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ON MULTIPLICATIVE SIDON SETS

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Abstract

Fix integers $b > a \geq 1$ with $g := \gcd(a, b)$. A set $S \subseteq \mathbb{N}$ is $\{a, b\}$*-multiplicative* if $ax \neq by$ for all $x, y \in S$. For all n, we determine an $\{a, b\}$-multiplicative set with maximum cardinality in $[n]$, and conclude that the maximum density of an $\{a, b\}$-multiplicative set is $\frac{b}{b+g}$. For $A, B \subseteq \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is $\{A, B\}$-multiplicative if for all $a \in A$ and $b \in B$ and $x, y \in S$, the only solutions to $ax = by$ have $a = b$ and $x = y$. For $1 < a < b < c$ and a, b, c coprime, we give a $\mathcal{O}(1)$ time algorithm to approximate the maximum density of an $\{\{a\}, \{b, c\}\}$-multiplicative set to arbitrary given precision.

1. Introduction

Erdős [3], Erdős [4], Erdős [5] defined a set $S \subseteq \mathbb{N}$ to be *multiplicative Sidon*[2] if $ab = cd$ implies $\{a, b\} = \{c, d\}$ for all $a, b, c, d \in S$; see [9, 10, 11]. In a similar direction, Wang [14] defined a set $S \subseteq \mathbb{N}$ to be *double-free* if $x \neq 2y$ for all $x, y \in S$, and proved that the maximum density of a double-free set is $\frac{2}{3}$; see [1] for related results. Here $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[n] := \{1, 2, \dots, n\}$, and the *density* of $S \subseteq \mathbb{N}$ is

$$\lim_{n\to\infty} \frac{|S \cap [n]|}{n} \ .$$

Motivated by some questions in graph colouring, Pór and Wood [8] generalised the notion of double-free sets as follows. For $k \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is *k-multiplicative*

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[2] Additive Sidon sets have been more widely studied; see the classical papers [6, 12, 13] and the survey by O'Bryant [7].

(*Sidon*) if $ax = by$ implies $a = b$ and $x = y$ for all $a, b \in [k]$ and $x, y \in S$. Pór and Wood [8] proved that the maximum density of a k-multiplicative set is $\Theta(\frac{1}{\log k})$.

Here we study the following alternative generalization of double-free sets. For distinct $a, b \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is *$\{a, b\}$-multiplicative* if $ax \neq by$ for all $x, y \in S$. Our first result is to determine the maximum density of an $\{a, b\}$-multiplicative set. Assume that $a < b$ throughout.

Say $x \in \mathbb{N}$ is an *i-th subpower* of b if $x = b^i y$ for some $y \not\equiv 0 \pmod{b}$. If x is an i-th subpower of b for some even/odd i then x is an *even/odd* subpower of b. The following table gives the even subpowers of $b \in \{2, 3, 4\}$ and the corresponding entry in *The On-Line Encyclopedia of Integer Sequences*.

$b = 2$	$1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, \dots$	[A003159]
$b = 3$	$1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 22, \dots$	[A007417]
$b = 4$	$1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, \dots$	[A171948]

We prove the following result:

Theorem 1. *Fix integers $b > a \geqslant 1$. Let $g := \gcd(a, b)$. Then for every integer $n \in \mathbb{N}$, the even subpowers of $\frac{b}{g}$ in $[n]$ are an $\{a, b\}$-multiplicative set in $[n]$ with maximum cardinality. And the even subpowers of $\frac{b}{g}$ are an $\{a, b\}$-multiplicative set with density $\frac{b}{b+g}$, which is maximum.*

Note that if $g = a$ then $b \geqslant 2g$ and $b + g \leqslant \frac{3}{2}b$, and if $g < a$ then $a \geqslant 2g$ and $b + g \leqslant b + a < \frac{3}{2}b$. In both cases the density bound $\frac{b}{b+g}$ in Theorem 1 is at least $\frac{2}{3}$, which is the bound obtained by Wang [14] for the $a = 1$ and $b = 2$ case.

We propose a further generalization of double-free sets. Let $A, B \subseteq \mathbb{N}$. Say $S \subseteq \mathbb{N}$ is *$\{A, B\}$-multiplicative* if $ax = by$ implies $a = b$ and $x = y$ for all $a \in A$ and $b \in B$, and $x, y \in S$. One case is easily dealt with. If $B := \{b\}$ and b is coprime to each element of A, and there is some element $a \in A$ such that $a < b$, then, by the reasoning above, the even subpowers of b form an $\{A, B\}$-multiplicative set of (maximum) density $\frac{b}{b+1}$.

The simplest nontrivial case (not covered by Theorem 1) is $\{A, B\}$-multiplicativity for $A = \{a\}$, $B = \{b, c\}$, $1 < a < b < c$, with a, b, c pairwise coprime. We have the following theorem:

Theorem 2. *Consider $a, b, c \in \mathbb{N}$ pairwise coprime, with $1 < a < b < c$. For all fixed $\epsilon > 0$, there is a $\mathcal{O}(1)$ time algorithm that computes the maximum density of an $\{\{a\}, \{b, c\}\}$-multiplicative set to within ϵ.*

2. Proof of Theorem 1

First suppose that $\gcd(a, b) = 1$. Let T be the set of even subpowers of b. We now prove that T is an $\{a, b\}$-multiplicative set with maximum density. In fact, for all $[n]$, we prove that $T_n := T \cap [n]$ has maximum cardinality out of all $\{a, b\}$-multiplicative sets contained in $[n]$.

The key to our proof is to model the problem using a directed graph. Let G be the directed graph with $V(G) := [n]$ where $(x, y) \in E(G)$ whenever $bx = ay$ (implying $x < y$). Thus $S \subseteq [n]$ is $\{a, b\}$-multiplicative if and only if S is an independent set in G. If (x, y, z) is a directed path in G, then $x = \frac{a}{b}y$ and $z = \frac{b}{a}y$. Thus each vertex y has indegree and outdegree at most 1. Since $(x, y) \in E(G)$ implies $x < y$, G contains no directed cycles. Thus G is a collection of disjoint directed paths. Hence a maximum independent set in G is obtained by taking all the vertices at even distance from the source vertices[3], where a vertex y is a source (indegree 0) if and only if $\frac{a}{b}y$ is not an integer; that is, if $y \not\equiv 0 \pmod b$.

We now prove that the vertices at distance d from a source vertex are precisely the d-th subpowers of b. We proceed by induction on $d \geqslant 0$. Each vertex y of G has an incoming edge if and only if $\frac{a}{b}y \in \mathbb{N}$, which occurs if and only if $y \equiv 0 \pmod b$ since $\gcd(a, b) = 1$. Thus the source vertices of G are precisely the 0-th subpowers of b. This proves the $d = 0$ case of the induction hypothesis. Now consider a vertex y at distance d from a source vertex. Thus $y = \frac{b}{a}x$ for some vertex x at distance $d - 1$ from a source vertex. By induction, x is a $(d - 1)$-th subpower of b. That is, $x = b^{d-1}z$ for some $z \not\equiv 0 \pmod b$. Thus $y = b^d \frac{z}{a}$, which, since $\gcd(a, b) = 1$, implies that $\frac{z}{a}$ is an integer. Hence $\frac{z}{a} \not\equiv 0 \pmod b$ and y is a d-th subpower of b, as claimed.

This proves that the even subpowers of b form a maximum independent set in G. That is, T_n is an $\{a, b\}$-multiplicative set of maximum cardinality in $[n]$. To illustrate this proof, the following table shows two examples of the graph G with $b = 3$. Observe that the i-th row consists of the i-th subpowers of 3 regardless of a.

$a = 1$ and $b = 3$								
1	2	4	5	7	8	10	11	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
3	6	12	15	21	24	30	33	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
9	18	36	45	63	72	90	99	$\cdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
27	48	108	135	189	216	270	297	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$a = 2$ and $b = 3$											
1	2	4	5	7	8	10	11	13	14	16	$\cdots$
	$\downarrow$	$\downarrow$			$\downarrow$	$\downarrow$			$\downarrow$	$\downarrow$	
	3	6			12	15			21	24	$\cdots$
		$\downarrow$			$\downarrow$					$\downarrow$	
		9			18					36	$\cdots$
					$\downarrow$					$\downarrow$	
					27					48	$\cdots$
										$\vdots$	

[3]Note that this is not necessarily the only maximum independent set—for a path component with odd length, we may take the vertices at odd distance from the source of this path. This observation readily leads to a characterization of all maximum independent sets in G, and thus of all $\{a, b\}$-multiplicative sets in $[n]$ with maximum cardinality. Details omitted.

We now bound $|T_n|$ from above. Observe that

$$T_n = \left\{ b^{2i}y : 0 \leqslant i \leqslant \tfrac{1}{2}\log_b n,\ 1 \leqslant y \leqslant \frac{n}{b^{2i}},\ y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$\begin{aligned} |T_n| &\leqslant \sum_{i=0}^{\lfloor (\log_b n)/2 \rfloor} \left\lceil \frac{b-1}{b}\frac{n}{b^{2i}} \right\rceil \\ &\leqslant 1 + \tfrac{1}{2}(\log_b n) + \frac{(b-1)n}{b}\sum_{i \geqslant 0}\frac{1}{b^{2i}} \\ &\leqslant 1 + \tfrac{1}{2}(\log_b n) + \frac{(b-1)n}{b}\frac{b^2}{b^2-1} \\ &= 1 + \tfrac{1}{2}(\log_b n) + \frac{b}{b+1}n \ . \end{aligned}$$

We now bound $|T_n|$ from below. Observe that

$$T_n = [n] \setminus \left\{ b^{2i+1}y : 0 \leqslant i \leqslant \tfrac{1}{2}((\log_b n) - 1),\ 1 \leqslant y \leqslant \frac{n}{b^{2i+1}},\ y \not\equiv 0 \pmod{b} \right\} .$$

Thus

$$\begin{aligned} |T_n| &\geqslant n - \sum_{i=0}^{\lfloor ((\log_b n)-1)/2 \rfloor} \left\lceil \frac{b-1}{b}\frac{n}{b^{2i+1}} \right\rceil \\ &\geqslant n - \tfrac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2}\sum_{i \geqslant 0}\frac{1}{b^{2i}} \\ &\geqslant n - \tfrac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2}\frac{b^2}{b^2-1} \\ &= n - \tfrac{1}{2}((\log_b n) + 1) - \frac{n}{b+1} \\ &= \frac{b}{b+1}n - \tfrac{1}{2}((\log_b n) + 1) \ . \end{aligned}$$

These upper and lower bounds on $|T_n|$ imply that

$$|T_n| = \frac{b}{b+1}n + \Theta(\log_b n) \ .$$

Hence the density of T is $\frac{b}{b+1}$, and because T_n is optimal for each n, no $\{a, b\}$-multiplicative set has density greater than $\frac{b}{b+1}$.

We now drop the assumption that $\gcd(a, b) = 1$. Let $g := \gcd(a, b)$. Since $ax = by$ if and only if $\frac{a}{g}x = \frac{b}{g}y$, a set S is $\{a, b\}$-multiplicative if and only if S is $\{\frac{a}{g}, \frac{b}{g}\}$-multiplicative. Since $\frac{b/g}{b/g+1} = \frac{b}{b+g}$, the theorem is proved.

3. Proof of Theorem 2

Fix $A = \{a\}$ and $B = \{b, c\}$, where $1 < a < b < c$, and a, b, c are pairwise coprime. For convenience, we use the infinite graph G with vertex set $\mathbb{N}$ and edge set

$$E(G) = \{\{x, y\} : bx = ay \text{ or } cx = ay, \text{ and } x, y \in \mathbb{N}\}.$$

Let G_n denote the subgraph of G induced by the vertex set $[n]$. Let δ be the maximum density of an $\{\{a\}, \{b, c\}\}$-multiplicative set. Then

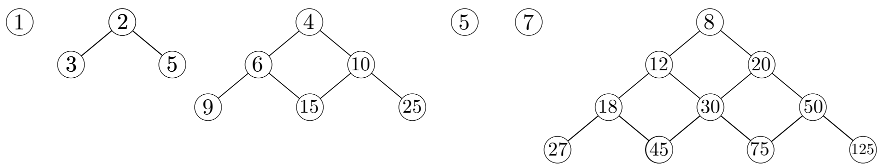
$$\delta = \lim_{n\to\infty} \frac{\alpha(G_n)}{n} \ ,$$

where $\alpha(G_n)$ is the size of a maximum independent set in G_n.

The infinite graph G has components $C_{p,q}$ with vertex set

$$V(C_{p,q}) = \{a^{p-x-y} b^x c^y q : x, y \in \mathbb{N}_0\}$$

for all $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, and q not divisible by a, b, or c. Note that each $C_{p,q}$ is finite. Define p as the *height* of the component, and subsets of constant $x + y$ as *rows*. Note that the maximum and minimum vertices in $C_{p,q}$ are $c^p q$ and $a^p q$ respectively. The first few components of G for $a = 2$, $b = 3$, and $c = 5$ are shown below:



For a, b, c as above and fixed $\epsilon > 0$, let d be a non-negative integer $d \in \mathbb{N}_0$, to be specified later. Basically, d is a cutoff height which allows us to partition the components of G_n into three types, for any given $n \in \mathbb{N}$. The first are *complete* components $C_{p,q}$ where $n > c^p q$. The second are *small* incomplete components $S_{p,q}$ where $p \leqslant d$ and $a^p q \leqslant n < c^p q$. The third are *large* incomplete components $L_{p,q}$ with $p > d$ and $a^p q \leqslant n < c^p q$.

Let $\alpha_T(G_n)$ denote the size of a maximum independent set in the components of type T in G_n, for $T \in \{C, S, L\}$. We clearly have

$$\alpha(G_n) = \alpha_C(G_n) + \alpha_S(G_n) + \alpha_L(G_n) \ .$$

Thus,

$$\delta = \lim_{n\to\infty} \frac{\alpha_C(G_n)}{n} + \lim_{n\to\infty} \frac{\alpha_S(G_n)}{n} + \lim_{n\to\infty} \frac{\alpha_L(G_n)}{n} = \delta_C + \delta_S + \delta_L$$

where

$$\delta_T = \lim_{n\to\infty} \frac{\alpha_T(G_n)}{n}.$$

Below we show that these limits exist, and we determine δ_C and δ_S explicitly. Then we show that, for any $\epsilon > 0$, we can choose d so that $\delta_L < \epsilon$. Hence, we can calculate δ to arbitrary precision.

3.1. Complete Components

We require the following lemma about independent sets in grid-like graphs by Cassaigne and Zimmerman [2].

Lemma 1. *Define a graph H by $V(H) := \mathbb{N}_0 \times \mathbb{N}_0$ and*

$$E(H) := \{\{\mathbf{v}, \mathbf{w}\} : \mathbf{v}, \mathbf{w} \in V(H), |v_1 - w_1| + |v_2 - w_2| = 1\}.$$

Suppose that F is a finite subgraph of H such that $(x, y) \in V(F)$ implies $(x-1, y) \in V(F)$ unless $x = 0$, and $(x, y-1) \in V(F)$ unless $y = 0$. Then one of the sets

$$O := \{(x, y) \in V(F) : x + y \text{ is odd}\} \text{ or}$$
$$E := \{(x, y) \in V(F) : x + y \text{ is even}\}$$

is a maximum independent set in F.

Now, consider a complete component $C_{p,1}$ of G_n. Note that every complete component $C_{p,q}$ of height q is isomorphic to $C_{p,1}$, and can be obtained by multiplying each vertex by q. Thus, we call $C_{p,q}$ a *q-copy* of $C_{p,1}$. In general, we use this terminology for isomorphic components of any type obtained by multiplying each vertex by q.

Observe that we can apply Lemma 1 to $C_{p,1}$, since it is isomorphic to a subgraph of H with the required properties. Define a function $\varphi : V(C_{p,1}) \to \mathbb{N}_0 \times \mathbb{N}_0$ by

$$\varphi(a^{p-x-y} b^x c^y) = (x, y).$$

If $a^{p-x-y}b^x c^y$ is adjacent to $a^{p-x'-y'}b^{x'}c^{y'}$, then $|x - x'| + |y - y'| = 1$ since they must differ by a factor of b/a or c/a. Thus, since φ is injective, it defines an isomorphism from $C_{p,1}$ to a subgraph of H. Assume $a^{p-x-y}b^x c^y \in V(C_{p,1})$. Then $a^{p-x-y+1}b^{x-1}c^y \in V(C_{p,1})$ unless $x = 0$, and similarly $a^{p-x-y+1}b^x c^{y-1} \in V(C_{p,1})$ unless $y = 0$. Under φ, these are clearly equivalent to the conditions required for Lemma 1.

Hence, by Lemma 1 and the definition of φ, a maximum independent set in $C_{p,1}$ is given by choosing all rows with $x + y$ even, or all rows with $x + y$ odd. In fact, it is clear that a maximum independent set is obtained by choosing the bottom row first, then alternating between remaining rows. Thus, if $p = 2i - 1$, then $\alpha(C_{p,1}) = i(i + 1)$. If $p = 2i$, then $\alpha(C_{p,1}) = (i + 1)^2$. Since the largest vertex in such a component is c^p, we must have $p \leqslant \log_c n$ for the component $C_{p,1}$ to be complete. Hence, the maximum height of a complete component is $\lfloor \log_c n \rfloor$.

Now we multiply by the number of components of height p that are complete. For a given p, we require $1 \leqslant q \leqslant nc^{-p}$. Since a, b, c are pairwise coprime, the density of numbers not divisible by a, b, or c is

$$\frac{(a-1)(b-1)(c-1)}{abc} \ ,$$

the number of components of height p in G_n is

$$\frac{(a-1)(b-1)(c-1)n}{c^p abc} + o(n) \ .$$

Let $M(n) = \frac{1}{2}\lfloor \log_c n \rfloor$. The total number of vertices in a maximum independent set in complete components is therefore

$$\alpha_C(G_n) = \frac{(a-1)(b-1)(c-1)n}{abc} \sum_{i=0}^{M(n)} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] + o(n) \ .$$

Thus, the density contribution is

$$\begin{aligned}
\delta_C = \lim_{n\to\infty} \frac{\alpha_C(G_n)}{n} &= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{(a-1)(b-1)(c-1)n}{abc} \sum_{i=0}^{M(n)} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] \\
&= \frac{(a-1)(b-1)(c-1)}{abc} \sum_{i=0}^{\infty} \left[\frac{i(i+1)}{c^{2i-1}} + \frac{(i+1)^2}{c^{2i}} \right] \\
&= \frac{(a-1)(b-1)(c-1)}{abc} \cdot \frac{c^4}{(c-1)^3(c+1)} \\
&= \frac{(a-1)(b-1)c^3}{ab(c-1)^2(c+1)}.
\end{aligned}$$

3.2. Small Incomplete Components

Now we consider the small incomplete components. Let $C_{p,1}[r]$ be the subgraph of $C_{p,1}$ induced by $[r]$. Define

$$f(p, r) := \alpha(C_{p,1}[r])$$

for $r \in \mathbb{N}$. We can calculate all f for $p \leqslant d$ in $\mathcal{O}(c^d)$ time with a computer, again using Lemma 1. (In fact, these components have bounded size, so any exponential time maximum independent set algorithm runs in $O(1)$ time.) Note that $C_{p,q}[n]$ is a q-copy of $C_{p,1}[\lfloor n/q \rfloor]$, and therefore $\alpha(C_{p,q}[n]) = f(p, \lfloor n/q \rfloor)$. So we can find the size of maximum independent sets in the small components using the f's.

More precisely, given $p \leqslant d$ and n, for how many values of q is $C_{p,q}[n]$ a q-copy of $C_{p,1}[r]$, where $r = \lfloor n/q \rfloor$? First note that

$$\frac{n}{r+1} < q \leqslant \frac{n}{r} \ .$$

Thus, there are

$$\frac{(a-1)(b-1)(c-1)n}{abc}\left(\frac{1}{r}-\frac{1}{r+1}\right)+o(n)=\frac{(a-1)(b-1)(c-1)n}{abcr(r+1)}+o(n)$$

q-copies of $C_{p,1}[r]$. The only restriction on r is that $a^p \leqslant r \leqslant c^p - 1$. Hence, the size of a maximum independent set in components of type S is

$$\sum_{p=0}^{d}\sum_{r=a^p}^{c^p-1}\frac{(a-1)(b-1)(c-1)n}{abcr(r+1)}f(p,r)+o(n)\ .$$

As $n \to \infty$, the density contribution of small components is therefore

$$\begin{aligned}\delta_S &= \lim_{n\to\infty}\frac{1}{n}\sum_{p=0}^{d}\sum_{r=a^p}^{c^p-1}\frac{(a-1)(b-1)(c-1)n}{abcr(r+1)}f(p,r)\\ &= \sum_{p=0}^{d}\sum_{r=a^p}^{c^p-1}\frac{(a-1)(b-1)(c-1)}{abcr(r+1)}f(p,r).\end{aligned}$$

Since a, b and c are constants and d will be chosen so that it is bounded by a function of a, b and c (see the next section for details), δ_S can be computed in $\mathcal{O}(1)$ time.

3.3. Large Incomplete Components

Finally, we show that we can choose d so that the density of a maximum independent set in components of type L is less than ϵ. For large components,

$$p > d \quad \text{and} \quad a^p q \leqslant n < c^p q\ .$$

The latter implies $c^{-p}n < q \leqslant a^{-p}n$. From the density of q, the number of large incomplete components $L_{p,q}$ for a given $p > d$ is

$$\frac{(a-1)(b-1)(c-1)n}{abc}\left(\frac{1}{a^p}-\frac{1}{c^p}\right)+o(n)\ .$$

Since there are less than p^2 vertices in a component of height p,

$$\begin{aligned}\alpha_L(G_n) &\leqslant \sum_{p=d}^{\infty}p^2\cdot\frac{(a-1)(b-1)(c-1)n}{abc}\left(\frac{1}{a^p}-\frac{1}{c^p}\right)\\ &\leqslant \frac{(a-1)(b-1)(c-1)n}{abc}\sum_{p=d}^{\infty}\frac{p^2}{a^p}\\ &= \frac{(a-1)(b-1)(c-1)n}{abc}\cdot\frac{a^{1-d}((a-1)^2d^2+2(a-1)d+a+1)}{(a-1)^3}\end{aligned}$$

$$\leqslant \frac{(b-1)(c-1)n}{bc} \cdot a^{-d/2}$$

where the last inequality holds for $d \geqslant 22$. Define $\beta := (b-1)(c-1)/bc$. Hence,

$$\delta_L = \lim_{n\to\infty} \frac{\alpha_L(G_n)}{n} \leqslant \lim_{n\to\infty} \frac{1}{n} \cdot \beta n \cdot a^{-d/2} = \beta a^{-d/2} \; .$$

So, to obtain a precision of ϵ in the approximation $\delta \approx \delta_C + \delta_S$, we pick

$$d = \max\{2\log_a(\beta/\epsilon), 22\}$$

which is a function of a, b, c, and ϵ. This completes the proof of Theorem 2.

The following table gives approximate values of δ for small a, b, and c:

a	b	c	δ
2	3	5	0.7292
2	3	7	0.7407
2	5	7	0.8235
2	5	9	0.8187
2	7	9	0.8709
3	4	5	0.7093
3	4	7	0.7934
3	5	7	0.8239
3	5	8	0.8212
3	7	8	0.8727

These results were obtained by incrementing d and looking for convergence to 4 decimal places. We also approximated δ_S using a naive algorithm (based on Lemma 1) for large n. Numerical convergence occurred at values of d slightly lower than the bound given above.

References

[1] Jean-Paul Allouche, André Arnold, Jean Berstel, Srečko Brlek, William Jockusch, Simon Plouffe, and Bruce E. Sagan. A relative of the Thue-Morse sequence. *Discrete Math.*, 139(1–3):455–461, 1995. doi:10.1016/0012-365X(93)00147-W.

[2] Julien Cassaigne and Paul Zimmerman. Numerical evaluation of the strongly triple-free set constant, 1996. http://iml.univ-mrs.fr/~cassaign/publis/cz.ps.gz.

[3] Paul Erdős. On sequences of integers no one of which divides the product of two others and some related problems. *Izvestiya Naustno-Issl. Inst. Mat. i Meh. Tomsk*, 2:74–82, 1938. http://www.renyi.hu/~p_erdos/1938-07.pdf.

[4] Paul Erdős. On some applications of graph theory to number theoretic problems. *Publ. Ramanujan Inst. No.*, 1:131–136, 1968/1969. http://www.renyi.hu/~p_erdos/1968-09.pdf.

[5] Paul Erdős. Some applications of graph theory to number theory. In *The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968)*, pp. 77–82. Springer, Berlin, 1969. http://www.renyi.hu/~p_erdos/1969-14.pdf.

[6] Paul Erdős and Pál Turán. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941. doi:10.1112/jlms/s1-16.4.212.

[7] Kevin O'Bryant. A complete annotated bibliography of work related to Sidon sequences. *Electron. J. Combin.*, DS11, 2004. http://www.combinatorics.org/ojs/index.php/eljc/article/view/ds11.

[8] Attila Pór and David R. Wood. Colourings of the Cartesian product of graphs and multiplicative Sidon sets. *Combinatorica*, 29(4):449–466, 2009. doi:10.1007/s00493-009-2257-0.

[9] Imre Z. Ruzsa. Erdős and the integers. *J. Number Theory*, 79(1):115–163, 1999. doi:10.1006/jnth.1999.2395.

[10] Imre Z. Ruzsa. Additive and multiplicative Sidon sets. *Acta Math. Hungar.*, 112(4):345–354, 2006. doi:10.1007/s10474-006-0102-0.

[11] András Sárközy. Unsolved problems in number theory. *Period. Math. Hungar.*, 42(1-2):17–35, 2001. doi:10.1023/A:1015236305093.

[12] Simon Sidon. Ein Satz űber trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen. *Math. Ann.*, 106:536–539, 1932. MR: 1512772.

[13] James Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938. doi:10.2307/1990067.

[14] Edward T. H. Wang. On double-free sets of integers. *Ars Combin.*, 28:97–100, 1989. MR: 1039135.