On the Upward Planarity of Mixed Plane Graphs*

Fabrizio Frati[1], Michael Kaufmann[2], János Pach[3], Csaba D. Tóth[4], and David R. Wood

[1] School of Information Technologies, The University of Sydney, Australia
brillo@it.usyd.edu.au
[2] Wilhelm-Schickard-Institut für Informatik, Universität Tübingen, Germany
mk@informatik.uni.tuebingen.de
[3] EPFL, Lausanne, Switzerland and Rényi Institute, Budapest, Hungary
pach@cims.nyu.edu
[4] California State University Northridge, USA and University of Calgary, Canada
cdtoth@acm.org
[5] School of Mathematical Sciences, Monash University, Melbourne, Australia
david.wood@monash.edu

Abstract. A *mixed plane graph* is a plane graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. An *orientation* of a mixed plane graph G is an assignment of directions to the undirected edges of G resulting in a directed plane graph $\boldsymbol{G}$. In this paper, we study the computational complexity of testing whether a given mixed plane graph G is *upward planar*, i.e., whether it admits an orientation resulting in a directed plane graph $\boldsymbol{G}$ such that $\boldsymbol{G}$ admits a planar drawing in which each edge is represented by a curve monotonically increasing in the y-direction according to its orientation.

Our contribution is threefold. First, we show that the upward planarity testing problem is solvable in cubic time for *mixed outerplane graphs*. Second, we show that the problem of testing the upward planarity of mixed plane graphs reduces in quadratic time to the problem of testing the upward planarity of *mixed plane triangulations*. Third, we exhibit linear-time testing algorithms for two classes of mixed plane triangulations, namely *mixed plane* 3*-trees* and mixed plane triangulations in which *the undirected edges induce a forest*.

1 Introduction

Upward planarity is the natural extension of planarity to directed graphs. When visualizing a directed graph, one usually requires an *upward drawing*, that is, a drawing in which the directed edges flow monotonically in the y-direction. A drawing is *upward planar* if it is planar and upward. Testing whether a directed graph $\boldsymbol{G}$ admits an upward planar drawing is NP-hard [9], however, it is polynomial-time solvable if $\boldsymbol{G}$ has a *fixed planar embedding* [2], if it has a *single-source* [3,13], if it is *outerplanar* [15], or if it is

* Pach was supported by Hungarian Science Foundation EuroGIGA Grant OTKA NN 102029, by Swiss National Science Foundation Grants 200020-144531 and 200021-137574, and by NSF Grant CCF-08-30272. Tóth was supported in part by NSERC (RGPIN 35586) and NSF (CCF-0830734).

S. Wismath and A. Wolff (Eds.): GD 2013, LNCS 8242, pp. 1–12, 2013.

a *series-parallel graph* [7]. *Exponential-time algorithms* [1] and *FPT algorithms* [12] for upward planarity testing are known.

In this paper we deal with *mixed graphs*. A mixed graph is a graph whose edge set is partitioned into a set of directed edges and a set of undirected edges. Mixed graphs unify the expressive power of directed and undirected graphs, as they allow one to simultaneously represent hierarchical and non-hierarchical relationships. A number of problems on mixed graphs have been studied, e.g., *coloring mixed graphs* [11,17] and *orienting mixed graphs to satisfy connectivity requirements* [5,6].

Upward planarity generalizes to mixed graphs as follows. A drawing of a mixed graph is *upward planar* if it is planar, every undirected edge is a y-monotone curve, and every directed edge is an arc with monotonically increasing y-coordinates. Hence, testing the upward planarity of a mixed graph is equivalent to testing whether its undirected edges can be oriented to produce an upward planar directed graph. Since the upward planarity testing problem is NP-hard for directed graphs [9], it is NP-hard for mixed graphs as well. Binucci and Didimo [4] studied the problem of testing the upward planarity of *mixed plane graphs*, that is, of mixed graphs with a given plane embedding. They describe an ILP formulation for the problem and present experiments showing the efficiency of their solution. Different graph drawing questions on mixed graphs (related to crossing and bend minimization) have been studied in [8,10].

We show the following results.

In Section 3 we show that the upward planarity testing problem can be solved in $O(n^3)$ time for n-vertex *mixed outerplane graphs*. Our dynamic programming algorithm uses a characterization for the upward planarity of directed plane graphs due to Bertolazzi et al. [2], and it tests the upward planarity of a mixed outerplane graph G based on the upward planarity of two subgraphs of G.

In Section 4 we show that, for every n-vertex mixed plane graph G, there exists an $O(n^2)$-vertex *mixed plane triangulation* G' such that G is upward planar if and only if G' is upward planar. As a consequence, the problem of testing the upward planarity of mixed plane graphs is polynomial-time solvable (NP-hard) if and only if the problem of testing the upward planarity of mixed plane triangulations is polynomial-time solvable (resp., NP-hard).

In Section 5, motivated by the previous result, we present linear-time algorithms to test the upward planarity of two classes of mixed plane triangulations, namely *mixed plane* 3*-trees* and mixed plane triangulations in which *the undirected edges induce a forest*. The former algorithm uses dynamic programming, while the latter algorithm uses induction on the number of undirected edges in the mixed plane triangulation.

Because of space limitations, some proofs are omitted or sketched in this extended abstract. Complete proofs are available in the full version of the paper.

2 Preliminaries

A planar drawing of a graph determines a circular ordering of the edges incident to each vertex. Two planar drawings of the same graph are *equivalent* if they determine the same circular orderings around each vertex. A *planar embedding* is an equivalence class of planar drawings. A planar drawing partitions the plane into topologically connected regions, called *faces*. The unbounded face is the *outer face* and the bounded faces are the *internal faces*. An edge of G incident to the outer face (not incident to the outer face) is called *external* (resp., *internal*). Two planar drawings with the same planar embedding have the same faces. However, they could still differ in their outer faces. A *plane embedding* is a planar embedding together with a choice for the outer face. A *plane graph* is a graph with a given plane embedding. An *outerplane graph* is a plane graph whose vertices are all incident to the outer face. A *plane triangulation* is a plane graph whose faces are delimited by 3-cycles. An *outerplane triangulation* is an outerplane graph whose internal faces are delimited by 3-cycles.

A *block* of a graph $G(V, E)$ is a maximal (both in terms of vertices and in terms of edges) 2-connected subgraph of G; in particular, an edge of G whose removal disconnects G is considered as a block of G. In this paper, when talking about the connectivity of mixed graphs or directed graphs, we always refer to the connectivity of their underlying undirected graphs.

A vertex v in a directed graph is a *sink* (*source*) if every edge incident to v is incoming at v (resp., outgoing at v). A vertex v in a directed plane graph is *bimodal* if the incoming edges at v are consecutive in the cyclic ordering of edges incident to v (which implies that the outgoing edges at v are also consecutive). A directed plane graph is *bimodal* if every vertex is bimodal. A vertex v in a 2-connected directed outerplane graph is a *sink-switch* (*source-switch*) if the two external edges incident to v are both incoming (resp., outgoing) at v.

Bertolazzi et al. [2] characterized the directed plane graphs that are upward planar. In this paper, we will use such a characterization when dealing with two specific classes of directed plane graphs, namely directed outerplane triangulations and directed plane triangulations. Thus, we state such a characterization directly for such graph classes.

Theorem 1 ([2]). *A directed outerplane triangulation* $\mathbf{G}$ *is upward planar if and only if it is acyclic, it is bimodal, and the number of sources plus the number of sinks in* $\mathbf{G}$ *equals the number of sink-switches (or source-switches) plus one.*

Theorem 2 ([2]). *A directed plane triangulation* $\mathbf{G}$ *is upward planar if and only if it is acyclic, it is bimodal, and* $\mathbf{G}$ *has exactly one source and one sink that are incident to the outer face of* $\mathbf{G}$.

A mixed plane graph is upward planar if and only if each of its connected components is upward planar. Thus, without loss of generality, we only consider connected mixed plane graphs. In the following lemma, we show that a stronger condition can in fact be assumed for each considered plane graph G, namely that G is 2*-connected*.

Lemma 1. *Every n-vertex mixed plane graph G can be augmented with new edges and vertices to a 2-connected mixed plane graph G' with $O(n)$ vertices such that G is upward planar if and only if G' is. If G is outerplane, than G' is also outerplane. Moreover, G' can be constructed from G in $O(n)$ time.*

Proof Sketch: While G has a cutvertex c that is incident to a face f (if G is outerplane, then f is its outer face), we consider two edges (v_1, c) and (u_2, c) that are consecutively incident to c in G and that belong to different blocks of G. We add a vertex w inside f and connect it to v_1 and u_2. The repetition of such an augmentation leads to a 2-connected mixed plane graph G' satisfying the conditions of the lemma. □

3 Upward Planarity Testing for Mixed Outerplane Graphs

This section is devoted to the proof of the following theorem.

Theorem 3. *The upward planarity of an n-vertex mixed outerplane graph can be tested in $O(n^3)$ time.*

Let G be any n-vertex mixed outerplane graph. By Lemma 1, an $O(n)$-vertex 2-connected mixed outerplane graph G^* can be constructed in $O(n)$ time such that G is upward planar if and only if G^* is.

We introduce some notation and terminology. Let u and v be distinct vertices of G^*. We denote by $G^* + (u, v)$ the graph obtained from G^* by adding edge (u, v) if it is not already in G^*, and by $G^* - u$ the graph obtained from G^* by deleting u and its incident edges. Consider an orientation $\boldsymbol{G}^*$ of G^*. A vertex is *sinky* (*sourcey*) in $\boldsymbol{G}^*$ if it is a sink-switch but not a sink (if it is a source-switch but not a source, resp.). A vertex that is neither a sink, a source, sinky, nor sourcey is *ordinary*; that is, v is ordinary if the two external edges incident to v are one incoming at v and one outgoing at v in $\boldsymbol{G}^*$. We say the *status* of a vertex of G^* in $\boldsymbol{G}^*$ is sink, source, sinky, sourcey, or ordinary.

First note that G^* is upward planar if and only if there is an upward planar directed outerplane triangulation T of G^*, that is, if and only if G^* can be augmented to a mixed outerplane triangulation, and the undirected edges of such a triangulation can be oriented in such a way that the resulting directed outerplane triangulation T is upward planar. The approach of our algorithm is to determine if there is such a T using recursion. The algorithm can be easily modified to produce T if it exists.

We observe that a directed outerplane triangulation T is acyclic if and only if every 3-cycle in T is acyclic. One direction is trivial. Conversely, suppose that T contains a directed cycle. Let C be a shortest directed cycle of T. If C is a 3-cycle, then we are done. Otherwise, an edge $(x, y) \notin C$ exists in T between two vertices x and y both in C. Thus, $C + (x, y)$ contains two shorter cycles, one of which is a directed cycle, contradicting the choice of C. Hence, to ensure the acyclicity of a directed outerplane triangulation, it suffices to ensure that its internal faces are acyclic.

A *potential edge* of G^* is a pair of distinct vertices x and y in G^* such that $G^*+(x, y)$ is outerplane, which is equivalent to saying that x and y are incident to a common internal face of G^* (notice that an edge of G^* is a potential edge of G^*). Fix some external edge r of G^*, called the *root edge*. Let $e = \{x, y\}$ be an internal potential edge of G^*. Then $\{x, y\}$ separates G^*, that is, G^* contains two subgraphs G_1^* and G_2^*, such that $G^* = G_1^* \cup G_2^*$ and $V(G_1^* \cap G_2^*) = \{x, y\}$. (Thus, there is no edge between $G_1^* - x - y$ and $G_2^* - x - y$.) W.l.o.g., $r \in E(G_1^*)$. Let $G_e^* := G_2^* + (x, y)$. Observe that G_e^* is a 2-connected mixed outerplane graph with e incident to the outer face. Also, let $e = \{x, y\} \neq r$ be an external potential edge of G^*. Then, we define G_e^* to be the 2-vertex graph containing the single edge (x, y). Further, let $G_r^* := G^*$. For any (internal or external) potential edge $e = \{x, y\}$ of G^* and for an orientation $\overrightarrow{xy}$ of e, let $G_{\overrightarrow{xy}}^*$ be G_e^* with e oriented $\overrightarrow{xy}$. Define a partial order $\prec$ on the potential edges of G^* as follows. For distinct potential edges e and f of G^*, say $e \prec f$ if both end-vertices of f are in G_e^*. Loosely speaking, $e \prec f$ if $G^* + e + f$ is outerplane and e is "between" r and f.

A *potential arc* of G^* is a potential edge that is assigned an orientation preserving its orientation in G^*. So if e is an undirected edge of G^* or a potential edge not in

G^*, then there are two potential arcs associated with e, while if e is a directed edge of G^*, then there is one potential arc associated with e. If a potential arc $\overrightarrow{xy}$ is part of a triangulation T of G^*, then x is a source, sourcey, or ordinary, and y is a sink, sinky, or ordinary in $G^*_{\overrightarrow{xy}}$. We define the *status* of $\overrightarrow{xy}$ in $G^*_{\overrightarrow{xy}}$ as an ordered pair S of $S(x) \in \{$source, sourcey, ordinary$\}$ and $S(y) \in \{$sink, sinky, ordinary$\}$.

We now define a function UP($\overrightarrow{xy}, S$), that takes as an input a potential arc $\overrightarrow{xy}$ and a status S of $\overrightarrow{xy}$, and has value "true" if and only if there is an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$; notice that, if $\overrightarrow{xy}$ is external and does not correspond to r, then $T_{\overrightarrow{xy}}$ is a single edge.

First, the values of UP($\overrightarrow{xy}, S$) can be computed in total $O(n)$ time for all the external potential arcs $\overrightarrow{xy}$ of G^* not corresponding to r and for all statuses of $\overrightarrow{xy}$. Indeed, UP($\overrightarrow{xy}, S$) is true if and only if $S(x) =$ source and $S(y) =$ sink.

We show below that, for each potential arc $\overrightarrow{xy}$ in G^* that is internal or that is external and corresponds to r, and for each status S of $\overrightarrow{xy}$, the value of UP($\overrightarrow{xy}, S$) can be computed in $O(n)$ time from values associated to potential arcs corresponding to potential edges e with $\{x, y\} \prec e$. Since there are at most $n(n+1)$ potential arcs and nine statuses for each potential arc, all the values of UP($\overrightarrow{xy}, S$) can be computed in $O(n^3)$ time by dynamic programming in reverse order to a linear extension of $\prec$. Then, there is an upward planar directed outerplane triangulation of G^* if and only if UP($\overrightarrow{xy}, S$) is true for some orientation $\overrightarrow{xy}$ of r and some status S of $\overrightarrow{xy}$.

Let $\overrightarrow{xy}$ be a potential arc that is internal to G^* or that corresponds to r. Let S be a status of $\overrightarrow{xy}$. Suppose that UP($\overrightarrow{xy}, S$) is true. Then, there is an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$. Such a triangulation contains a vertex $z \in V(G^*_{xy}) - x - y$ such that (x, y, z) is an internal face of $T_{\overrightarrow{xy}}$. Since $T_{\overrightarrow{xy}}$ has edge (x, y) oriented from x to y, then edges (x, z) and (y, z) cannot be simultaneously incoming at x and outgoing at y, respectively, as otherwise $T_{\overrightarrow{xy}}$ would contain a directed cycle, which is not possible by Theorem 1. Hence, edges (x, z) and (y, z) in $T_{\overrightarrow{xy}}$ are either outgoing at x and incoming at y, or outgoing at x and outgoing at y, or incoming at x and incoming at y, respectively.

Now, for any status S of $\overrightarrow{xy}$ and for a particular vertex $z \in V(G^*_{xy}) - x - y$, we characterize the conditions for which an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) oriented according to each of the three orientations described above.

Lemma 2. *There is an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$, that contains edge (x, z) outgoing at x, and that contains edge (z, y) incoming at y, if and only if $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{xz}$ and S_2 of $\overrightarrow{zy}$ such that the following conditions hold: (a) $S_1(x) = S(x) \in \{$source, sourcey, ordinary$\}$, (b) $S_2(y) = S(y) \in \{$sink, sinky, ordinary$\}$, (c) $S_1(z) \in \{$sink, ordinary$\}$, (d) $S_2(z) \in \{$source, ordinary$\}$, (e) $S_1(z) =$ sink or $S_2(z) =$ source, and (f) both UP($\overrightarrow{xz}, S_1$) and UP($\overrightarrow{zy}, S_2$) are true.*

Proof: ($\Longrightarrow$) Let $T_{\overrightarrow{xy}}$ be an upward planar directed outerplane triangulation of $G^*_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$, that contains edge (x, z) outgoing at x, and that contains edge (z, y) incoming at y. Then, $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are potential arcs of G^*. Further, $T_{\overrightarrow{xy}}$ determines upward planar directed outerplane triangulations $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$

respectively of $G^*_{\overrightarrow{xz}}$ and $G^*_{\overrightarrow{zy}}$ (where $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ are single edges if $\overrightarrow{xz}$ and $\overrightarrow{zy}$ are external, respectively), as well as statuses S_1 and S_2 of $\overrightarrow{xz}$ and $\overrightarrow{zy}$, respectively, such that (f) both UP($\overrightarrow{xz}, S_1$) and UP($\overrightarrow{zy}, S_2$) are true. Since $\overrightarrow{xy}$ and $\overrightarrow{xz}$ are consecutive outgoing arcs at x, we have (a) $S_1(x) = S(x) \in$ {source, sourcey, ordinary}. Similarly, (b) $S_2(y) = S(y) \in$ {sink, sinky, ordinary}. Since $\overrightarrow{xz}$ is incoming at z, we have $S_1(z) \in$ {sink, ordinary, sinky}. However, if $S_1(z) =$ sinky, then z is not bimodal in $T_{\overrightarrow{xy}}$. Thus (c) $S_1(z) \in$ {sink, ordinary}. Similarly, (d) $S_2(z) \in$ {source, ordinary}. Finally, if z is ordinary in both $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$, then z is not bimodal in $T_{\overrightarrow{xy}}$. Thus (e) $S_1(z) =$ sink or $S_2(z) =$ source.

($\Longleftarrow$) Let $T_{\overrightarrow{xz}}$ be an upward planar directed outerplane triangulation of $G^*_{\overrightarrow{xz}}$ respecting S_1 ($T_{\overrightarrow{xz}}$ is a single edge if $\overrightarrow{xz}$ is external). Let $T_{\overrightarrow{zy}}$ be an upward planar directed outerplane triangulation of $G^*_{\overrightarrow{zy}}$ respecting S_2 ($T_{\overrightarrow{zy}}$ is a single edge if $\overrightarrow{zy}$ is external). Such triangulations exist because UP($\overrightarrow{xz}, S_1$) and UP($\overrightarrow{zy}, S_2$) are true. Let $T_{\overrightarrow{xy}}$ be the triangulation of $G^*_{\overrightarrow{xy}}$ determined from $T_{\overrightarrow{xz}}$ and $T_{\overrightarrow{zy}}$ by adding the arc $\overrightarrow{xy}$. Since $T_{\overrightarrow{xz}}$, $T_{\overrightarrow{zy}}$, and (x, y, z) are acyclic, $T_{\overrightarrow{xy}}$ is acyclic. Since x is bimodal in $T_{\overrightarrow{xz}}$, it is bimodal in $T_{\overrightarrow{xy}}$. Similarly, y is bimodal in $T_{\overrightarrow{xy}}$. As described above, the conditions on $S_1(z)$ and $S_2(z)$ imply that z is bimodal in $T_{\overrightarrow{xy}}$. Every other vertex is bimodal in $T_{\overrightarrow{xy}}$ because it is bimodal in $T_{\overrightarrow{xz}}$ or in $T_{\overrightarrow{zy}}$. Hence, $T_{\overrightarrow{xy}}$ is bimodal.

Let s_1, t_1 and w_1 (s_2, t_2 and w_2; s, t and w) be the number of sources, sinks, and sink-switches in $T_{\overrightarrow{xz}}$ (resp., in $T_{\overrightarrow{zy}}$; resp., in $T_{\overrightarrow{xy}}$), respectively. By Theorem 1, $s_i + t_i = w_i + 1$, for $i \in \{1, 2\}$. If z is a sink in $T_{\overrightarrow{xz}}$ and ordinary in $T_{\overrightarrow{zy}}$, then $s = s_1 + s_2$, $t = t_1 + t_2 - 1$ (for z), and $w = w_1 + w_2$. If z is a source in $T_{\overrightarrow{zy}}$ and ordinary in $T_{\overrightarrow{xz}}$, then $s = s_1 + s_2 - 1$ (for z), $t = t_1 + t_2$, and $w = w_1 + w_2$. If z is a sink in $T_{\overrightarrow{xz}}$ and a source in $T_{\overrightarrow{zy}}$, then $s = s_1 + s_2 - 1$ (for z) and $t = t_1 + t_2 - 1$ (for z) and $w = w_1 + w_2 - 1$ (for z). In all three cases, it follows that $s + t = w + 1$.

By Theorem 1, $T_{\overrightarrow{xy}}$ is upward planar. By construction, $T_{\overrightarrow{xy}}$ respects $S(x)$ and $S(y)$ and contains edge (x, z) outgoing at x and edge (z, y) incoming at y. □

Lemma 3. *There is an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) incoming at z if and only if $\overrightarrow{xz}$ and $\overrightarrow{yz}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{xz}$ and S_2 of $\overrightarrow{yz}$ such that the following conditions hold: (a) $S_1(x) = S(x) \in$ {source, sourcey, ordinary}, (b) $S(y) \in$ {sinky, ordinary}, (c) $S_2(y) \in$ {source, ordinary}, (d) $S(y) =$ ordinary if and only if $S_2(y) =$ source, (e) $S(y) =$ sinky if and only if $S_2(y) =$ ordinary, (f) $S_1(z) \in$ {sink, sinky, ordinary}, (g) $S_2(z) \in$ {sink, sinky, ordinary}, (h) $S_1(z) \in$ {sink, ordinary} or $S_2(z) =$ sink, (i) $S_2(z) \in$ {sink, ordinary} or $S_1(z) =$ sink, and (j) both UP($\overrightarrow{xz}, S_1$) and UP($\overrightarrow{yz}, S_2$) are true.*

Lemma 4. *There is an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ that respects $S(x)$ and $S(y)$ and that contains edges (z, x) and (z, y) outgoing at z if and only if $\overrightarrow{zx}$ and $\overrightarrow{zy}$ are potential arcs of G^* and there are statuses S_1 of $\overrightarrow{zx}$ and S_2 of $\overrightarrow{zy}$ such that the following conditions hold: (a) $S_2(y) = S(y) \in$ {sink, sinky, ordinary}, (b) $S(x) \in$ {sourcey, ordinary}, (c) $S_1(x) \in$ {sink, ordinary}, (d) $S(x) =$ ordinary if and only if $S_1(x) =$ sink, (e) $S(x) =$ sourcey if and only if $S_1(x) =$ ordinary, (f) $S_1(z) \in$ {source, sourcey, ordinary}, (g) $S_2(z) \in$ {source, sourcey, ordinary}, (h)*

$S_1(z) \in \{source, ordinary\}$ or $S_2(z)$ = source, (i) $S_2(z) \in \{source, ordinary\}$ or $S_1(z)$=source, and (j) both UP($\overrightarrow{zx}, S_1$) and UP($\overrightarrow{zy}, S_2$) are true.

For any status S of $\overrightarrow{xy}$ and for a particular vertex $z \in V(G^*_{xy}) - x - y$, it can be checked in $O(1)$ time whether an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z) by checking whether the conditions in at least one of Lemmata 2-4 are satisfied. Further, UP($\overrightarrow{xy}, S$) is true if and only if there exists a vertex $z \in V(G^*_{xy}) - x - y$ such that an upward planar directed outerplane triangulation $T_{\overrightarrow{xy}}$ exists that respects $S(x)$ and $S(y)$ and that contains edges (x, z) and (y, z). Thus, we can determine UP($\overrightarrow{xy}, S$) in $O(n)$ time since there are less than n possible choices for z.

This completes the proof of Theorem 3. The time complexity analysis can be strengthened as follows. Suppose that every internal face of G^* has at most t vertices. Then each vertex v is incident to less than $t \cdot \deg_{G^*}(v)$ potential edges and the total number of potential arcs is less than $2\sum_v t \cdot \deg_{G^*}(v) \leq 8tn$. Since each potential arc has nine statuses, and since there are less than t choices for z, the time complexity is $O(t^2 n)$. In particular, if G^* is an outerplane triangulation, then the time complexity is $O(n)$.

4 Reducing Mixed Plane Graphs to Mixed Plane Triangulations

This section is devoted to the proof of the following theorem.

Theorem 4. *Let G be an n-vertex mixed plane graph. There exists an $O(n^2)$-vertex mixed plane triangulation G' such that G is upward planar if and only if G' is. Moreover, G' can be constructed from G in $O(n^2)$ time.*

Proof: By Lemma 1, an $O(n)$-vertex 2-connected mixed plane graph G^* can be constructed in $O(n)$ time such that G is upward planar if and only if G^* is.

We show how to construct a graph G' satisfying the statement of the theorem. In order to construct G', we augment G^* in several steps. At each step, vertices and edges are inserted inside a face f of G^* delimited by a cycle C_f with $n_f \geq 4$ vertices. Such an insertion is done in such a way that one of the faces that is created by the insertion of vertices and edges into f has $n_f - 1$ vertices, while all the other such faces have 3 vertices. The repetition of such an augmentation yields the desired graph G'.

We now describe how to augment G^*. Consider any face f of G^* delimited by a cycle C_f with $n_f \geq 4$ vertices. Let $(u_1, u_2, \dots, u_{n_f})$ be the clockwise order of the vertices along C_f starting at any vertex. Insert a cycle C'_f inside f with $n_f - 1$ vertices $v_1, v_2, \dots, v_{n_f-1}$ in this clockwise order along C'_f. For any $1 \leq i \leq n_f - 1$, insert edges (v_i, u_i) and (v_i, u_{i+1}) inside C_f and outside C'_f; also, insert edge (v_1, u_{n_f}) inside cycle $(u_{n_f}, u_1, v_1, v_{n_f-1})$. All the edges inserted in f are undirected. See Fig. 1. Denote by G'_f the graph consisting of cycle C_f together with the vertices and edges inserted in f. Observe that the face of G'_f delimited by C'_f has $n_f - 1$ vertices, while all the other faces into which f is split by the insertion of x_f and of its incident edges have 3 vertices.

We show that G^* before the augmentation is upward planar if and only if G^* after the augmentation is upward planar. One implication is trivial, given that G^* before the augmentation is a subgraph of G^* after the augmentation. For the other implication,

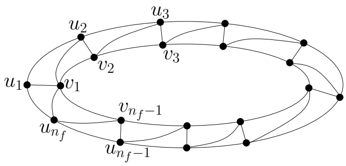

Fig. 1. Augmentation of a face f

it suffices to prove that, for any upward planar orientation $\boldsymbol{C}_f$ of C_f, there exists an upward planar orientation $\boldsymbol{G}'_f$ of G'_f that coincides with $\boldsymbol{C}_f$ when restricted to C_f.

Consider an upward planar drawing Γ_f of C_f with orientation $\boldsymbol{C}_f$. We shall place the vertices of C'_f inside f in Γ_f, thus obtaining a drawing Γ'_f of $\boldsymbol{G}'_f$.

Pach and Tóth [14] proved that any planar drawing of a graph G in which all the edges are y-monotone can be triangulated by the insertion of y-monotone edges inside the faces of G (the result in [14] states that the addition of a vertex might be needed to triangulate the outer face of G, which however is not the case if the outer face is bounded by a simple cycle, as in our case). Hence, there exists an index j, with $1 \leq j \leq n_f$, such that a y-monotone curve can be drawn connecting u_{j-1} and u_{j+1} inside f.

If $j < n_f$, then for $1 \leq i \leq j-1$, we place v_i inside f close to u_i, with $y(v_i) \neq y(u_i)$, so that y-monotone curves can be drawn inside f connecting v_i with u_{i-1}, with u_i, and with u_{i+1} (we draw y-monotone curves corresponding to edges of G'_f). Then, we place v_j inside f close to u_{j+1}, with $y(v_j) \neq y(u_{j+1})$, so that y-monotone curves can be drawn inside f connecting v_j with u_{j-1}, with u_j, with u_{j+1}, and with u_{j+2} (we in fact draw y-monotone curves corresponding to edges of G'_f). This is possible, since a y-monotone curve can be drawn inside f connecting v_j and u_j, by construction, and since a y-monotone curve can be drawn inside f connecting u_{j-1} and u_{j+1}, by assumption, hence a y-monotone curve can be drawn inside f connecting v_j and u_{j-1}. Then, for $j+1 \leq i \leq n_f-1$, we place v_i inside f close to u_{i+1}, with $y(v_i) \neq y(u_{i+1})$, so that y-monotone curves can be drawn inside f connecting v_i with u_i, with u_{i+1}, and with u_{i+2} (we in fact draw y-monotone curves corresponding to edges of G'_f). For any $1 \leq i \leq n_f-1$, since y-monotone curves can be drawn inside f connecting v_i with the vertices of C_f to which v_{i-1} and v_{i+1} are close, y-monotone curves can be drawn inside f representing the edges of C'_f (we in fact draw such curves). If $j = n_f$, the drawing is constructed analogously by placing v_i inside f close to u_i, for any $1 \leq j \leq n_f-1$.

The number of vertices of the mixed plane triangulation G' resulting from the augmentation is $O(n^2)$. Namely, the number of vertices inserted inside a face f of G^* with n_f vertices is $(n_f - 1) + (n_f - 2) + \cdots + 3$, hence the number of vertices of G' is $\sum_f (n_f(n_f-1)/2 - 3) = O(n^2)$, given that $\sum_f n_f \in O(n)$ (where the sums are over all the faces of G^*). Finally, the augmentation of G^* to G' can be easily performed in a time that is linear in the size of G', hence quadratic in the size of the input graph. □

Corollary 1. *The problem of testing the upward planarity of mixed plane graphs is polynomial-time equivalent to the problem of testing the upward planarity of mixed plane triangulations.*

5 Upward Planarity Testing of Mixed Plane Triangulations

In this section we show how to test in linear time the upward planarity of two classes of mixed plane triangulations.

A *plane* 3*-tree* is a plane triangulation that can be constructed as follows. Denote by H_{abc} a plane 3-tree whose outer face is delimited by a cycle (a, b, c), with vertices a, b, and c in this clockwise order along the cycle. A cycle (a, b, c) is the only plane 3-tree H_{abc} with three vertices. Any plane 3-tree H_{abc} with $n > 3$ vertices can be constructed from three plane 3-trees H_{abd}, H_{bcd}, and H_{cad} by identifying the vertices incident to their outer faces with the same label. See Fig. 2(a).

Theorem 5. *The upward planarity of an n-vertex mixed plane* 3*-tree can be tested in $O(n)$ time.*

Consider an n-vertex mixed plane 3-tree H_{uvw}. We define a function UP(xy, H_{abc}) as follows. For each graph H_{abc} in the construction of H_{uvw} and for any distinct $x, y \in \{a, b, c\}$ we have that UP(xy, H_{abc}) is true if and only if there exists an upward planar orientation of H_{abc} in which cycle (a, b, c) has x has a source and y as a sink.

Observe that H_{uvw} is upward planar if and only if UP(xy, H_{uvw}) is true for some $x, y \in \{u, v, w\}$ with $x \neq y$. The necessity comes from the fact that, in any upward planar orientation of H_{uvw}, the cycle delimiting the outer face of H_{uvw} has exactly one source x and one sink y, by Theorem 2. The sufficiency is trivial.

We show how to compute the value of UP(xy, H_{abc}), for each graph H_{abc} in the construction of H_{uvw}.

If $|H_{abc}| = 3$, then let $x, y, z \in \{a, b, c\}$ with $x \neq y$, $x \neq z$, and $y \neq z$. Then, UP(xy, H_{abc}) is true if and only if edges (x, y), (x, z), and (z, y) are not prescribed to be outgoing at y, outgoing at z, and outgoing at y, respectively. Hence, if $|H_{abc}| = 3$ the value of UP(xy, H_{abc}) can be computed in $O(1)$ time.

Second, if $|H_{abc}| > 3$, denote by H_{abd}, H_{bcd}, and H_{cad} the three graphs that compose H. We have the following:

Lemma 5. *For any distinct $x, y, z \in \{a, b, c\}$, UP(xy, H_{abc}) is true if and only if:*
(1) UP(xy, H_{xyd}), UP(xd, H_{zxd}), and UP(zy, H_{yzd}) are all true; or
(2) UP(xy, H_{xyd}), UP(xz, H_{zxd}), and UP(dy, H_{yzd}) are all true.

Proof Sketch: $(\Longrightarrow)$ Assume that H_{abc} has an upward planar orientation $\boldsymbol{H}_{abc}$ with x and y as a source and sink in $\{a, b, c\}$, respectively, (let $z \in \{a, b, c\}$ with $z \neq x, y$). Edge (z, d) might be outgoing or incoming at z, as in Figs. 2(b) and 2(c), respectively. In the first case, UP(xy, H_{xyd}), UP(zy, H_{yzd}), and UP(xd, H_{zxd}) are all true, while in the second case UP(xy, H_{xyd}), UP(dy, H_{yzd}), and UP(xz, H_{zxd}) are all true.

$(\Longleftarrow)$ Consider the case in which UP(xy, H_{xyd}), UP(xd, H_{zxd}), and UP(zy, H_{yzd}) are all true, the other case is analogous. Then, there exist upward planar orientations $\boldsymbol{H}_{xyd}$, $\boldsymbol{H}_{zxd}$, and $\boldsymbol{H}_{yzd}$ of H_{xyd}, H_{zxd}, and H_{yzd} with x and y, with x and d, and with z and y as a source and sink, respectively. Orientations $\boldsymbol{H}_{xyd}$, $\boldsymbol{H}_{zxd}$, and $\boldsymbol{H}_{yzd}$ together yield an orientation UP(xy, H_{xyz}) of $\boldsymbol{H}_{xyz}$, which is upward planar by Theorem 2. □

For each graph H_{abc} in the construction of H_{uvw} and for any distinct $x, y \in \{a, b, c\}$, the conditions in Lemma 5 can be computed in $O(1)$ time by dynamic programming. Thus, the running time of the algorithm is $O(n)$. This concludes the proof of Theorem 5.

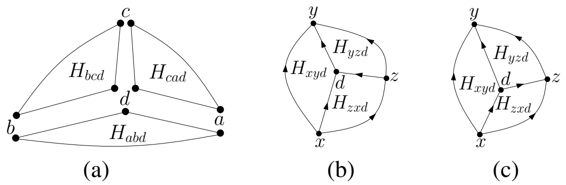

Fig. 2. (a) Construction of a plane 3-tree H_{abc} with $n > 3$ vertices. (b)-(c) Distinct orientations of edge (z, d) in two upward planar orientations of H_{abc}.

We now deal with mixed plane triangulations with no cycle of undirected edges.

Theorem 6. *The upward planarity of an n-vertex mixed plane triangulation in which the undirected edges induce a forest can be tested in $O(n)$ time.*

Proof: Let G be an n-vertex mixed plane triangulation. Let F be the set of undirected edges of G. We assume that F contains no external edge of G. Indeed, F contains at most two external edges: We can guess the orientation of all the external edges in F, and for each of the four possibilities, independently, test the upward planarity for the mixed graph G in which only the internal edges of F are undirected.

We prove the statement by induction on the size of F.

If $|F| = 0$, then G is a directed plane triangulation and its upward planarity can be tested in linear time by checking whether G satisfies the conditions in Theorem 2.

If $|F| > 0$, consider a leaf v in the forest whose edge set is F. Denote by (v, w) the only undirected edge of G incident to v. By the assumptions, (v, w) is an internal edge of G. Let (v, w, x_1) and (v, w, x_2) be the internal faces of G incident to edge (v, w).

Suppose that both edges (x_1, v) and (x_2, v) are incoming at v. If v has an outgoing incident edge, then by the bimodality condition in Theorem 2, edge (v, w) is incoming at v in every upward planar orientation of G. Suppose that v has no outgoing incident edge. If v is the sink of G (recall that the edges incident to the outer face of G are directed), then edge (v, w) is incoming at v in every upward planar orientation of G, by the single sink condition in Theorem 2. Otherwise, edge (v, w) is outgoing at v in every upward planar orientation of G, again by the single sink condition in Theorem 2.

Analogously, if both (x_1, v) and (x_2, v) are outgoing at v, the orientation of edge (v, w) can be decided without loss of generality.

Assume that (x_1, v) and (x_2, v) are incoming and outgoing at v, respectively, the case in which they are outgoing and incoming at v is analogous. We have two cases.

Case 1: (x_1, x_2) is an edge of G. By the acyclicity condition in Theorem 2, edge (x_1, x_2) is outgoing at x_1 in every upward planar orientation of G.

If $\deg(v) = 3$, then remove v and its incident edges from G, obtaining a mixed plane triangulation G' with one fewer undirected edge than G. Inductively test whether G' admits an upward planar orientation. If not, then G does not admit any upward planar orientation, either. If G' admits an upward planar orientation $\boldsymbol{G'}$, then construct an upward drawing Γ' of $\boldsymbol{G'}$; insert v in Γ' inside cycle (w, x_1, x_2), so that $y(v) > y(x_1)$, $y(v) < y(x_2)$, and $y(v) \neq y(w)$. Draw y-monotone curves connecting v with each of w, x_1, and x_2. The resulting drawing Γ of G is an upward planar orientation $\boldsymbol{G}$ of G, provided that it coincides with $\boldsymbol{G}'$ when restricted to G', the edges (x_1, v) and (x_2, v) are drawn as y-monotone curves according to their orientations, and the edge (v, w) is drawn as a y-monotone curve.

If $\deg(v) > 3$, then the cycle (w, x_1, x_2) does not delimit a face of G, and it contains non-empty sets V' and V'' of vertices in its interior and its exterior, respectively. Then, two upward planarity tests can be performed, namely one for the subgraph G' of G induced by $V' \cup \{w, x_1, x_2\}$, and one for the subgraph G'' of G induced by $V'' \cup \{w, x_1, x_2\}$. If one of the tests fails, then G admits no upward planar orientation. Otherwise, upward planar orientations $\boldsymbol{G}'$ of G' and $\boldsymbol{G}''$ of G'' together form an upward planar orientation $\boldsymbol{G}$ of G, provided that each edge of (w, x_1, x_2) has the same orientation in $\boldsymbol{G}'$ and in $\boldsymbol{G}''$.

Case 2: (x_1, x_2) is not an edge of G. Remove (v, w) from G and insert a directed edge (x_1, x_2) outgoing at x_1 inside face (x_1, v, x_2, w). This results in a graph G' with one fewer undirected edge than G. We show that G is upward planar iff G' is.

Suppose that G admits an upward planar orientation $\boldsymbol{G}$. Let Γ be an upward planar drawing of $\boldsymbol{G}$. Remove edge (v, w) from G in Γ. Draw edge (x_1, x_2) inside cycle $C_f = (x_1, v, x_2, w)$, thus ensuring the planarity of the resulting drawing Γ' of G', following closely the drawing of path (x_1, v, x_2), thus ensuring the upwardness of Γ'.

Suppose that G' admits an upward planar orientation $\boldsymbol{G}'$. Let Γ' be an upward planar drawing of $\boldsymbol{G}'$. Remove (x_1, x_2) from Γ'. Since $\boldsymbol{G}'$ is acyclic, C_f has three possible orientations in $\boldsymbol{G}'$. In Orientation 1, w is its source and x_2 its sink; in Orientation 2, x_1 is its source and w its sink; finally, in Orientation 3, x_1 is its source and x_2 its sink. If C_f is oriented in $\boldsymbol{G}'$ as in Orientation 1 (as in Orientation 2), then draw edge (v, w) inside C_f in Γ', thus ensuring the planarity of the resulting drawing Γ of G, following closely the drawing of path (w, x_1, v) (resp., of path (v, x_2, w)), thus ensuring the upwardness of Γ. If C_f is oriented in $\boldsymbol{G}'$ as in Orientation 3, slightly perturb the position of the vertices in Γ' so that $y(v) \neq y(w)$. Draw edge (v, w) in Γ' as follows. Suppose that $y(v) < y(w)$, the other case being analogous. Draw a line segment inside C_f starting at v and slightly increasing in the y-direction, until reaching path (x_1, w, x_2). Then, follow such a path to reach w. This results in an upward drawing of edge (v, w) inside C_f, hence in an upward planar drawing of G.

Finally, the running time of the described algorithm is clearly $O(n)$. □

6 Conclusions

We considered the problem of testing the upward planarity of mixed plane graphs. We proved that the upward planarity testing problem is $O(n^3)$-time solvable for mixed outerplane graphs. It would be interesting to investigate whether our techniques can be strengthened to deal with larger classes of mixed plane graphs, e.g. series-parallel plane graphs. Also, since testing upward planarity is a polynomial-time solvable problem for directed outerplanar graphs [15], it might be polynomial-time solvable for mixed outerplanar graphs without a prescribed plane embedding as well.

We proved that the upward planarity testing problem for mixed plane graphs is polynomial-time equivalent to the upward planarity testing problem for mixed plane triangulations (and showed two classes of mixed plane triangulations for which the problem can be solved efficiently). This, together with the characterization of the upward planarity of directed plane triangulations in terms of acyclicity, bimodality, and uniqueness of the sources and sinks (see [2] and Theorem 2), might indicate that a polynomial-time algorithm for testing the upward planarity of mixed plane triangulations should be pursued. On the other hand, Patrignani [16] proved that testing the existence of an acyclic and bimodal orientation for a mixed plane graph is NP-hard.

Acknowledgments. Thanks to Hooman Reisi Dehkordi, Peter Eades, Graham Farr, Seok-Hee Hong, and Brendan McKay for useful discussions on the problems considered in this paper.

References

1. Bertolazzi, P., Di Battista, G., Didimo, W.: Quasi-upward planarity. Algorithmica 32(3), 474–506 (2002)
2. Bertolazzi, P., Di Battista, G., Liotta, G., Mannino, C.: Upward drawings of triconnected digraphs. Algorithmica 12(6), 476–497 (1994)
3. Bertolazzi, P., Di Battista, G., Mannino, C., Tamassia, R.: Optimal upward planarity testing of single-source digraphs. SIAM J. Comput. 27(1), 132–169 (1998)
4. Binucci, C., Didimo, W.: Upward planarity testing of embedded mixed graphs. In: Speckmann, B. (ed.) GD 2011. LNCS, vol. 7034, pp. 427–432. Springer, Heidelberg (2011)
5. Boesch, F., Tindell, R.: Robbins's theorem for mixed multigraphs. Amer. Math. Monthly 87(9), 716–719 (1980)
6. Chung, F., Garey, M., Tarjan, R.: Strongly connected orientations of mixed multigraphs. Networks 15(4), 477–484 (1985)
7. Didimo, W., Giordano, F., Liotta, G.: Upward spirality and upward planarity testing. SIAM J. Discrete Math. 23(4), 1842–1899 (2009)
8. Eiglsperger, M., Kaufmann, M., Eppinger, F.: An approach for mixed upward planarization. J. Graph Algorithms Appl. 7(2), 203–220 (2003)
9. Garg, A., Tamassia, R.: On the computational complexity of upward and rectilinear planarity testing. SIAM J. Comput. 31(2), 601–625 (2001)
10. Gutwenger, C., Jünger, M., Klein, K., Kupke, J., Leipert, S., Mutzel, P.: A new approach for visualizing uml class diagrams. In: Diehl, S., Stasko, J., Spencer, S. (eds.) SOFTVIS, pp. 179–188. ACM (2003)
11. Hansen, P., Kuplinsky, J., de Werra, D.: Mixed graph colorings. Math. Meth. Oper. Res. 45(1), 145–160 (1997)
12. Healy, P., Lynch, K.: Two fixed-parameter tractable algorithms for testing upward planarity. Int. J. Found. Comput. Sci. 17(5), 1095–1114 (2006)
13. Hutton, M.D., Lubiw, A.: Upward planarity testing of single-source acyclic digraphs. SIAM J. Comput. 25(2), 291–311 (1996)
14. Pach, J., Tóth, G.: Monotone drawings of planar graphs. J. Graph Theory 46(1), 39–47 (2004)
15. Papakostas, A.: Upward planarity testing of outerplanar dags. In: Tamassia, R., Tollis, I.G. (eds.) GD 1994. LNCS, vol. 894, pp. 298–306. Springer, Heidelberg (1995)
16. Patrignani, M.: Finding bimodal and acyclic orientations of mixed planar graphs is NP-complete. Tech. Rep. RT-DIA-188-2011, Dept. Comp. Sci. Autom., Roma Tre Univ. (2011)
17. Sotskov, J.N., Tanaev, V.S.: Chromatic polynomial of a mixed graph. Vestsi Akademii Navuk BSSR. Ser. Fiz. Mat. Navuk 6, 20–23 (1976)