 74|S_8| + 15|S_9| \\ &\qquad + 12e(N) + 12e(A) + 12e(Z) + 12e(N,Z) + 12e(N,A) + 12e(A,Z) + 30g.\end{aligned} \tag{20}$$

Summing $12 \times (4)$ with (20) and since $e(S,N) = \sum_{i \in [4,9]} i|S_i|$, we next obtain

$$\begin{aligned}&2|S_7| + 22|S_8| + 93|S_9| + 57|A| + 3|A_8| + 6|A_{10}| + 36|N| + 60|Z| \\ &\quad \leqslant 66|S_4| + 45|S_5| + 24|S_6| + 36\big|V(G)\big| + 66g - 72.\end{aligned}$$

Combining this with $4|S_8| + 54|S_9| + 3|A_8| + 6|A_{10}| + 3|Z| \geqslant 0$, it follows that

$$\begin{aligned}&2|S_7| + 18|S_8| + 39|S_9| + 57|A| + 36|N| + 57|Z| \\ &\quad \leqslant 66|S_4| + 45|S_5| + 24|S_6| + 36\big|V(G)\big| + 66g - 72.\end{aligned} \tag{21}$$

Summing $21 \times (6)$ with (21) and since $|S| = \sum_{i \in [4,9]} |S_i|$, we derive

$$5|S_7| + 57|A| + 57|N| + 57|Z| \leqslant 150|S| + 36\big|V(G)\big| + 66g - 72.$$

Since $5|S_7| \geqslant 0$ and $|V(G)| = |S| + |N| + |A| + |Z|$, we obtain

$$21\big|V(G)\big| \leqslant 207|S| + 66g - 72. \tag{22}$$

Summing $207 \times (5)$ with (22) gives $21|V(G)| \leqslant 273g - 72$. That is, $|V(G)| \leqslant 13g - \frac{24}{7}$, as claimed. To conclude the proof of Theorem 1, note that $\frac{24}{7} > 3$, implying that $|V(G)| \leqslant 13g - 4$ since $|V(G)|$ and g are both integers.

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