Majority Colourings of Digraphs

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Abstract. We prove that every digraph has a vertex 4-colouring such that for each vertex v, at most half the out-neighbours of v receive the same colour as v. We then obtain several results related to the conjecture obtained by replacing 4 by 3.

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1 Introduction

A *majority colouring* of a digraph is a function that assigns each vertex v a colour, such that at most half the out-neighbours of v receive the same colour as v. In other words, more than half the out-neighbours of v receive a colour different from v (hence the name 'majority'). Whether every digraph has a majority colouring with a bounded number of colours was posed as an open problem on mathoverflow [7]. In response, Ilya Bogdanov proved that a bounded number of colours suffice for tournaments. The following is our main result.

Theorem 1. *Every digraph has a majority 4-colouring.*

Proof. Fix a vertex ordering. First, 2-colour the vertices left-to-right so that for each vertex v, at most half the out-neighbours of v to the left of v in the ordering receive the same colour as v. Second, 2-colour the vertices right-to-left so that for each vertex v, at most half the out-neighbours of v to the right of v in the ordering receive the same colour as v. The product colouring is a majority 4-colouring. □

Note that this proof implicitly uses two facts: (1) every digraph has an edge-partition into two acyclic subgraphs, and (2) every acyclic digraph has a majority 2-colouring.

The following conjecture naturally arises:

Conjecture 2. *Every digraph has a majority 3-colouring.*

September 1, 2016

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Research supported by the Australian Research Council.

This conjecture would be best possible. For example, a majority colouring of an odd directed cycle is proper (since each vertex has out-degree 1), and therefore three colours are necessary. There are examples with large outdegree as well. For odd $k \geqslant 1$ and prime $n \gg k$, let G be the directed graph with $V(G) = \{v_0, \dots, v_{n-1}\}$ where $N^+_G(v_i) = \{v_{i+1}, \dots, v_{i+k}\}$ and vertex indices are taken modulo n. Suppose that G has a majority 2-colouring. If some sequence $v_i, v_{i+1}, \dots, v_{i+k}$ contains more than $\frac{k+1}{2}$ vertices of one colour, say red, and v_i is the leftmost red vertex in this sequence, then more than $\frac{k-1}{2}$ out-neighbours of v_i are red, which is not allowed. Thus each sequence $v_i, v_{i+1}, \dots, v_{i+k}$ contains exactly $\frac{k+1}{2}$ vertices of each colour. This implies that v_i and v_{i+k+1} receive the same colour, as otherwise the sequence $v_{i+1}, \dots, v_{i+k+1}$ would contain more than $\frac{k+1}{2}$ vertices of the colour assigned to v_{i+k+1}. For all vertices v_i and v_j, if $\ell = \frac{j-i}{k+1}$ in the finite field $\mathbb{Z}_n$, then $j = i + \ell(k+1)$ and $v_i, v_{i+(k+1)}, v_{i+2(k+1)}, \dots, v_{i+\ell(k+1)} = v_j$ all receive the same colour. Thus all the vertices receive the same colour, which is a contradiction. Hence the claimed 2-colouring does not exist.

Note that being majority c-colourable is not closed under taking induced subgraphs. For example, let G be the digraph with $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc, ca, cd\}$. Then G has a majority 2-colouring: colour a and c by 1 and colour b and d by 2. But the subdigraph induced by $\{a, b, c\}$ is a directed 3-cycle, which has no majority 2-colouring.

The remainder of the paper takes a probabilistic approach to Conjecture 2, proving several results that provide evidence for Conjecture 2. A probabilistic approach is reasonable, since in a random 3-colouring, one would expect that a third of the out-neighbours of each vertex v receive the same colour as v. So one might hope that there is enough slack to prove that for *every* vertex v, at most half the out-neighbours of v receive the same colour as v. Section 2 proves Conjecture 2 for digraphs with very large minimum outdegree (at least logarithmic in the number of vertcies), and then for digraphs with large minimum outdegree (at least a constant) and not extremely large maximum indegree. Section 3 shows that large minimum outdegree (at least a constant) is sufficient to prove the existence of one of the colour classes in Conjecture 2. Section 4 discusses multi-colour generalisations of Conjecture 2.

Before proceeding, we mention some related topics in the literature:

- For undirected graphs, the situation is much simpler. Lovász [4] proved that for every undirected graph G and integer $k \geqslant 1$, there is a k-colouring of G such that every vertex v has at most $\frac{1}{k}\deg(v)$ neighbours receiving the same colour as v. The proof is simple. Consider a k-colouring of G that minimises the number of monochromatic edges. Suppose that some vertex v coloured i has greater than $\frac{1}{k}\deg(v)$ neighbours coloured i. Thus less than $\frac{k-1}{k}\deg(v)$ neighbours of v are not coloured i, and less than $\frac{1}{k}\deg(v)$ neighbours of v receive some colour $j \neq i$. Thus, if v is recoloured j, then the number of monochromatic edges decreases. Hence no vertex v has greater than $\frac{1}{k}\deg(v)$ neighbours with the same colour as v.

- Seymour [6] considered digraph colourings such that every non-sink vertex receives a colour different from some outneighbour, and proved that a strongly-connected digraph G admits a 2-colouring with this property if and only G has an even directed cycle. The proof shows that every digraph has such a 3-colouring, which we repeat here: We may assume that G is strongly connected. In particular, there are no sink vertices. Choose a maximal set X of vertices such that $G[X]$ admits a 3-colouring where every vertex has a colour different from some outneighbour. Since any directed cycle admits such a colouring, $X \neq \emptyset$. If $X \neq V(G)$, then choose an edge uv entering X and colour u different from the colour of v, contradicting the maximality of X. So $X = V(G)$. (The same proof show two colours suffice if you start with an even cycle.)

- Alon [1, 2] posed the following problem: Is there a constant c such that every digraph with minimum outdegree at least c can be vertex-partitioned into two induced digraphs, one with minimum outdegree at least 2, and the other with minimum outdegree at least 1?

- Wood [8] proved the following edge-colouring variant of majority colourings: For every digraph G and integer $k \geqslant 2$, there is a partition of $E(G)$ into k acyclic subgraphs such that each vertex v of G has outdegree at most $\lceil \frac{\deg^+(v)}{k-1} \rceil$ in each subgraph. The bound $\lceil \frac{\deg^+(v)}{k-1} \rceil$ is best possible, since in each acyclic subgraph at least one vertex has outdegree 0.

2 Large Outdegree

We now show that minimum outdegree at least logarithmic in the number of vertices is sufficient to guarantee a majority 3-colouring. All logarithms are natural.

Theorem 3. *Every graph G with n vertices and minimum outdegree $\delta > 72\log(3n)$ has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.*

Proof. Randomly and independently colour each vertex of G with one of three colours $\{1, 2, 3\}$. Consider a vertex v with out-degree d_v. Let $X(v, c)$ be the random variable that counts the number of out-neighbours of v coloured c. Of course, $\mathbf{E}(X(v, c)) = d_v/3$. Let $A(v, c)$ be the event that $X(v, c) > d_v/2$. Note that $X(v, c)$ is determined by d_v independent trials and changing the outcome of any one trial changes $X(v, c)$ by at most 1. By the simple concentration bound[1],

$$\mathbf{P}(A(v, c)) \leqslant \exp(-(d_v/6)^2/2d_v) = \exp(-d_v/72) \leqslant \exp(-\delta/72).$$

[1] The simple concentration bound says that if X is a random variable determined by d independent trials, such that changing the outcome of any one trial can affect X by at most c, then $\mathsf{P}(X > \mathsf{E}(X) + t) \leqslant \exp(-t^2/2c^2d)$; see [5, Chapter 10]. With $\mathsf{E}(X_v) = d_v/3$ and $t = d_v/6$ and $c = 1$ we obtain the desired upper bound on $\mathsf{P}(X_v > d_v/2)$.

The expected number of events $A(v,c)$ that hold is

$$\sum_{v\in V(G)}\sum_{c\in\{1,2,3\}}\mathbf{P}(A(v,c))\leqslant 3n\exp(-\delta/72)<1,$$

where the last inequality holds since $\delta > 72\log(3n)$. Thus there exists colour choices such that no event $A(v,c)$ holds. That is, a majority 3-colouring exists. □

The following result shows that large outdegree (at least a constant) and not extremely large indegree is sufficient to guarantee a majority 3-colouring.

Theorem 4. *Every digraph with minimum out-degree $\delta \geqslant 1200$ and maximum in-degree at most $\exp(\delta/72)/12\delta$ has a majority 3-colouring. Moreover, at most half the out-neighbours of each vertex receive the same colour.*

Proof. We assume $\delta \geqslant 1200$, as otherwise the minimum out-degree δ is greater than the maximum in-degree $\exp(\delta/72)/12\delta$, which does not make sense.

We use the following weighted version of the Local Lemma [3, 5]: Let $\mathcal{A} := \{A_1,\dots,A_n\}$ be a set of 'bad' events, such that each A_i is mutually independent of $\mathcal{A}\setminus(D_i\cup\{A_i\})$, for some subset $D_i \subseteq A$. Assume there are numbers $t_1,\dots,t_n \geqslant 1$ and a real number $p\in[0,\frac{1}{4}]$ such that for $1\leqslant i\leqslant n$,

$$(a)\ \mathbf{P}(A_i)\leqslant p^{t_i}\quad\text{and}\quad (b)\sum_{A_j\in\mathcal{D}_i}(2p)^{t_j}\leqslant t_i/2.$$

Then with positive probability no event A_i occurs.

Define $p := \exp(-\delta/72)$. Since $\delta \geqslant 1200$ we have $p \in [0, \frac{1}{4}]$. Randomly and independently colour each vertex of G with one of three colours $\{1, 2, 3\}$. Consider a vertex v with out-degree d_v. Let $X(v,c)$ be the random variable that counts the number of out-neighbours of v coloured c. Of course, $\mathbf{E}(X(v,c)) = d_v/3$. Let $A(v,c)$ be the event that $X(v,c) > d_v/2$. Let $\mathcal{A} := \{A(v,c) : v\in V(G), c\in\{1,2,3\}\}$ be our set of events. Let $t(v,c) := t_v := d_v/\delta$ be the associated weight. Then $t_v \geqslant 1$. It suffices to prove that conditions (a) and (b) hold.

Note that $X(v,c)$ is determined by d_v independent trials and changing the outcome of any one trial changes $X(v,c)$ by at most 1. By the simple concentration bound,

$$\mathbf{P}(A(v,c))\leqslant \exp(-(d_v/6)^2/2d_v)=\exp(-d_v/72)=\exp(-\delta t_v/72)=p^{t_v}.$$

Thus condition (a) is satisfied. For each event $A(v,c)$ let $D(v,c)$ be the set of all events $A(w,c')\in\mathcal{A}$ such that v and w have a common out-neighbour. Then $A(v,c)$ is mutually independent of $\mathcal{A} \setminus (D(v,c) \cup \{A(v,c)\})$. Since $t_w \geqslant 1$,

$$\sum_{A(w,c')\in D(v,c)} (2p)^{t_w} \leqslant \sum_{A(w,c')\in D(v,c)} (2p)^1 = 2p|D(v,c)|.$$

Since each out-neighbour of v has in-degree at most $\exp(\delta/72)/12\delta$, we have $|D(v,c)| \leqslant d_v \exp(\delta/72)/4\delta$ and

$$\sum_{A(w,c')\in D(v,c)} (2p)^{t_w} \leqslant pd_v \exp(\delta/72)/2\delta = \exp(-\delta/72)t_v \exp(\delta/72)/2 = t_v/2.$$

Thus condition (b) is satisfied. By the local lemma, with positive probability, no event $A(v,c)$ occurs. That is, a majority 3-colouring exists. □

Note that the conclusion in Theorem 3 and Theorem 4 is stronger than in Conjecture 2. We now show that such a conclusion is impossible (without some extra degree assumption).

Lemma 5. *For all integers k and δ, there are infinitely many digraphs G with minimum outdegree δ, such that for every vertex k-colouring of G, there is a vertex v such that all the out-neighbours of v receive the same colour.*

Proof. Start with a digraph G_0 with at least $k\delta$ vertices and minimum outdegree δ. For each set S of δ vertices in G_0, add a new vertex with out-neighbourhood S. Let G be the digraph obtained. In every k-colouring of G, at least δ vertices in G_0 receive the same colour, which implies that for some vertex $v \in V(G) \setminus V(G_0)$, all the out-neighbours of v receive the same colour. □

3 Stable Sets

A set T of vertices in a digraph G is a *stable set* if for each vertex $v \in T$, at most half the out-neighbours of v are also in T. A majority colouring is a partition into stable sets. Of course, if a digraph has a majority 3-colouring, then it contains a stable set with at least one third of the vertices. The next lemma provides a sufficient condition for the existence of such a set.

Theorem 6. *Every digraph G with n vertices and minimum outdegree at least 22 has a stable set with at least $\frac{n}{3}$ vertices.*

Theorem 6 is proved via the following more general lemma.

Lemma 7. *For $0 < \alpha < p < \beta < 1$, every digraph G with minimum outdegree at least*

$$\delta := \left\lceil \frac{(\beta + p)\log\left(\frac{p}{p-\alpha}\right)}{(\beta - p)^2} \right\rceil$$

contains a set T of at least αn vertices, such that $|N_G^+(v) \cap T| \leqslant \beta |N_G^+(v)|$ for every vertex $v \in T$.

Proof. Let $d_v := |N_G^+(v)|$ be the outdegree of each vertex v of G. Initialise $S := \emptyset$. For each vertex v of G, add v to S independently and randomly with probability p. Let $X_v := |N_G^+(v) \cap S|$. Note that $X_v \sim \text{Bin}(d_v, p)$ and

$$\mathbf{P}(X_v > \beta d_v) = \sum_{k \geqslant \lfloor \beta d_v \rfloor + 1}^{d_v} \binom{d_v}{k} p^k (1-p)^{d_v - k}. \tag{1}$$

By the Chernoff bound[2],

$$\mathbf{P}(X_v > \beta d_v) \leqslant \exp\left(-\frac{(\beta - p)^2}{\beta + p} d_v\right) \leqslant \exp\left(-\frac{(\beta - p)^2}{\beta + p} \delta\right) \leqslant \frac{p - \alpha}{p}. \tag{2}$$

where the last inequality follows from the definition of δ. Let $B := \{v \in S : X_v > \beta d_v\}$. Then

$$\mathbf{E}(|B|) = \sum_{v \in V(G)} \mathbf{P}(v \in S \text{ and } X_v > \beta d_v).$$

Since the events $v \in S$ and $X_v > \beta d_v$ are independent,

$$\mathbf{E}(|B|) = \sum_{v \in V(G)} \mathbf{P}(v \in S)\, \mathbf{P}(X_v > \beta d_v) = p \sum_{v \in V(G)} \mathbf{P}(X_v > \beta d_v) \leqslant (p - \alpha) n.$$

Let $T := S \setminus B$. Thus $|N_G^+(v) \cap T| \leqslant \beta d_v$ for each vertex $v \in T$, as desired. By the linearity of expectation,

$$\mathbf{E}(|T|) = \mathbf{E}(|S|) - \mathbf{E}(|B|) = pn - \mathbf{E}(|B|) \geqslant \alpha n.$$

Thus there exists the desired set T. □

Proof of Theorem 6. The proof follows that of Lemma 7 with one change. Let $\alpha := \frac{1}{3}$ and $\beta := \frac{1}{2}$ and $p := 0.38$. Then $\delta = 129$. If $22 \leqslant d_v \leqslant 128$ then direct calculation of the formula in (1) verifies that $\mathbf{P}(X_v > \beta d_v) \leqslant \frac{p-\alpha}{p}$, as in (2). For $d_v \geqslant 129$ the Chernoff bound proves (2). The rest of the proof is the same as in Lemma 7. □

[2] The Chernoff bound implies that if $X \sim \text{Bin}(d, p)$ then $\mathbf{P}(X \geqslant (1+\epsilon)pd) \leqslant \exp(-\frac{\epsilon^2}{2+\epsilon}\, pd)$ for $\epsilon \geqslant 0$. With $\epsilon = \frac{\beta}{p} - 1$ we have $\mathbf{P}(X > \beta d) \leqslant \exp(-\frac{(\beta - p)^2}{p + \beta}\, d)$.

Note the following corollary of Lemma 7 obtained with $\alpha = \frac{1}{2} - \epsilon$ and $p = \frac{1}{2} - \frac{\epsilon}{2}$. This says that graphs with large minimum outdegree have a stable set with close to half the vertices.

Proposition 8. *For $0 < \epsilon < \frac{1}{2}$, every n-vertex digraph G with minimum outdegree at least $2\epsilon^{-2}(2-\epsilon)\log(\frac{1-\epsilon}{\epsilon})$ contains a stable set of at least $(\frac{1}{2} - \epsilon)n$ vertices.*

4 Multi-Colour Generalisation

The following natural generalisation of Conjecture 2 arises.

Conjecture 9. *For $k \geqslant 2$, every digraph has a vertex $(k+1)$-colouring such that for each vertex v, at most $\frac{1}{k}\deg^+(v)$ out-neighbours of v receive the same colour as v.*

The proof of Theorem 1 generalises to give an upper bound of k^2 on the number of colours in Conjecture 9. It is open whether the number of colours is $O(k)$. This conjecture would be best possible, as shown by the following example. Let G be the k-th power of an n-cycle, with arcs oriented clockwise, where $n \geqslant 2k+3$ and $n \not\equiv 0 \pmod{k+1}$. Each vertex has outdegree k. Say G has a vertex $(k+1)$-colouring such that for each vertex v, at most ϵk out-neighbours of v receive the same colour as v. If $\epsilon k < 1$ then the underlying undirected graph of G is properly coloured, which is only possible if $n \equiv 0 \pmod{k+1}$. Hence $\epsilon \geqslant \frac{1}{k}$.

Lemma 7 with $\alpha = \frac{1}{k} - \epsilon$ and $\beta = \frac{1}{k}$ and $p = \frac{1}{k} - \frac{\epsilon}{2}$ implies the following 'stable set' version of Conjecture 9 for digraphs with large minimum outdegree.

Proposition 10. *For $k \geqslant 2$ and $\epsilon \in (0, \frac{1}{k})$, every n-vertex digraph G with minimum outdegree at least $2\epsilon^{-2}(\frac{4}{k} - \epsilon)\log\left(\frac{2}{\epsilon k} - 1\right)$ contains a set T of at least $(\frac{1}{k} - \epsilon)n$ vertices, such that for every vertex $v \in T$, at most $\frac{1}{k}\deg^+(v)$ out-neighbours of v are also in T.*

5 Open Problems

In addition to resolving Conjecture 2, the following open problems arise from this paper:

1. Is there a constant $\beta < 1$ for which every digraph has a 3-colouring, such that for every vertex v, at most $\beta\deg^+(v)$ out-neighbours receive the same colour as v?
2. Does every tournament have a majority 3-colouring?
3. Does every Eulerian digraph have a majority 3-colouring? Note that for an Eulerian digraph G, if each vertex v has in-degree and out-degree $\deg(v)$, then by the result for undirected graphs mentioned in Section 1, the underlying undirected graph of G has a

4-colouring such that each vertex v has at most $\frac{1}{2}\deg(v)$ in- or- out-neighbours with the same colour as v. In particular, G has a majority 4-colouring. By an analogous argument every Eulerian digraph has a 3-colouring such that each vertex v has at most $\frac{2}{3}\deg(v)$ in- or- out-neighbours with the same colour as v, thus proving a special case of the first question above.

4. Does every digraph in which every vertex has in-degree and out-degree k have a majority 3-colouring? A variant of Theorem 4 proves this result for $k \geqslant 144$.

5. Is there a characterisation of digraphs that have a majority 2-colouring (or a polynomial time algorithm to recognise such digraphs)?

6. Does every digraph have a $O(k)$-colouring such that for each vertex v, at most $\frac{1}{k}\deg^+(v)$ out-neighbours receive the same colour as v (for all $k \geqslant 2$)?

7. A digraph G is *majority c-choosable* if for every function $L : V(G) \to \mathbb{Z}$ with $|L(v)| \geqslant c$ for each vertex $v \in V(G)$, there is a majority colouring of G with each vertex v coloured from $L(v)$. Is every digraph majority c-choosable for some constant c? The proof of Theorem 1 shows that acyclic digraphs are majority 2-choosable, and obviously Theorem 3 and Theorem 4 extend to the setting of choosability.

8. Consider the following fractional setting. Let $S(G)$ be the set of all stable sets of a digraph G. Let $S(G,v)$ be the set of all stable sets containing v. A *fractional majority colouring* is a function that assigns each stable set $T \in S(G)$ a weight $x_T \geqslant 0$ such that $\sum_{T\in S(G,v)} x_T \geqslant 1$ for each vertex v of G. What is the minimum number k such that every digraph G has a fractional majority colouring with total weight $\sum_{T\in S(G)} x_T \leqslant k$? Perhaps it is less than 3.

Acknowledgements

This research was initiated at the Workshop on Graph Theory at Bellairs Research Institute (March 25 – April 1, 2016).

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