Layered separators for queue layouts, 3D graph drawing and nonrepetitive coloring

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***Abstract*—Graph separators are a ubiquitous tool in graph theory and computer science. However, in some applications, their usefulness is limited by the fact that the separator can be as large as $\Omega(\sqrt{n})$ in graphs with n vertices. This is the case for planar graphs, and more generally, for proper minor-closed families. We study a special type of graph separator, called a *layered separator*, which may have linear size in n, but has bounded size with respect to a different measure, called the *breadth*. We prove that a wide class of graphs admit layered separators of bounded breadth, including graphs of bounded Euler genus.**

We use layered separators to prove $\mathcal{O}(\log n)$ bounds for a number of problems where $\mathcal{O}(\sqrt{n})$ was a long standing previous best bound. This includes the *nonrepetitive chromatic number* and *queue-number* of graphs with bounded Euler genus. We extend these results to all proper minor-closed families, with a $\mathcal{O}(\log n)$ bound on the nonrepetitive chromatic number, and a $\log^{\mathcal{O}(1)} n$ bound on the queue-number. Only for planar graphs were $\log^{\mathcal{O}(1)} n$ bounds previously known. Our results imply that every graph from a proper minor-closed class has a *3-dimensional grid drawing* with $n \log^{\mathcal{O}(1)} n$ volume, whereas the previous best bound was $\mathcal{O}(n^{3/2})$.

Readers interested in the full details should consult arXiv:1302.0304 and arXiv:1306.1595, rather than the current extended abstract.

***Keywords*-graph; separator; minor; graph drawing; queue layout, track layout; nonrepetitive colouring**

I. Introduction

Graph separators are a ubiquitous tool in graph theory and computer science since they are key to many divide and conquer and dynamic programming algorithms. Typically, the smaller the separator the better the results obtained. For instance, many problems that are $\mathcal{NP}$-complete for general graphs have polynomial time solutions for classes of graphs that have bounded size separators—that is, graphs of bounded treewidth.

By the classical result of Lipton and Tarjan [1], every n-vertex planar graph has a separator with $\mathcal{O}(\sqrt{n})$ vertices. More generally, the same is true for all proper minor-closed families[1], as proved by Alon *et al.* [2]. While these results have found widespread use, separators of size $\Theta(\sqrt{n})$, or non-constant separators in general, are not small enough to be useful in some applications.

In this paper we study a type of graph separator, called layered separators, that may have $\Omega(n)$ vertices but have constant size with respect to a different measure. In particular, layered separators intersect each layer of some predefined vertex layering in a constant number of vertices. We prove that many classes of graphs admit such separators, and we show how they can be used to obtain logarithmic bounds for a variety of applications for which $\mathcal{O}(\sqrt{n})$ was the best known long-standing bound. These applications include nonrepetitive graph colourings, track layouts, queue layouts and 3-dimensional grid drawings of graphs. In addition, layered separators lend themselves to simple proofs.

For example, our results imply that every graph of bounded Euler genus[2] has $\mathcal{O}(\log n)$ queue-number. Except for planar graphs, the previously best known bound was $\mathcal{O}(\sqrt{n})$. For planar graphs, our result improves the $\mathcal{O}(\log^2 n)$ bound by Di Battista *et al.* [3], and, more importantly, replaces their long and complex proof by a much simpler proof. Finally, our result generalises to prove that graphs in a proper minor-closed family have queue-number at most $\log^{\mathcal{O}(1)} n$.

In the remainder of the introduction, we define layered separators, and describe our results on the classes of graphs that admit them. Following that, we describe the implications that these results have on the above-mentioned applications.

A. Layered Separations

A *layering* of a graph G is a partition $(V_0, V_1, \dots, V_t)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i-j| \leq 1$. Each set V_i is called a *layer*. For example, for a vertex r of a connected graph G, if V_i is the set of vertices at distance i from r, then $(V_0, V_1, \dots)$ is a layering of G, called the *bfs layering* of G starting from r. A *bfs tree* of G rooted at r is a spanning tree of G such that for every vertex v of G, the distance between v and r in G

Research of Dujmović and Morin supported by NSERC. Research of Wood supported by the Australian Research Council.

[1]A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if $H \in \mathcal{G}$ for every minor H of G for every graph $G \in \mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs.

[2]The *Euler genus* of a surface Σ is $2-\chi$, where χ is the Euler characteristic of Σ. Thus the orientable surface with h handles has Euler genus $2h$, and the non-orientable surface with c cross-caps has Euler genus c. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds.

0272-5428/13 $26.00 © 2013 IEEE
DOI 10.1109/FOCS.2013.38

equals the distance between v and r in T. Thus, if $v \in V_i$ then the vr-path in T contains exactly one vertex from layer V_j for $0 \leq j \leq i$.

A *separation* of a graph G is a pair (G_1, G_2) of subgraphs of G, such that $G = G_1 \cup G_2$ and there is no edge of G between $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$. The set $V(G_1 \cap G_2)$ is called a *separator*. The *order* of a separation (G_1, G_2) is $|V(G_1 \cap G_2)|$.

A graph G *admits layered separations of breadth* ℓ with respect to a layering $(V_0, V_1, \dots, V_t)$ of G if for every set $S \subseteq V(G)$, there is a separation (G_1, G_2) of G such that:

- each layer V_i contains at most ℓ vertices in $V(G_1) \cap V(G_2) \cap S$, and
- both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $\frac{2}{3}|S|$ vertices in S.

Layered separations were first explicitly defined in [4], although they are implicit in many earlier papers, including the seminal work of Lipton and Tarjan [1] on separators in planar graphs. Dujmović *et al.* [4] showed that a result of Lipton and Tarjan [1] implies the following lemma, which was used by Lipton and Tarjan as a subroutine in their $\mathcal{O}(\sqrt{n})$ separator result.

Lemma 1 ([4], [1]). *Every planar graph admits layered separations of breadth* 2.

In this paper we prove, for example, that all graphs of bounded Euler genus admit layered separations of bounded breadth.

Theorem 2. *Every graph with Euler genus at most g admits layered separations of breadth* $3(g + 1)$.

We generalise Theorem 2 by exploiting Robertson and Seymour's graph minor structure theorem. Roughly speaking, a graph G is almost embeddable in a surface Σ if by deleting a bounded number of 'apex' vertices, the remaining graph can be embedded in Σ, except for a bounded number of 'vortices', where crossings are allowed in a well-structured way; see Section IV where all these terms are defined. Robertson and Seymour proved that every graph from a proper minor-closed class can be obtained from clique-sums of graphs that are almost embeddable in a surface of bound Euler genus. Here, apex vertices can be adjacent to any vertex in the graph. However, such freedom is not possible for graphs that admit layered separations of bounded breadth. For example, if the planar $\sqrt{n} \times \sqrt{n}$ grid plus one dominant vertex admits layered separations of breadth ℓ, then $\ell \in \Omega(\sqrt{n})$; see Section IV. We define the notion of strongly almost embeddable graphs, in which apex vertices are only allowed to be adjacent to vortices and other apex vertices. With this restriction, we prove that graphs obtained from clique-sums of strongly almost embeddable graphs admits layered separations of bound breadth. This is the most general class of graphs known to admit layered separations of bound breadth. Then, in all the applications that we consider, we deal with (unrestricted) apex vertices separately, leading to $\mathcal{O}(\log n)$ or $\log^{\mathcal{O}(1)} n$ bounds for all proper minor-closed families.

B. Queue-number and 3-Dimensional Grid Drawings

Let G be a graph. In a linear ordering $\preceq$ of $V(G)$, two edges vw and xy are *nested* if $v \prec x \prec y \prec w$. A *$k$-queue layout* of a graph G consists of a linear ordering $\preceq$ of $V(G)$ and a partition $E_1, \dots, E_k$ of $E(G)$, such that no two edges in each set E_i are nested with respect to $\preceq$. The *queue-number* of a graph G is the minimum integer k such that G has a k-queue layout, and is denoted by $\text{qn}(G)$. Queue layouts were introduced by Heath and Rosenberg [5], [6] and have since been widely studied. They have applications in parallel process scheduling, fault-tolerant processing, matrix computations, and sorting networks; see [7], [8] for surveys.

The dual concept of a queue layout is a *stack layout*, commonly called a *book embedding*. It is defined similarly, except that no two edges in the same set are allowed to cross with respect to the vertex ordering. *Stack number* (also known as *book thickness* or *page-number*) is bounded for planar graphs, for graphs of bounded Euler genus, and for all proper minor-closed graph families; see the survey [8]. No such bounds are known for the queue-number of these graph families. Heath *et al.* [6], [5] conjectured that planar graphs have $\mathcal{O}(1)$ queue-number—this question remains open.

In Section V, we prove that every n-vertex graph that admits layered separations of breadth ℓ has $\mathcal{O}(\ell \log n)$ queue-number. This implies that all the graph families described in Section I-A, such as graphs with bounded Euler genus, have $\mathcal{O}(\log n)$ queue-number. In addition, we extend this result to all proper minor-closed families with an upper bound of $\log^{\mathcal{O}(1)} n$. The previously best known bound for all these families, except for planar graphs, was $\mathcal{O}(\sqrt{n})$. Until recently, the best known upper bound for the queue-number of planar graphs was also $\mathcal{O}(\sqrt{n})$. This upper bound follows easily from the fact that planar graphs have pathwidth at most $\mathcal{O}(\sqrt{n})$. In a breakthrough result, the queue-number upper bound for planar graphs was reduced to $\mathcal{O}(\log^2 n)$ by Di Battista *et al.* [3][3]. Pemmaraju [7] conjectured that planar graphs have $\mathcal{O}(\log n)$ queue-number; our result establishes the truth of this conjecture. Pemmaraju also conjectured that this is the correct lower bound. To date, however, the best known lower bound is a constant.

One motivation for studying queue layouts is their connection with 3-dimensional graph drawing. A *3-dimensional grid drawing* of a graph G represents the vertices of G by distinct grid points in $\mathbb{Z}^3$ and represents each edge of G by the open segment between its endpoints, such that no two edges intersect. The *volume* of a 3-dimensional grid drawing

[3]The original bound on the queue-number of planar graphs proved by Di Battista *et al.* [3] was $\mathcal{O}(\log^4 n)$, which was subsequently improved to $\mathcal{O}(\log^2 n)$ [private communication, Fabrizio Frati, 2012].

is the number of grid points in the smallest axis-aligned grid-box that encloses the drawing. For example, Cohen *et al.* [9] proved that the complete graph K_n has a 3-dimensional grid drawing with volume $\mathcal{O}(n^3)$ and this bound is optimal. Dujmović *et al.* [10] proved that every graph with bounded maximum degree has a 3-dimensional grid drawing with volume $\mathcal{O}(n^{3/2})$, and the same bound holds for graphs from a proper minor-closed class. In fact, every graph with bounded degeneracy has a 3-dimensional grid drawing with $\mathcal{O}(n^{3/2})$ volume [11]. Dujmović *et al.* [12] proved that every graph with bounded treewidth has a 3-dimensional grid drawing with volume $\mathcal{O}(n)$. Whether planar graphs have 3-dimensional grid drawings with $\mathcal{O}(n)$ volume is a major open problem due to Felsner *et al.* [13]. We prove the best known bound of $\mathcal{O}(n \log n)$ for this problem. This improves upon the best previous $\mathcal{O}(n \log^8 n)$ bound by Di Battista *et al.* [3]. More generally, our results imply a $\mathcal{O}(n \log n)$ volume bound for all families of graphs that admit layered separations of bounded breadth, such as graphs of bounded Euler genus. More generally, we prove an $n \log^{\mathcal{O}(1)} n$ volume bound for all proper minor-closed families.

C. Nonrepetitive Graph Colourings

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a k*-colouring* of a graph G is a function ψ that assigns one of k colours to each vertex of G. A path $(v_1, v_2, \dots, v_{2t})$ of even order in G is *repetitively* coloured by ψ if $\psi(v_i) = \psi(v_{t+i})$ for $1 \leq i \leq t$. A colouring ψ of G is *nonrepetitive* if no path of G is repetitively coloured by ψ. Observe that a nonrepetitive colouring is *proper*, in the sense that adjacent vertices are coloured differently. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer k such that G admits a nonrepetitive k-colouring.

The seminal result in this area is by Thue, who in 1906 proved that every path is nonrepetitively 3-colourable. Nonrepetitive colourings have recently been widely studied; see the survey [15]. A number of graph classes are known to have bounded nonrepetitive chromatic number. In particular, trees are nonrepetitively 4-colourable [16], [17], outerplanar graphs are nonrepetitively 12-colourable [17], [18], and more generally, every graph with treewidth k is nonrepetitively 4^k-colourable [17]. Graphs with maximum degree Δ are nonrepetitively $\mathcal{O}(\Delta^2)$-colourable [19].

Perhaps the most important open problem in the field of nonrepetitive colourings is whether planar graphs have bounded nonrepetitive chromatic number [19]. The best known lower bound is 11, due to Ochem [4]. Dujmović *et al.* [4] showed that layered separations can be used to construct nonrepetitive colourings with $\mathcal{O}(\log n)$ colours.

Lemma 3 ([4])**.** *If an n-vertex graph G admits layered separations of breadth ℓ then*

$$\pi(G) \leq 4\ell(1 + \log_{3/2} n) \ .$$

Applying Lemma 1, Dujmović *et al.* [4] concluded that $\pi(G) \leq 8(1 + \log_{3/2} n)$ for every n-vertex planar graph G. Theorem 2 and Lemma 3 imply the following generalisation:

Theorem 4. *For every n-vertex graph G with Euler genus g,*

$$\pi(G) \leq 12(g+1)(1 + \log_{3/2} n) \ .$$

The previous best bound for graphs of bounded genus was $\mathcal{O}(\sqrt{n})$. In Theorem 25 below, we extend Theorem 4 to a $\mathcal{O}(\log n)$ bound for arbitrary proper minor-closed classes.

II. Tree Decompositions and Separations

For graphs G and H, an H*-decomposition* of G is a set $\{B_x \subseteq V(G) : x \in V(H)\}$ of sets of vertices in G (called *bags*) indexed by the vertices of H, such that:

(1) for every edge vw of G, some bag B_x contains both v and w, and
(2) for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H.

The *width* of a decomposition is the size of the largest bag minus 1. If H is a tree, then an H-decomposition is called a *tree decomposition*. The *treewidth* of a graph G is the minimum width of any tree decomposition of G. Separators and treewidth are closely connected.

Lemma 5 (Robertson and Seymour [20], (2.5) & (2.6))**.** *If S is a set of vertices in a graph G, then for every tree decomposition of G there is a bag B such that each connected component of $G - B$ contains at most $\frac{1}{2}|S|$ vertices in S, which implies that G has a separation (G_1, G_2) with $V(G_1) \cap V(G_2) = B$ and both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $\frac{2}{3}|S|$ vertices in S.*

The *breadth* of an H-decomposition is defined as follows. An H-decomposition $\{B_x : x \in V(H)\}$ of a graph G has *breadth* ℓ if there is a layering $(V_0, V_1, \dots, V_t)$ of G such that each bag B_x contains at most ℓ vertices in each layer V_i. Lemma 5 implies:

Lemma 6. *If a graph G has a tree decomposition with breadth ℓ then G admits layered separations of breadth ℓ.*

Tree decompositions of bounded breadth lead to tree decompositions of bounded width for graphs of bounded diameter.

Lemma 7. *If a connected graph G of diameter d has a tree decomposition with breadth ℓ, then G has treewidth at most $\ell(d+1) - 1$.*

Proof: The number of layers in any layering of G is at most $d + 1$. So each bag in the tree decomposition contains at most $\ell(d+1)$ vertices. ■

We have the following lower bound on the breadth of layered separations.

Lemma 8. *Let G be a connected graph with diameter d and treewidth k. If G admits layered separations of breadth ℓ, then $\ell \geq \frac{k}{4d+4}$.*

Proof: The number of layers in any layering of G is at most $d+1$. Since G admits layered separations of breadth ℓ, for every set $S \subseteq V(G)$, there is a separation (G_1, G_2) of G such that $V(G_1) \cap V(G_2)$ contains at most ℓ vertices in each layer, implying that $|V(G_1) \cap V(G_2)| \leq \ell(d+1)$, and both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $\frac{2}{3}|S|$ vertices in S. By a result of Reed [21, Fact 2.7], $k \leq 4\ell(d+1)$. ∎

The following useful observation describes how to produce a tree decomposition from a general H-decomposition.

Lemma 9. *Let H be a connected graph with n vertices and $n-1+c$ edges. Assume there is an H-decomposition of a graph G with breadth ℓ. Let T be a spanning tree of H. Then there is a T-decomposition of G with breadth $\ell(c+1)$.*

Proof: Let $\{B_v : v \in V(H)\}$ be an H-decomposition of G with breadth ℓ. For each vertex v of H, let B'_v be the bag obtained from B_v by adding $B_x \cap B_y$ for each of the c edges $xy \in E(H) \setminus E(T)$. This adds at most ℓc vertices in each layer to each bag. Removing such an edge xy from H breaks condition (2) of the H-decomposition precisely for vertices in $B_x \cap B_y$. Thus, by adding $B_x \cap B_y$ to every bag, it is easily seen that $\{B'_v : v \in V(H)\}$ is a T-decomposition of G. It has breadth at most $\ell(c+1)$. ∎

III. Surfaces and Clique Sums

This section constructs tree decompositions and layered separations of bounded breadth in graphs of bounded Euler genus. These results are then extended to more general graph classes via the clique-sum operation. For compatibility with this operation, we introduce the following concept that is a little stronger than saying that a graph has a tree decomposition of bounded breadth. Say a graph G is *ℓ-good* if for every minor H of G and for every vertex r of H there is a tree decomposition of H of breadth ℓ with respect to some layering of H in which $\{r\}$ is the first layer.

Theorem 10. *Every graph G with Euler genus at most g is $3(g+1)$-good.*

Proof: Since the class of graphs with Euler genus at most g is minor-closed, we may assume that $H = G$ (in the definition of good). We may assume that G is a triangulation of a surface with Euler genus at most g. Let $F(G)$ be the set of faces of G. Say G has n vertices. By Euler's formula, $|F(G)| = 2n+2g-4$ and $|E(G)| = 3n+3g-6$. Let r be the given vertex of G (in the definition of good). Let $(V_0, V_1, \dots, V_t)$ be the bfs layering of G starting from r; thus $V_0 = \{r\}$. Let T be a bfs tree of G rooted at r. Let D be the graph with vertex set $F(G)$, where two vertices are adjacent if the corresponding faces share an edge not in T. So $|V(D)| = |F(G)| = 2n+2g-4$ and $|E(D)| = (3n+3g-6)-(n-1) = 2n+3g-5$. From an Eulerian tour of the graph obtained from T by doubling each edge, it is easy to construct a walk in D that visits every face of G. Thus D is connected.

For each face $f = xyz$ of G, let B_f be the union of the xr-path in T, the yr-path in T, and the zr-path in T. Thus, B_f contains at most three vertices in each layer V_i. We claim that $\{B_f : f \in F(G)\}$ is a D-decomposition of G. For every edge vw of G, if f is a face incident to vw then v and w are in the bag B_f. It remains to show that for each vertex v of G, if F' is the set of faces f of G such that v is in B_f, then the induced subgraph $D[F']$ is connected (since F' is clearly non-empty). Let T' be the subtree of T rooted at v. Then a face f of G is in F' if and only if f is incident with a vertex in T'. From an Eulerian tour of the graph obtained from T' by doubling each edge, it is easy to construct a walk in $D[F']$ that visits every face in F'. Thus $D[F']$ is connected, and $\{B_f : f \in F(G)\}$ is a D-decomposition of G with breadth at most 3.

Let D' be a spanning tree of D. Thus D' has $|V(D)|-1 = 2n+2g-5$ edges, and D contains $(2n+3g-5)-(2n+2g-5) = g$ edges that are not in D'. By Lemma 9, there is a D'-decomposition of G with breadth $3(g+1)$ with respect to $(V_0, \dots, V_t)$. Therefore G is $3(g+1)$-good. ∎

Observe that Lemma 6 and Theorem 10 imply Theorem 2. Some historical notes on the proof of Theorem 10 are in order. A spanning tree in an embedded graph with an 'interdigitating' spanning tree in the dual is sometimes called a *tree-cotree decomposition*, and was introduced for planar graphs by von Staudt [22]. This decomposition was generalised for orientable surfaces [23] and for non-orientable surfaces [24]. Aleksandrov and Djidjev [25] used similar ideas to construct separators (they call D a *separation graph*). Eppstein [26] used this approach to show that every planar graph with radius r has treewidth at most $3r$.

A *k-clique* is a set of k pairwise adjacent vertices in a graph. Let $C_1 = \{v_1, \dots, v_k\}$ be a k-clique in a graph G_1. Let $C_2 = \{w_1, \dots, w_k\}$ be a k-clique in a graph G_2. Let G be the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for $1 \leq i \leq k$, and possibly deleting some edges in C_1 $(= C_2)$. Then G is a *k-clique sum* of G_1 and G_2. If $k \leq \ell$ then G is a *$(\leq \ell)$-clique sum* of G_1 and G_2

Lemma 11. *For $\ell \geq k$, if G is a $(\leq k)$-clique-sum of ℓ-good graphs G_1 and G_2, then G is ℓ-good.*

Proof: Let H be the given minor of G, and let r be the given vertex in H. Then H is a clique-sum of graphs H_1 and H_2, where H_1 and H_2 are minors of G_1 and G_2

respectively. Let $K := V(H_1 \cap H_2)$. We may assume that K is a clique in H since a subgraph of an ℓ-good graph is also ℓ-good. Without loss of generality, r is in H_1. Since G_1 is ℓ-good, there is a tree decomposition T_1 of H_1 of breadth at most ℓ with respect to some layering of H_1 in which $\{r\}$ is the first layer. Observe that K is contained in at most two consecutive layers of this layering of H_1. Let K' be the subset of K in the first of these two layers. Note that if r is in K then $K' = \{r\}$. Let H_2' be the graph obtained from H_2 by contracting K' into a new vertex w. Since G_2 is ℓ-good and H_2' is a minor of G_2, there is a tree decomposition T_2 of H_2' with breadth at most ℓ with respect to some layering of H_2 in which $\{w\}$ is the first layer. Replacing layer $\{w\}$ by K' gives a layering of H_2, whose first layer is K' and whose second layer contains $K \setminus K'$. Prior to this replacement, each bag in T_2 contains at most one vertex in the first layer and at most ℓ vertices in each remaining layer. After this replacement, each bag in T_2 contains at most $|K'|$ vertices in the first layer and at most ℓ vertices in each remaining layer. Since $|K'| \leq k \leq \ell$, each bag in T_2 contains at most ℓ vertices in each layer. That is, T_2 has breadth ℓ. Now, the layerings of H_1 and H_2 can be overlaid, with the layer containing K' in common, and the layer containing $K \setminus K'$ in common. By the definition of K', it is still the case that the first layer is $\{r\}$. Let T be the tree decomposition of H obtained from the disjoint union of T_1 and T_2 by adding an edge between a bag in T_1 containing K and a bag in T_2 containing K. (Each clique is contained in some bag of a tree decomposition.) For each bag B of T the intersection of B with a single layer consists of the same set of vertices as the intersection of B and the corresponding layer in the layering of H_1 or H_2. Hence T has breadth ℓ. ■

Lemma 11 is immediately applicable for H-minor-free graphs, where H has a drawing in the plane with at most one crossing; see the full paper for details.

IV. The Graph Minor Structure Theorem

This section introduces the graph minor structure theorem of Robertson and Seymour. This theorem shows that every graph in a proper minor-closed class can be constructed using four ingredients: graphs on surfaces, vortices, apex vertices, and clique-sums. We show that, with a restriction on the apex vertices, every graph that can be constructed using these ingredients has a tree decomposition of bounded breadth, and thus admits layered separations of bounded breadth.

Let G be a graph embedded in a surface Σ. Let F be a facial cycle of G (thought of as a subgraph of G). An *F-vortex* is an F-decomposition $\{V_x \subseteq V(H) : x \in V(F)\}$ of a graph H such that $V(G \cap H) = V(F)$ and $x \in V_x$ for each $x \in V(F)$. For $g, p, a \geq 0$ and $k \geq 1$, a graph G is *(g, p, k, a)-almost-embeddable* if for some set $A \subseteq V(G)$ with $|A| \leq a$, there are graphs $G_0, G_1, \dots, G_q$ with $q \leq p$ such that:

- $G - A = G_0 \cup G_1 \cup \dots \cup G_q$,
- $G_1, \dots, G_q$ are pairwise vertex disjoint;
- G_0 is embeddable in a surface of Euler genus $\leq g$,
- there are q pairwise disjoint facial cycles $F_1, \dots, F_q$ of G_0, and
- for $1 \leq i \leq q$, there is an F_i-vortex $\{V_x \subseteq V(G_i) : x \in V(F_i)\}$ of G_i of width at most k.

The vertices in A are called *apex* vertices. They can be adjacent to any vertex in G.

A graph is *k-almost-embeddable* if it is (k, k, k, k)-almost-embeddable. The following graph minor structure theorem by Robertson and Seymour is at the heart of graph minor theory. In a tree decomposition $\{B_x \subseteq V(G) : x \in V(T)\}$, the *torso* of a bag B_x is the subgraph obtained from $G[B_x]$ by adding all edges vw where $v, w \in B_x \cap B_y$ for some edge $xy \in E(T)$.

Theorem 12 (Robertson and Seymour [27])**.** *For every fixed graph H there is a constant $k = k(H)$ such that every H-minor-free graph is obtained by clique-sums of k-almost-embeddable graphs. Alternatively, every H-minor-free graph has a tree decomposition in which each torso is k-almost embeddable.*

As stated earlier, it is not the case that all graphs described by the graph minor structure theorem admit layered separations of bounded breadth. For example, let G be the graph obtained from the $\sqrt{n} \times \sqrt{n}$ grid by adding one dominant vertex. Thus, G has diameter 2, contains no K_6-minor, and has treewidth at least $\sqrt{n}$. By Lemma 8, if G admits layered separations of breadth ℓ, then $\ell \geq \Omega(\sqrt{n})$.

We will show that the following restriction to the definition of almost-embeddable leads to graph classes that admit layered separations of bounded breadth. A graph G is *strongly (g, p, k, a)-almost-embeddable* if it is (g, p, k, a)-almost-embeddable and there is no edge between an apex vertex and a vertex in $G_0 - (G_1 \cup \dots \cup G_q)$. That is, each apex vertex is only adjacent to other apex vertices or vertices strictly in the vortices. A graph is *strongly k-almost-embeddable* if it is strongly (k, k, k, k)-almost-embeddable.

Theorem 13. *Every strongly (g, p, k, a)-almost-embeddable graph G is $(g+p+1)(a+3k+3)$-good (and thus has a tree decomposition and admits layered separations of breadth $(g + p + 1)(a + 3k + 3)$).*

The proof of Theorem 13, which builds on the proof of Theorem 10, can be found in the full version of the paper.

Theorem 13 and Lemmas 6 and 7 imply:

Corollary 14. *Let G be a graph obtained by clique-sums of strongly k-almost-embeddable graphs. Then:*

(a) G has a tree decomposition and admits layered separations of breadth $(2k + 1)(4k + 3)$,

(b) if G has diameter d then G has treewidth at most $(2k+1)(4k+3)(d+1)-1$.

Corollary 14(b) improves upon a result by Grohe [28, Proposition 10] who proved an upper bound on the treewidth of $f(k)\,d$, where $f(k) \approx k^k$. Moreover, this result of Grohe [28] assumes there are no apex vertices. That is, it is for clique-sums of $(k,k,k,0)$-almost-embeddable graphs.

V. Track and Queue Layouts

The main result of this section is expressed in terms of track layouts of graphs, a type of graph layout that is closely related to queue layouts and 3-dimensional grid drawings. A *vertex $|I|$-colouring* of a graph G is a partition $\{V_i : i \in I\}$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. The elements of the set I are *colours*, and each set V_i is a *colour class*. Suppose that $\preceq_i$ is a total order on each colour class V_i. Then each pair $(V_i, \preceq_i)$ is a *track*, and $\{(V_i, \preceq_i) : i \in I\}$ is an $|I|$-*track assignment* of G. The *span* of an edge vw in a track assignment $\{(V_i, \preceq_i) : 1 \leq i \leq t\}$ is $|i-j|$ where $v \in V_i$ and $w \in V_j$.

An *X-crossing* in a track assignment consists of two edges vw and xy such that $v \prec_i x$ and $y \prec_j w$, for distinct colours i and j. A t-track assignment of G that has no X-crossings is called a *t-track layout* of G. The minimum t such that a graph G has t-track layout is called the *track-number* of G, denoted by $\mathrm{tn}(G)$. Queue and track layouts are closely related, in that Dujmović *et al.* [12] proved that $\mathrm{qn}(G) \leq \mathrm{tn}(G)-1$. Conversely, Dujmović *et al.* [29] proved that $\mathrm{tn}(G) \leq f(\mathrm{qn}(G))$ for some function f. In this sense, queue-number and track-number are tied. The main result of this section is the following connection between layered separators and track layouts.

Theorem 15. *If a graph G admits layered separations of breadth ℓ then*

$$\mathrm{qn}(G) < \mathrm{tn}(G) \leq 3\ell(\lceil \log_{3/2} n \rceil + 1) \ .$$

Proof: Let $L = (V_0, V_1, \ldots, V_t)$ be the corresponding layering of G. Let S be a subset of $V(G)$. By assumption, there is a separation (G_1, G_2) of G such that each layer V_i contains at most ℓ vertices in $V(G_1) \cap V(G_2) \cap S$, and both $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ contain at most $\frac{2}{3}|S|$ vertices in S. We call $V(G_1) \cap V(G_2)$ an (ℓ, L)-separator of $G[S]$.

With $S = V(G)$, removing such a separator from G splits G into connected components each of which has at most $\frac{2}{3}|V(G)|$ vertices and its own (ℓ, L)–separator. Thus the process can continue until each connected component is an (ℓ, L)–separator of itself. This process naturally defines a rooted tree S and a mapping of $V(G)$ to the nodes of S, as follows. The root of S is a node to which the vertices of an (ℓ, L)–separator of G are mapped. The root has $c \geq 1$ children in S, one for each connected component G_j, $j \in [1, c]$, obtained by removing the (ℓ, L)–separator from G. The vertices of an (ℓ, L)–separator of G_j, $j \in [1, c]$, are mapped to a child of the root. The process continues until each component is an (ℓ, L)–separator of itself, or more specifically until each component has at most ℓ-vertices in each layer of L. In that case, such a component is an (ℓ, L)–separator of itself and its vertices are mapped to a leaf of S. This defines a rooted tree S and a partition of V to the nodes of S. One important observation, is that the height of S is most $\lceil \log_{3/2} n \rceil + 1$. For a node s of S, let $s(G)$ denote the set of vertices of G that are mapped to s and let $G[s]$ denote the graph induced by $s(G)$ in G. Note that for each node s of S, $s(G)$ has at most ℓ vertices in any layer of L.

To prove our result we first create a track layout T of G with possibly many tracks. We then modify that layout in order to reduce the number of tracks to $\mathcal{O}(\ell \log n)$.

To ease the notation, for a track $(V_r, \preceq_r)$, indexed by colour r, in a track assignment R, we denote that track by (r) when the ordering on each colour class is implicit. Also we sometimes write $v \preceq_R w$. This indicates that v and w are on a same track r of R and that $v \preceq_r w$.

Throughout this proof, it is important to keep in mind that a *layer* is a subset of vertices of G defined by the layering L and that a *track* is an (ordered) subset of vertices of G defined by a track assignment of G.

We first define a track assignment T of G; see Figure 1. Each vertex v of $V(G)$ is assigned to a track whose colour is defined by three indices (d, i, k). Let s_v denote the node of the tree S that v is mapped to. The first index is the depth of s_v in S. The root is considered to have depth 1. Thus the first index, d, ranges from 1 to $\lceil \log_{3/2} n \rceil + 1$. The second index is the layer of L that contains v. Thus the second index, i, can be as big as $\Omega(n)$. Finally, $s_v(G)$ contains at most ℓ vertices from layer i in L. Label these, at most ℓ, vertices arbitrarily from 1 to ℓ and let the third index k of each of them be determined by this label. Consider the tracks themselves to be lexicographically ordered.

To complete the track assignment we need to define the ordering of vertices in the same track. To do that we first define a simple track layout of the tree S. Consider a natural way to draw S in the plane without crossings such that all the nodes of S that are at the same distance from the root are drawn on the same horizontal line. This defines a track layout T_S of S where each horizontal line is a track and the ordering of the nodes within each track is implied by the crossing free drawing of S.

To complete the track assignment T, we need to define the total order of vertices that are in the same track of T. For any two vertices v and w of G that are assigned to the same track (d, i, k) in T, let $v <_T w$ if the node s_v that v maps to in S appears in T_S to the left of the node s_w that w maps to in S, that is, if $s_v <_{T_S} s_w$. Since v and w are in the same track of T only if they are mapped two distinct nodes of S that are the the same distance from the root of S, this defines a total order of each track in T. Figure 1 depicts the

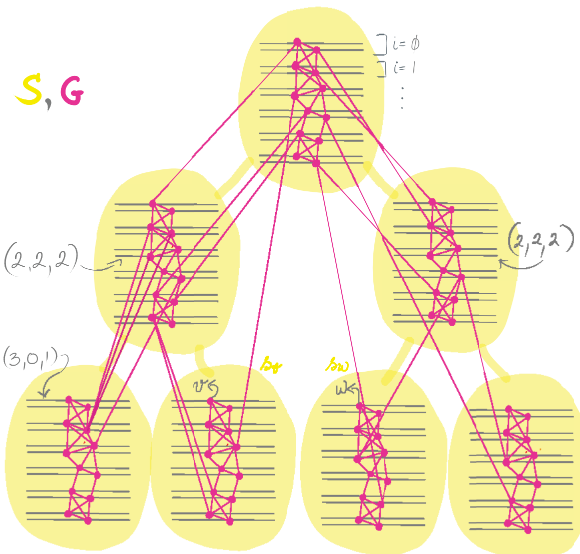

Figure 1. A track layout T of a graph G which has a layered ($\ell = 2$)–separator.

resulting track assignment T of G.

It is not difficult to verify that T is indeed a track layout of G, that is, T does not have X-crossings. This track layout however may have $\Omega(n)$ tracks. We now modify T to reduce the number of tracks to the claimed number. For a vertex v of G, let (d_v, i_v, k_v) denote the track of v in T.

Dujmović *et al.* [29], inspired by Felsner *et al.* [30], proved that a track layout with maximum span s can be wrapped into a $(2s+1)$–track layout. Unfortunately, the track layout T of G does not have bounded span—its span can be $\Omega(n)$. (Since the tracks of T are ordered by lexicographical ordering, span is well defined in T.) However parts of the layout do have bounded span. In particular, consider the graph, G_d induced by the vertices of G that are assigned to the tracks of T that have the same first index, d. For each d, the tracks of T with that first index equal to d, define a track layout T_d of G_d, as illustrated, for $d = 2$ case, in the top part of Figure 2. Recall that G_d is comprised of disjoint layered (l, L)-separators (see Figure 1). Since each edge in a layered (l, L)-separator either connects two vertices in the same layer of L or two vertices from two consecutive layers of L, the span of an edge of G_d in T_d is at most $2\ell - 1$.

For each d, we now wrap the track layout T_d into a 3ℓ-track layout T'_d of G_d, as illustrated in Figure 2.

Lemma 16. *[30], [29] Let T denote a track layout of a graph G with tracks in T indexed by (i, k) where $i \in \{0, \ldots, p\}$ and $k \in \{1, \ldots, \ell\}$ and such that for each edge vw of G, with v in track (i_v, k_v) and w in (i_w, k_w), $|i_v - i_w| \leq 1$. Then T can be modified (wrapped) into a 3ℓ track layout T' of G as follows: Each vertex v of G is assigned to a track $(i_v \mod 3, k_v)$ and two vertices v and*

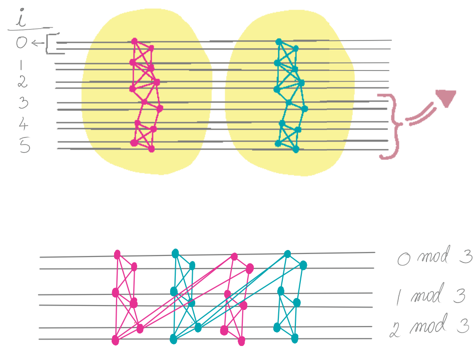

Figure 2. Top figure: the track layout T_2 of G_2. Bottom figure: the track layout T'_2 obtained by wrapping T_2.

x that are in the same track of T' are ordered as follows. Let $i_v \leq i_x$.
(1) If $i_v < i_x$, then $v <_{T'} x$.
(2) Otherwise, ($i_v = i_x$), v and x are ordered in T' as in T.

The proof of Lemma 16 mimics the wrapping lemmas of Felsner *et al.* [30] and Dujmović *et al.* [29], and is omitted. This defines a track assignment T' of G. Lemma 16 implies that for all d, T'_d has the following useful properties. Consider two vertices a and b that are in the same track $f' = (d_a, i_a \mod 3, k_a) = (d_b, i_b \mod 3, k_b)$ in T'_d. Then if $a <_{f'} b$ in T'_d and

(1) $i_a \neq i_b$ then $i_b \geq i_a + 3$,
(2) otherwise, ($i_a = i_b$), a and b were in the same track f in T and $a <_f b$. (This is because the wrapping does not change the ordering of vertices that were already in the same track in T).

Since $d \leq \lceil \log_{3/2} n \rceil + 1$, $i \mod 3 \leq 3$ and $k \leq \ell$, the track assignment T' of G has at most $3\ell \lceil \log_{3/2} n \rceil + 1$ tracks, as claimed. It remains to prove that T' is in fact a track layout of G, that is, there are no X-crossings in the track assignment T'

Assume by contradiction that there are two edges vw and xy that form an X-crossing in T'. Let v and x belong to a same track in T' and let y and w belong to a same track in T'. If $d_v = d_w = d_x = d_y$, then v, w, x and y belong to the the same graph G_d and thus they do not form and X-crossing since T'_d does not have X-crossings by the wrapping lemma.

Thus $d_v = d_x = d_1$ and $d_w = d_y = d_2$ and $d_1 \neq d_2$. Let without loss of generality $d_1 < d_2$ and $v <_{T'} x$ and $y <_{T'} w$ in T'. Since w and y are in the same track, $d_y = d_w$, $k_y = k_w$ and either $i_y = i_w$ or $i_w \geq i_y + 3$ by properties (1) and (2). There are thus two cases to consider. First consider the case that $i_w \geq i_y + 3$. Since w is adjacent to v, $i_v = \{i_w - 1, i_w, i_w + 1\}$ and similarly $i_x = \{i_y - 1, i_y, i_y + 1\}$. Thus $i_v \geq i_w - 1 \geq i_y + 2$ and $i_x \leq i_y + 1$. Thus, $i_v > i_x$ and property (1) applies to v and x. This contradicts the assumption that $v <_{T'} x$, since property (1), implies that $i_x > i_v$.

Finally, consider the case that $i_y = i_w$. Then y and w are in the same track in T and their ordering, $y <_{T'} w$, in T' is the same as in T, $y <_T w$, by property (2). Since v and x are in the same track in T' and $v <_{T'} x$, either $i_v = i_x$ or $i_x \geq i_v + 3$, by properties (1) and (2). Thus again, since w is adjacent to v, $i_v = \{i_w - 1, i_w, i_w + 1\}$ and similarly $i_x = \{i_y - 1, i_y, i_y + 1\}$. Since $i_y = i_w$, no pair of these indices differs by at least 3 and thus $i_v = i_x$ by property (1). That implies that v and x are in the same track in T and thus by property (2) their ordering in T', $v <_{T'} x$, is the same as in T, $v <_T x$. This implies that vw and xy form an X-crossing in T thus providing the desired contradiction. This completes the proof of Theorem 15. ■

Theorem 15 and Lemma 1 imply:

Theorem 17. *Every n-vertex planar graph has*

$$\mathrm{qn}(G) < \mathrm{tn}(G) \leq 6\lceil \log_{3/2} n \rceil + 6 \ .$$

This bound on $\mathrm{qn}(G)$ was improved to $\mathrm{qn}(G) \leq 4\lceil \log_{3/2} n \rceil$ by Fabrzio Frati [personal communication, 2013]. Theorem 2 and Theorem 15 implies the following generalisation of these results.

Theorem 18. *For every n-vertex graph with Euler genus g,*

$$\mathrm{qn}(G) < \mathrm{tn}(G) \leq 9(g+1)(\lceil \log_{3/2} n \rceil + 1) \ .$$

Theorem 18 is extended to arbitrary minor-closed classes in Section VI.

Our results for 3-dimensional graph drawings are based on the following connection with track layouts.

Lemma 19 ([12], [10]). *If a c-colourable n-vertex graph G has a t-track layout then G has 3-dimensional grid drawings with $\mathcal{O}(t^2 n)$ volume and with $\mathcal{O}(c^7 tn)$ volume.*

Every graph with Euler genus g is $\mathcal{O}(\sqrt{g})$-colourable. Thus Theorem 18 and Lemma 19 imply:

Theorem 20. *Every n-vertex graph with Euler genus g has a 3-dimensional grid drawing with volume $\mathcal{O}(g^{9/2} n \log n)$.*

The best previous upper bound on the volume of 3-dimensional grid drawings of graph with bounded Euler genus was $\mathcal{O}(n^{3/2})$ by Dujmović *et al.* [10]. Theorem 20 is extended to arbitrary minor-closed classes with an $n \log^{\mathcal{O}(1)} n$ volume bound in Section VI.

VI. Arbitrary Minor-Closed Classes

As observed in Section IV, it is not the case that graphs in any proper minor-closed class admit layered separations of bounded breadth. However, in this section we extend our methods from previous sections to prove that graphs from any proper minor-closed class have nonrepetitive chromatic number $\mathcal{O}(\log n)$, track/queue-number $\log^{\mathcal{O}(1)} n$, and 3-dimensional grid drawings with $n \log^{\mathcal{O}(1)} n$ volume.

In a layering $(V_0, V_1, \dots, V_t)$ of a graph G the *shadow* of a subgraph H of $G[V_i]$ is the set of vertices in V_{i-1} adjacent to H, and $(V_0, V_1, \dots, V_t)$ is *shadow complete* if for each layer V_i and each connected component H of $G[V_i]$, the shadow of H is a clique. This concept was introduced by Kündgen and Pelsmajer [17] and implicitly by Dujmović etal [12]. It is a key to the proof that graphs of bounded treewidth have bounded nonrepetitive chromatic number [17] and bounded track-number [12].

A tree decomposition $(B_x \subseteq V(G) : x \in V(T))$ of a graph G is *k-rich* if $B_x \cap B_y$ is a clique in G on at most k vertices, for each edge $xy \in E(T)$. The following lemma generalises a result by Kündgen and Pelsmajer [17], who proved it when each bag of the tree decomposition is a clique (that is, for chordal graphs). We allow bags to induce more general graphs. For example, in Theorems 23 and 25 below each bag induces an ℓ-almost embeddable graph.

For a subgraph H of a graph G, a tree decomposition $(C_y \subseteq V(H) : y \in V(F))$ of H is *contained in* a tree decomposition $(B_x \subseteq V(G) : x \in V(T))$ of G if for each bag C_y there is bag B_x such that $C_y \subseteq B_x$.

Lemma 21. *Let G be a graph with a k-rich tree decomposition $\mathcal{T}$. Then G has a shadow complete layering $(V_0, \dots, V_t)$ such that for each layer V_i, the subgraph $G[V_i]$ has a $(k-1)$-rich forest decomposition contained in $\mathcal{T}$.*

Proof: We may assume that G is connected with at least one edge. Say $\mathcal{T} = (B_x \subseteq V(G) : x \in V(T))$ is a k-rich tree decomposition of G. If $B_x \subseteq B_y$ for some edge $xy \in E(T)$, then contracting xy into y (and keeping bag B_y) gives a new k-rich tree decomposition of G. Moreover, if a tree decomposition of a subgraph of G is contained in the new tree decomposition of G, then it is contained in the original. Thus, we may assume that $B_x \not\subseteq B_y$ and $B_y \not\subseteq B_x$ for each edge $xy \in V(T)$.

Let G' be the graph obtained from G by adding an edge between every pair of vertices in a common bag (if the edge does not already exist). Let r be a vertex of G. Let α be a node of T such that $r \in B_\alpha$. Root T at α. Now every non-root node of T has a parent node. Let $V_0 := \{r\}$. Let t be the eccentricity of r in G'. For $1 \leq i \leq t$, let V_i be the set of vertices of G at distance i from r in G'. Since G is connected, G' is connected. Thus $(V_0, V_1, \dots, V_t)$ is a layering of G' and also of G (since $G \subseteq G'$).

Since each bag B_x is a clique in G', V_1 is the set of vertices of G in bags that contain r (not counting r itself). More generally, V_i is the set of vertices of G in bags that intersects V_{i-1} but are not in $V_0 \cup \dots \cup V_{i-1}$.

Define $B'_\alpha := B_\alpha \setminus \{r\}$ and $B''_\alpha := \{r\}$. For a non-root node $x \in V(T)$ with parent node y, define $B'_x := B_x \setminus B_y$ and $B''_x := B_x \cap B_y$. Since $B_x \not\subseteq B_y$, we have $B'_x \neq \emptyset$.

Consider a node x of T. Since B_x is a clique in G', B_x is contained in at most two consecutive layers. Consider (not necessarily distinct) vertices $u, v \in B'_x$, which is not empty.

Then the distance between u and r in G' equals the distance between v and r in G'. Thus B'_x is contained in one layer, say $V_{\ell(x)}$. Let w be the neighbour of v in some shortest path between B'_x and r in G'. Then w is in $B''_x \cap V_{\ell(x)-1}$. In conclusion, each bag B_x is contained in precisely two consecutive layers, $V_{\ell(x)-1} \cup V_{\ell(x)}$, such that $\emptyset \neq B'_x \subseteq V_{\ell(x)}$ and $B_x \cap V_{\ell(x)-1} \subseteq B''_x \neq \emptyset$. Also, observe that if y is an ancestor of x in T, then $\ell(y) \leq \ell(x)$. Call this property $(\star)$.

The claim in the lemma is trivial for $i = 0$. So assume $1 \leq i \leq t$. Let T_i be the subforest of T induced by the nodes x such that $\ell(x) = i$. We claim that $\mathcal{T}_i := \{B_x \cap V_i : x \in V(T_i)\}$ is a T_i-decomposition of $G[V_i]$. First we prove that each vertex $v \in V_i$ is in some bag of $\mathcal{T}_i$. Let x be the node of T closest to α, such that $v \in B_x$. Then $v \in B'_x$ and $\ell(x) = i$. Hence v is in the bag $B_x \cap V_i$ of $\mathcal{T}_i$, as desired. Now we prove that for each edge $vw \in E(G[V_i])$, both v and w are in a common bag of $\mathcal{T}_i$. Let x be the node of T closest to α, such that $v \in B_x$. Let y be the node of T closest to α, such that $w \in B_y$. Since v and w appear in a common bag of $\mathcal{T}$, without loss of generality, x is on the $y\alpha$-path in T. Thus $w \in B'_y$ and $y \in V(T_i)$. Moreover, v is also in B_y (since v and w are in a common bag of $\mathcal{T}$). Thus, v and w are in the bag $B_y \cap V(H)$ of $\mathcal{F}$, as desired. Finally, we prove that for each vertex $v \in V_i$, the set of bags in $\mathcal{T}_i$ that contain v correspond to a (connected) subtree of T_i. By assumption, this property holds in T. Let X be the subtree of T whose corresponding bags in $\mathcal{T}$ contain v. Let x be the root of X. Then $v \in B'_x$ and $\ell(x) = i$. By property $(\star)$, $\ell(z) \in \{i, i+1\}$ for each node z in X. Moreover, deleting from X the nodes z such that $\ell(z) = i+1$ leaves a connected subtree of X, which is precisely the subtree of T_i whose bags in $\mathcal{T}_i$ contain v. Hence $\mathcal{T}_i := \{B_x \cap V_i : x \in V(T_i)\}$ is a T_i-decomposition of $G[V_i]$. By definition, $\mathcal{T}_i$ is contained in $\mathcal{T}$.

We now prove that $\mathcal{T}_i$ is $(k-1)$-rich. Consider an edge $xy \in E(T_i)$. Without loss of generality, y is the parent of x in T_i. Our goal is to prove that $B_x \cap B_y \cap V_i$ is a clique on at most $k-1$ vertices. Certainly, it is a clique on at most k vertices, since $\mathcal{T}$ is k-rich. Now, $\ell(x) = i$ (since $x \in V(T_i)$). Thus $B'_x \subseteq V_i$ and $B'_x \neq \emptyset$. Let v be a vertex in B'_x. Let w be the neighbour of v on a shortest path in G' between v and r. Thus w is in $B''_x \cap V_{i-1}$. Thus $|B''_x \cap V_i| \leq k-1$, as desired. Hence $\mathcal{T}_i$ is $(k-1)$-rich.

We now prove that $(V_0, V_1, \ldots, V_t)$ is shadow complete. Consider a layer V_i where $1 \leq i \leq t$. Let H be a connected component of $G[V_i]$. Let X be the subtree of T_i whose corresponding bags in $\mathcal{T}_i$ intersect $V(H)$. Since H is connected, X is indeed a connected subtree of T_i. By construction, $\ell(z) = i$ for each node $z \in V(X)$. Let x be the root of X. Let v be a vertex of H, and let w be a neighbour of v in V_{i-1}. (That is, w is in the shadow of H.) Let y be the node closest to x in X, such that $v \in B_y$. Then $v \in B'_y$ and $w \in B''_y$. Since $\ell(z) = i$ for each node z in the yx-path in X, we have $w \in B''_z$ for each such node z. In particular, $w \in B''_x$. Since B''_x is a clique, the shadow of H Is a clique. Hence $(V_0, V_1, \ldots, V_t)$ is shadow complete. ∎

To apply Lemma 21 in the construction of track layouts we use the following lemma, which is implicit in [12].

Lemma 22 ([12])**.** *For some number c, let $\mathcal{G}_0$ be the class of graphs with track-number at most c. For $k \geq 1$, let $\mathcal{G}_k$ be the class of graphs that have a shadow complete layering with the property that each layer induces a graph in $\mathcal{G}_{k-1}$. Then every graph in $\mathcal{G}_k$ has track-number and queue-number at most $6^k\, c^{(k+1)!}$.*

Theorem 23. *For every fixed graph H, every H-minor-free n-vertex graph has track-number and queue-number at most $\log^{\mathcal{O}(1)} n$.*

Proof: By Theorem 12, there are constants $k \geq 1$ and $\ell \geq 1$ depending only on H, such that every H-minor-free graph is a subgraph of a graph in $\mathcal{G}_k$, where $\mathcal{G}_k$ is the class of graphs that have a k-rich tree decomposition such that each bag induces an ℓ-almost embeddable subgraph.

Consider a graph $G \in \mathcal{G}_0$ with at most n vertices. Then G is the disjoint union of ℓ-almost embeddable graphs. To layout one ℓ-almost embeddable graph, put each of the at most ℓ apex vertices on its own track, and layout the remaining graph with $3(2\ell+1)(4\ell+3)(\lceil \log_{3/2} n \rceil + 1)$ colours by Corollary 14 andTheorem 15. (Here we do not use the clique-sums in Corollary 14.) Of course, the track-number of a graph is the maximum track-number of its connected components. Thus G has track-number at most $\ell + 3(2\ell+1)(4\ell+3)(\lceil \log_{3/2} n \rceil + 1)$.

Let G be an n-vertex graph in $\mathcal{G}_k$. Let $\mathcal{T}$ be a k-rich tree decomposition of G such that each bag induces an ℓ-almost embeddable subgraph. By Lemma 21, G has a shadow complete layering $(V_0, \ldots, V_t)$ such that for each layer V_i, the induced subgraph $G[V_i]$ has a $(k-1)$-rich tree decomposition $\mathcal{T}_i$ contained in $\mathcal{T}$. Since $\mathcal{T}_i$ is contained in $\mathcal{T}$, each bag of $\mathcal{T}_i$ induces an ℓ-almost embeddable subgraph. That is, each layer V_i induces a graph in $\mathcal{G}_{k-1}$. By Lemma 22 with $c = \ell + 3(2\ell+1)(4\ell+3)(\lceil \log_{3/2} n \rceil + 1)$, our graph G has track-number at most $6^k(\ell + 3(2\ell+1)(4\ell+3)(\lceil \log_{3/2} n \rceil + 1))^{(k+1)!}$, which is in $\mathcal{O}(\log^p n)$ for some constant p depending only on H. ∎

Lemma 19 and Theorem 23 imply:

Theorem 24. *For every fixed graph H, every H-minor-free n-vertex graph has a 3-dimensional grid drawing with volume $n \log^{\mathcal{O}(1)} n$.*

The best previous volume bound for H-minor-free graphs was $\mathcal{O}(n^{3/2})$ [10].

The next theorem is our main result about nonrepetitive colourings. Its proof is analogous to that of Theorem 23 using Lemma 26 below in place of Lemma 22.

Theorem 25. *For every fixed graph H, every H-minor-free n-vertex graph is nonrepetitively $\mathcal{O}(\log n)$-colourable.*

The following lemma is implicit in the work of Kündgen and Pelsmajer [17].

Lemma 26 ([17])**.** *For some number c, let $\mathcal{G}_0$ be the class of graphs with nonrepetitive chromatic number at most c. For $k \geq 1$, let $\mathcal{G}_k$ be the class of graphs that have a shadow complete layering with the property that each layer induces a graph in $\mathcal{G}_{k-1}$. Then every graph in $\mathcal{G}_k$ has nonrepetitive chromatic number at most $c4^k$.*

ACKNOWLEDGMENT

Thanks to Gwenaël Joret, Bruce Reed and Paul Seymour for helpful discussions. This research was partially completed at Bellairs Research Institute in Barbados.

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