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CLUSTERED COLOURING IN MINOR-CLOSED CLASSES

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The *clustered chromatic number* of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k-colourable with monochromatic components of size at most c. We prove that for every graph H, the clustered chromatic number of the class of H-minor-free graphs is tied to the tree-depth of H. In particular, if H is connected with tree-depth t, then every H-minor-free graph is $(2^{t+1}-4)$-colourable with monochromatic components of size at most $c(H)$. This provides the first evidence for a conjecture of Ossona de Mendez, Oum and Wood (2016) about defective colouring of H-minor-free graphs. If $t=3$, then we prove that 4 colours suffice, which is best possible. We also determine those minor-closed graph classes with clustered chromatic number 2. Finally, we develop a conjecture for the clustered chromatic number of an arbitrary minor-closed class.

1. Introduction

In a vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. A graph G is *k-colourable with clustering c* if each vertex can be assigned one of k colours such that each monochromatic component has at most c vertices. We shall consider such colourings, where the first priority is to minimise the number of colours, with small clustering as a secondary goal. With this

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viewpoint the following definition arises. The *clustered chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\star(\mathcal{G})$, is the minimum integer k such that, for some integer c, every graph in $\mathcal{G}$ has a k-colouring with clustering c. See [24] for a survey on clustered graph colouring.

This paper studies clustered colouring in minor-closed classes of graphs. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges. A class of graphs $\mathcal{M}$ is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of G is in $\mathcal{M}$, and some graph is not in $\mathcal{M}$. For a graph H, let $\mathcal{M}_H$ be the class of H-minor-free graphs (that is, not containing H as a minor). Note that we only consider simple finite graphs.

As a starting point, consider Hadwiger's Conjecture, which states that every graph containing no K_t-minor is properly $(t-1)$-colourable. This conjecture is easy for $t \leqslant 4$, is equivalent to the 4-colour theorem for $t=5$, is true for $t=6$ [19], and is open for $t \geqslant 7$. The best known upper bound on the chromatic number is $O(t\sqrt{\log t})$, independently due to Kostochka [10,11] and Thomason [21,22]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [20] for a survey.

Clustered colourings of K_t-minor-free graphs provide an avenue for attacking Hadwiger's Conjecture. Kawarabayashi and Mohar [9] first proved an $O(t)$ upper bound on $\chi_\star(\mathcal{M}_{K_t})$. In particular, they proved that every K_t-minor-free graph is $\lceil \frac{31}{2}t \rceil$-colourable with clustering $f(t)$, for some function f. The number of colours in this result was improved to $\lceil \frac{7t-3}{2} \rceil$ by Wood [23], to $4t-4$ by Edwards, Kang, Kim, Oum and Seymour [5], to $3t-3$ by Liu and Oum [13], and to $2t-2$ by Norin [15]. Thus $\chi_\star(\mathcal{M}_{K_t}) \leqslant 2t-2$. See [8,7] for analogous results for graphs excluding odd minors. For all of these results, the function $f(t)$ is very large, often depending on constants from the Graph Minor Structure Theorem. Van den Heuvel and Wood [6] proved the first such result with $f(t)$ explicit. In particular, they proved that every K_t-minor-free graph is $(2t-2)$-colourable with clustering $\lceil \frac{t-2}{2} \rceil$. The result of Edwards et al. [5] mentioned below implies that $\chi_\star(\mathcal{M}_{K_t}) \geqslant t-1$. Dvořák and Norin [4] have announced a proof that $\chi_\star(\mathcal{M}_{K_t}) = t-1$.

Now consider the class $\mathcal{M}_H$ of H-minor-free graphs for an arbitrary graph H. The maximum chromatic number of a graph in $\mathcal{M}_H$ is at most $O(|V(H)|\sqrt{\log|V(H)|})$ and is at least $|V(H)|-1$ (since $K_{|V(H)|-1}$ is H-minor-free), and Hadwiger's Conjecture would imply that $|V(H)|-1$ is the answer. However, for clustered colourings, fewer colours often suffice. For example, Dvořák and Norin [4] proved that graphs embeddable on any fixed surface are 4-colourable with bounded clustering, whereas the chromatic number is $\Theta(\sqrt{g})$ for surfaces of Euler genus g. Van den Heuvel and Wood [6] proved that $K_{2,t}$-minor-free graphs are 3-colourable with clustering $t-1$, and that $K_{3,t}$-minor-free graphs are 6-colourable with clustering $2t$. These results show that $\chi_\star(\mathcal{M}_H)$ depends on the structure of H, unlike the usual chromatic number which only depends on $|V(H)|$.

At the heart of this paper is the following question: what property of H determines $\chi_\star(\mathcal{M}_H)$? The following definitions help to answer this question. Let T be a rooted tree. The *depth* of T is the maximum number of vertices on a root–to–leaf path in T. The *closure* of T is obtained from T by adding an edge between every ancestor and descendent in T. The *connected tree-depth* of a graph H, denoted by $\overline{\mathrm{td}}(H)$, is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T. This definition is a variant of the more commonly used definition of the *tree-depth* of H, denoted by $\mathrm{td}(H)$, which equals the maximum connected tree-depth of the connected components of H. See [14] for background on tree-depth. If H is connected, then $\mathrm{td}(H) = \overline{\mathrm{td}}(H)$. In fact, $\mathrm{td}(H) = \overline{\mathrm{td}}(H)$ unless H has two connected components H_1 and H_2 with $\mathrm{td}(H_1) = \mathrm{td}(H_2) = \mathrm{td}(H)$, in which case $\overline{\mathrm{td}}(H) = \mathrm{td}(H) + 1$. We choose to work with connected tree-depth to avoid this distinction.

The following result is the primary contribution of this paper; it is proved in Section 2.

Theorem 1. *For every graph H, $\chi_\star(\mathcal{M}_H)$ is tied to the (connected) tree-depth of H. In particular,*

$$\overline{\mathrm{td}}(H) - 1 \leqslant \chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\mathrm{td}}(H)+1} - 4.$$

The upper bound in Theorem 1 gives evidence for, and was inspired by, a conjecture of Ossona de Mendez, Oum and Wood [16], which we now introduce. A graph G is *k-colourable with defect d* if each vertex of G can be assigned one of k colours so that each vertex is adjacent to at most d neighbours of the same colour; that is, each monochromatic component has maximum degree at most d. The *defective chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer k such that, for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d. Every colouring of a graph with clustering c has defect $c-1$. Thus, the defective chromatic number of a graph class is at most its clustered chromatic number. Ossona de Mendez et al. [16] conjectured the following behaviour for the defective chromatic number of $\mathcal{M}_H$.

Conjecture 2 ([16]). *For every graph H,*

$$\chi_\Delta(\mathcal{M}_H) = \overline{\mathrm{td}}(H) - 1.$$

Ossona de Mendez et al. [16] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geqslant \overline{\mathrm{td}}(H)-1$, in Conjecture 2. This follows from the observation that the closure of the rooted complete c-ary tree of depth k is not $(k-1)$-colourable with clustering c. The lower bound in Theorem 1 follows since $\chi_\Delta \leqslant \chi_\star$ for every class. The upper bound in Conjecture 2 is known to hold in some special cases. Edwards et al. [5] proved it if $H=K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t})=t-1$, which can be thought of as a defective version of Hadwiger's Conjecture. Ossona de Mendez et al. [16] proved the upper bound in Conjecture 2 if $\overline{\mathrm{td}}(H)\leqslant 3$ or if H is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}})=\min\{s,t\}$.

Theorem 1 provides some evidence for Conjecture 2 by showing that $\chi_\Delta(\mathcal{M}_H)$ and $\chi_\star(\mathcal{M}_H)$ are bounded from above by some function of $\overline{\mathrm{td}}(H)$. This was previously not known to be true.

While it is conjectured that $\chi_\Delta(\mathcal{M}_H) = \overline{\mathrm{td}}(H) - 1$, the following lower bound, proved in Section 2.3, shows that $\chi_\star(\mathcal{M}_H)$ might be larger, thus providing some distinction between defective and clustered colourings.

Theorem 3. *For each $k\geqslant 2$, there is a graph H_k with $\overline{\mathrm{td}}(H_k)=\mathrm{td}(H_k)=k$ such that*

$$\chi_\star(\mathcal{M}_{H_k}) \geqslant 2k-2.$$

We conjecture an analogous upper bound:

Conjecture 4. *For every graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\,\overline{\mathrm{td}}(H)-2.$$

A further contribution of the paper is to precisely determine the minor-closed graph classes with clustered chromatic number 2. This result is introduced and proved in Conjecture 3. Section 4 studies clustered colourings of graph classes excluding so-called fat stars as a minor. This leads to a proof of Conjecture 4 in the $\overline{\mathrm{td}}(H)=3$ case. We conclude in Section 5 with a conjecture about the clustered chromatic number of an arbitrary minor-closed class that generalises Conjecture 4.

2. Tree-depth Bounds

The main goal of this section is to prove that $\chi_\star(\mathcal{M}_H)$ is bounded from above by some function of $\overline{\mathrm{td}}(H)$. We actually provide two proofs. The first proof depends on deep results from graph structure theory and gives no explicit bound on the clustering. The second proof is self-contained, but gives a worse upper bound on the number of colours. Both proofs have their own merits, so we include both.

Let $C\langle h,k\rangle$ be the closure of the rooted complete k-ary tree of depth h. (Here each non-leaf node has exactly k children.)

If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \operatorname{dist}_G(v,r) = i\}$ for $i \geqslant 0$, then $V_0, V_1, \ldots$ is called the *BFS layering* of G starting at r.

2.1. First Proof

The first proof depends on the following Erdős-Pósa Theorem by Robertson and Seymour [18]. For a graph H and integer $p \geqslant 1$, let pH be the disjoint union of p copies of H.

Theorem 5 ([18]; see [17, Lemma 3.10]). *For every non-empty graph H with c connected components and for all integers $p, w \geqslant 1$, for every graph G with treewidth at most w and containing no pH minor, there is a set $X \subseteq V(G)$ of size at most pwc such that $G - X$ has no H minor.*

The next lemma is the heart of our proof.

Lemma 6. *For all integers $h, k, w \geqslant 1$, every $C\langle h,k\rangle$-minor-free graph G of treewidth at most w is $(2^h - 2)$-colourable with clustering kw.*

Proof. We proceed by induction on $h \geqslant 1$, with w and k fixed. The case $h = 1$ is trivial since $C\langle 1,k\rangle$ is the 1-vertex graph, so only the empty graph has no $C\langle 1,k\rangle$ minor, and the empty graph is 0-colourable with clustering 0. Now assume that $h \geqslant 2$, the claim holds for $h-1$, and G is a $C\langle h,k\rangle$-minor-free graph with treewidth at most w. Let $V_0, V_1, \ldots$ be the BFS layering of G starting at some vertex r.

Fix $i \geqslant 1$. Then $G[V_i]$ contains no $kC\langle h-1,k\rangle$ as a minor, as otherwise contracting $V_0 \cup \cdots \cup V_{i-1}$ to a single vertex gives a $C\langle h,k\rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}). Since G has treewidth at most w, so does $G[V_i]$. By Theorem 5 with $H = C\langle h-1,k\rangle$ and $c = 1$, there is a set $X_i \subseteq V_i$ of size at most kw, such that $G[V_i \setminus X_i]$ has no $C\langle h-1,k\rangle$ minor. By induction, $G[V_i \setminus X_i]$ is $(2^{h-1} - 2)$-colourable with clustering kw. Use one new colour for X_i. Thus $G[V_i]$ is $(2^{h-1} - 1)$-colourable with clustering kw.

Use disjoint sets of colours for even and odd i, and colour r by one of the colours used for even i. No edge joins V_i with V_j for $j \geqslant i+2$. Thus G is $(2^h - 2)$-coloured with clustering kw. ∎

To drop the assumption of bounded treewidth, we use the following result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan [3], the proof of which depends on the graph minor structure theorem.

Theorem 7 ([3]). *For every graph H there is an integer w such that for every graph G containing no H-minor, there is a partition V_1, V_2 of $V(G)$ such that $G[V_i]$ has treewidth at most w, for $i \in \{1,2\}$.*

Lemma 6 and Theorem 7 imply:

Lemma 8. *For all integers $h,k \geqslant 1$, there is an integer $g(h,k)$, such that every $C\langle h,k\rangle$-minor-free graph G is $(2^{h+1}-4)$-colourable with clustering at most $g(h,k)$.*

Fix a graph H. By definition, H is a subgraph of $C\langle \overline{\text{td}}(H), |V(H)|\rangle$. Thus every H-minor-free graph contains no $C(\overline{\text{td}}(H), |V(H)|)$-minor. Hence, Lemma 8 implies

$$\chi_\star(\mathcal{M}_H) \leqslant 2^{\overline{\text{td}}(H)+1} - 4,$$

which is the upper bound in Theorem 1.

Note Theorem 26 below improves the $h=3$ case in Lemma 6, which leads to a small constant-factor improvement in Theorem 1 for $h \geqslant 3$.

2.2. Second Proof

We now present our second proof that $\chi_\star(\mathcal{M}_H)$ is bounded from above by some function of $\overline{\text{td}}(H)$. This proof is self-contained (not using Theorems 5 and 7).

Let T be a rooted tree. Recall that the *closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is an ancestor of the other in T. The *weak closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is a leaf and the other is one of its ancestors. For $h,k \geqslant 1$, let $T\langle h,k\rangle$ be the rooted complete k-ary tree of depth h. Let $W\langle h,k\rangle$ be the weak closure of $T\langle h,k\rangle$.

Lemma 9. *For $h,k \geqslant 2$, the graph $W\langle h,k\rangle$ contains $C\langle h,k-1\rangle$ as a minor.*

Proof. Let r be the root vertex. Colour r blue. For each non-leaf vertex v, colour $k-1$ children of v *blue* and colour the other child of v *red*. Let X be the set of blue vertices v in $T\langle h,k\rangle$, such that every ancestor of v is blue. Note that X induces a copy of $T\langle h,k-1\rangle$ in $T\langle h,k\rangle$. Let v be a non-leaf vertex in X. Let w be the red child of v, and let T_v be the subtree of $T\langle h,k\rangle$ rooted at w. Then every leaf of T_v is adjacent in $W\langle h,k\rangle$ to v and to every ancestor of v. Contract T_v and the edge vw into v. Now v is adjacent to every ancestor of v in X. Do this for each non-leaf vertex in X. Note that T_u and T_v are disjoint for distinct non-leaf vertices $u, v \in X$. Thus, we obtain $C\langle h,k-1\rangle$ as a minor of $W\langle h,k\rangle$. ∎

A *model* of a graph H in a graph G is a collection $\{J_x : x \in V(H)\}$ of pairwise disjoint subtrees of G such that for every $xy \in E(H)$ there is an edge of G with one end in $V(J_x)$ and the other end in $V(J_y)$. Observe that a graph contains H as a minor if and only if it contains a model of H.

Lemma 10. *For $h \geqslant 2$ and $k \geqslant 1$, if a graph G contains $W\langle h, 6k\rangle$ as a minor, then G contains subgraphs G' and G'', both containing $W\langle h, k\rangle$ as a minor, such that $|V(G') \cap V(G'')| \leqslant 1$.*

Proof. Let $\{J_x : x \in V(W\langle h, 6k\rangle)\}$ be a model of $W\langle h, 6k\rangle$ in G. Let r be the root vertex of $W\langle h, 6k\rangle$. We may assume that for each leaf vertex x of $T\langle h, 6k\rangle$, there is exactly one edge between J_x and J_r.

Let Q be a tree obtained from J_r by splitting vertices, where:

- Q has maximum degree at most 3,
- J_r is a minor of Q; let $\{Q_v : v \in V(J_r)\}$ be the model of J_r in Q, so each edge vw of J_r corresponds to an edge of Q between Q_v and Q_w,
- there is a set L of leaf vertices in Q, and a bijection ϕ from L to the set of leaves of $T\langle h, 6k\rangle$, such that for each leaf x of $T\langle h, 6k\rangle$, if the edge between J_x and J_r in G is incident with vertex v in J_r, then $\phi^{-1}(x)$ is a vertex z in $L \cap Q_v$, in which case we say x and z are *associated*.

Let $L' \subseteq L$. Apply the following 'propagation' process in $T\langle h, 6k\rangle$. Initially, say that the vertices in $\phi(L')$ are *alive* with respect to L'. For each parent vertex y of leaves in $T\langle h, 6k\rangle$, if at least $2k$ of its $6k$ children are alive with respect to L', then y is also alive with respect to L'. Now propagate up $T\langle h, 6k\rangle$, so that a non-leaf vertex y of $T\langle h, 6k\rangle$ is *alive* if and only if at least $2k$ of its children are alive with respect to L'. Say L' is *good* if r is alive with respect to L'.

For an edge vw of Q let L_{vw} be the set of vertices in L in the subtree of $Q - vw$ containing v, and let L_{wv} be the set of vertices in L in the subtree of $Q - vw$ containing w. Since L is the disjoint union of L_{vw} and L_{wv}, every leaf vertex of $T\langle h, 6k\rangle$ is in exactly one of $\phi(L_{vw})$ or $\phi(L_{wv})$. By induction, every vertex in $T\langle h, 6k\rangle$ is alive with respect to L_{vw} or L_{wv} (possibly both). In particular, L_{vw} or L_{wv} is good (possibly both).

Suppose that both L_{vw} and L_{wv} are good. Then at least $2k$ children of r are alive with respect to L_{vw}, and at least $2k$ children of r are alive with respect to L_{wv}. Thus there are disjoint sets A and B, each consisting of k children of r, where every vertex in A is alive with respect to L_{vw}, and every vertex in B is alive with respect to L_{wv}. We now define a set of vertices, said to be *chosen* by v, all of which are alive with respect to L_{vw}. First, each vertex in A is *chosen* by v. Then for each non-leaf vertex z chosen by

v, choose k children of z that are also alive with respect to L_{vw}, and say they are *chosen* by v. Continue this process down to the leaves of $T\langle h, 6k\rangle$. We now define the graph G', which is initially empty. For each vertex z chosen by v, add the subgraph J_z to G'. Furthermore, for each leaf vertex z of $T\langle h, 6k\rangle$ chosen by v and for each ancestor y of z chosen by v, add the edge in G between J_z and J_y to G'. Define G'' analogously with respect to B and L_{wv}. At this point, G' and G'' are disjoint.

The edge vw in Q either corresponds to an edge or a vertex of J_r. First suppose that vw corresponds to an edge ab of J_r, where v is in Q_a and w is in Q_b. Let J_r^1 be the subtree of $J_r - ab$ containing a. Add J_r^1 to G', plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by v. Similarly, let J_r^2 be the subtree of $J_r - ab$ containing b, and add J_r^2 to G'', plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by w. Observe that G' and G'' are disjoint, and they both contain $W\langle h, k\rangle$ as a minor, as desired.

Now consider the case in which vw corresponds to a vertex z in J_r; that is, v and w are both in Q_z. Let J_r^1 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing v (which includes z). Add J_r^1 to G', plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by v. Similarly, let J_r^2 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing w (which includes z). Add J_r^2 to G'', plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k\rangle$ chosen by w. Observe that both G' and G'' contain $W\langle h, k\rangle$ as a minor, and $V(G_1) \cap V(G_2) = \{z\}$, as desired.

We may therefore assume that for each edge vw of Q, exactly one of L_{vw} and L_{wv} is good. Orient vw towards v if L_{vw} is good, and towards w if L_{wv} is good. Since at most one leaf of $T\langle h, 6k\rangle$ is associated with each leaf of Q, each edge incident with a leaf of Q is oriented away from the leaf. Since Q is a tree, Q contains a sink vertex v, which is therefore not a leaf. Let w_1, w_2 and possibly w_3 be the neighbours of v in Q. Let L_i be the set of vertices in L in the subtree of $Q - vw_i$ containing w_i. Since vw_i is oriented towards v, with respect to vw_i, the set L_i is not good. Since no leaf of $T\langle h, 6k\rangle$ is associated with v, the sets $\phi(L_1)$, $\phi(L_2)$ and $\phi(L_3)$ partition the leaves of $T\langle h, 6k\rangle$. Since each non-leaf vertex y in $T\langle h, 6k\rangle$ has $6k$ children, y is alive with respect to at least one of L_1, L_2 or L_3. In particular, at least one of L_1, L_2 or L_3 is good. This is a contradiction. ∎

Theorem 11. *Let $f(h) := \frac{1}{6}(4^h - 4)$ for every $h \geqslant 1$. Then there is a function $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $k \geqslant 1$, every graph either contains $W\langle h, k\rangle$ as a minor or is $f(h)$-colourable with clustering $g(h, k)$.*

Proof. We proceed by induction on $h \geqslant 1$. In the base case, $h=1$, since $W\langle 1,k\rangle$ is the 1-vertex graph, the result holds with $f(1)=g(1,k)=0$. Now assume that $h\geqslant 2$ and the result holds for $h-1$ and all k.

Let G be a graph, which we may assume is connected. Let $V_0,V_1,\dots$ be a BFS layering of G.

Fix $i\geqslant 1$. Let s be the maximum integer such that $G[V_i]$ contains s disjoint subgraphs $G_1,\dots,G_s$ each containing a $W\langle h-1,\max\{1,6^{k-s}\}k\rangle$ minor. First suppose that $s\geqslant k$. Then $G[V_i]$ contains k disjoint subgraphs each containing a $W\langle h-1,k\rangle$ minor. Contracting $V_0\cup\dots\cup V_{i-1}$ to a single vertex gives a $W\langle h,k\rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}), and we are done. Now assume that $s\leqslant k-1$.

If $s=0$, then $G[V_i]$ contains no $W\langle h-1,6^{k-1}k\rangle$ minor. By induction, $G[V_i]$ is $f(h-1)$-colourable with clustering $g(h-1,6^{k-1}k)$.

Now consider the case that $s\in[1,k-1]$. Apply Lemma 10 to G_j for each $j\in[1,r]$. Thus G_j contains subgraphs G'_j and G''_j, both containing $W\langle h-1,6^{k-s-1}k\rangle$ as a minor, such that $|V(G'_j)\cap V(G''_j)|\leqslant 1$. Let $X:=\bigcup_{j=1}^{s}(V(G'_j)\cap V(G''_j))$. Thus $|X|\leqslant s\leqslant k-1$. Let $A:=G[V_i]-\bigcup_{j=1}^{s}V(G'_j)$ and $B:=G[V_i]-\bigcup_{j=1}^{s}V(G''_j)$. By the maximality of s, the subgraph A contains no $W\langle h-1,6^{k-s-1}k\rangle$ minor (as otherwise $A,G'_1,\dots,G'_s$ would give $s{+}1$ pairwise disjoint subgraphs satisfying the requirements). By induction, A is $f(h-1)$-colourable with clustering $g(h-1,6^kk)$ since $6^{k-s-1}k\leqslant 6^kk$. Similarly, B is $f(h-1)$-colourable with clustering $g(h-1,6^kk)$. By construction, each vertex in $G[V_i]$ is in at least one of X, A or B. Use one new colour for X, which has size at most $s\leqslant k-1$.

In both cases, $G[V_i]$ is $(2f(h-1)+1)$-colourable with clustering $\max\{g(h-1,6^kk),k-1\}$. Use a different set of $2f(h-1)+1$ colours for even i and for odd i, and colour r by one of the colours used for even i. No edge joins V_i with V_j for $j\geqslant i+2$. Since $f(h)=4f(h-1)+2$, G is $f(h)$-colourable with clustering $g(h,k):=\max\{g(h-1,6^kk),k-1\}$. ∎

Note that the clustering function $g(h,k)$ in Theorem 11 satisfies

$$g(h,k)\leqslant k6^{k6^{k6^{\cdot^{\cdot^{\cdot^{k6^k}}}}}},$$

where the number of ks is h.

Theorem 12. *For every graph H,*

$$\chi_\star(\mathcal{M}_H)\leqslant \tfrac{1}{6}(4^{\overline{\operatorname{td}}(H)}-4).$$

Proof. Let G be a graph not containing H as a minor. By definition, H is a subgraph of $C\langle\overline{\mathrm{td}}(H),|V(H)|\rangle$. Thus G does not contain $C\langle\overline{\mathrm{td}}(H),|V(H)|\rangle$ as a minor. By Lemma 9, G does not contain $W\langle\overline{\mathrm{td}}(H),|V(H)|+1\rangle$ as a minor. By Theorem 11, there is a constant $c=c(H)$, such that G is $\frac{1}{6}(4^{\overline{\mathrm{td}}(H)}-4)$-colourable with clustering at most c. ∎

2.3. Lower Bound

We now prove Theorem 3, where $H_k:=C\langle k,3\rangle$, the closure of the complete ternary tree of depth k (which has tree-depth and connected tree-depth k).

Lemma 13. *$\chi_\star(\mathcal{M}_{C\langle k,3\rangle})\geqslant 2k-2$ for $k\geqslant 2$.*

Proof. Fix an integer c. We now recursively define graphs G_k (depending on c), and show by induction on k that G_k has no $(2k-3)$-colouring with clustering c, and $C\langle k,3\rangle$ is not a minor of G_k.

For the base case $k=2$, let G_2 be the path on $c+1$ vertices. Then G_2 has no $C\langle 2,3\rangle=K_{1,3}$ minor, and G_2 has no 1-colouring with clustering c.

Assume G_{k-1} is defined for some $k\geqslant 3$, that G_{k-1} has no $(2k$–$5)$-colouring with clustering c, and $C\langle k-1,3\rangle$ is not a minor of G_{k-1}. As illustrated in Figure 1, let G_k be obtained from a path $(v_1,\dots,v_{c+1})$ as follows: for $i\in\{1,\dots,c\}$ add $2c$–1 pairwise disjoint copies of G_{k-1} complete to $\{v_i,v_{i+1}\}$.

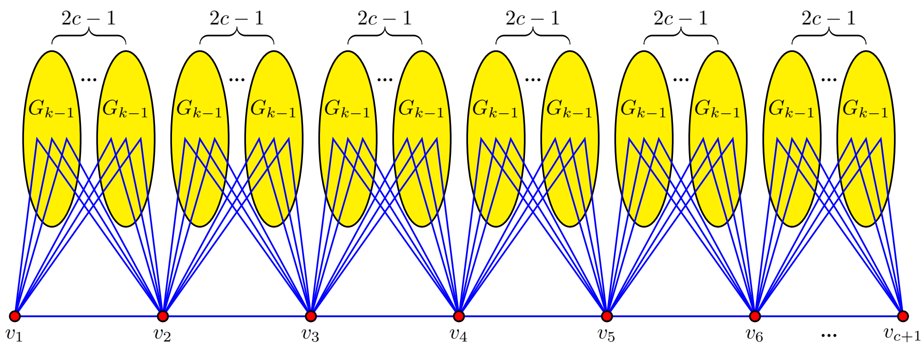

Figure 1. Construction of G_k

Suppose that G_k has a $(2k-3)$-colouring with clustering c. Then v_i and v_{i+1} receive distinct colours for some $i\in\{1,\dots,c\}$. Consider the $2c$–1 copies of G_{k-1} complete to $\{v_i,v_{i+1}\}$. At most $c-1$ such copies contain a vertex assigned the same colour as v_i, and at most $c-1$ such copies contain a vertex assigned the same colour as v_{i+1}. Thus some copy avoids both colours. Hence G_{k-1} is $(2k–5)$-coloured with clustering c, which is a contradiction. Therefore G_k has no $(2k-3)$-colouring with clustering c.

It remains to show that $C\langle k,3\rangle$ is not a minor of G_k. Suppose that G_k contains a model $\{J_x\colon x \in V(C\langle k,3\rangle)\}$ of $C\langle k,3\rangle$. Let r be the root vertex in $C\langle k,3\rangle$. Choose the $C\langle k,3\rangle$-model to minimise $\sum_{x\in V(C\langle k,3\rangle)} |V(J_x)|$. Since $\{v_1,\dots,v_{c+1}\}$ induces a connected dominating subgraph in G_k, by the minimality of the model, J_r is a connected subgraph of $(v_1,\dots,v_{c+1})$. Say $J_r=(v_i,\dots,v_j)$. Note that $C\langle k,3\rangle-r$ consists of three pairwise disjoint copies of $C\langle k-1,3\rangle$. The model X of one such copy avoids v_{i-1} and v_{j+1} (if these vertices are defined). Since $C\langle k-1,3\rangle$ is connected, X is contained in a component of $G_k-\{v_{i-1},\dots,v_{j+1}\}$ and is adjacent to $(v_i,\dots,v_j)$. Each such component is a copy of G_{k-1}. Thus $C\langle k-1,3\rangle$ is a minor of G_{k-1}, which is a contradiction. Thus $C\langle k,3\rangle$ is not a minor of G_k. ∎

3. 2-Colouring with Bounded Clustering

This section considers the following question: which minor-closed graph classes have clustered chromatic number 2? To answer this question we introduce three classes of graphs that are not 2-colourable with bounded clustering, as illustrated in Figure 2.

The first example is the *n-fan*, which is the graph obtained from the n-vertex path by adding one dominant vertex. If the n-fan is 2-colourable with clustering c, then the underlying path contains at most $c-1$ vertices of the same colour as the dominant vertex, implying that the other colour has at most c monochromatic components each with at most c vertices, and $n \leqslant c^2+c-1$. That is, if $n \geqslant c^2+c$, then the n-fan is not 2-colourable with clustering c.

The second example is the *n-fat star*, which is the graph obtained from the n-star (the star with n leaves) as follows: for each edge vw in the n-star, add n degree-2 vertices adjacent to v and w. Note that the n-fat star is $C\langle 3,n\rangle$. Suppose that the n-fat star has a 2-colouring with clustering $c \leqslant n$. Deleting the dominant vertex in the n-fat star gives n disjoint n-stars. Since $n \geqslant c$, in at least one of these n-stars, no vertex receives the same colour as the dominant vertex, implying there is a monochromatic component on $n+1 \geqslant c+1$ vertices. Thus, for $n \geqslant c$ there is no 2-colouring of the n-fat star with clustering c.

The third example is the *n-fat path*, which is the graph obtained from the n-vertex path as follows: for each edge vw of the n-vertex path, add n

degree-2 vertices adjacent to v and w. If $n \geqslant 2c-1$, then in every 2-colouring of the n-fat path with clustering c, adjacent vertices in the underlying path receive the same colour, implying that the underlying path is contained in a monochromatic component with more than c vertices. Thus, for $n \geqslant 2c-1$ there is no 2-colouring of the n-fat path with clustering c.

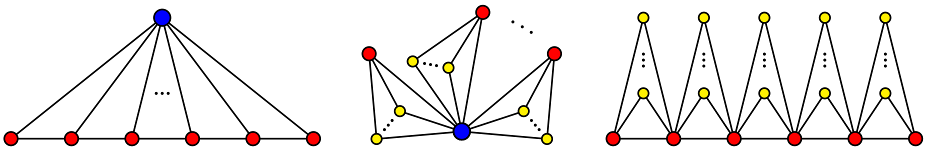

Figure 2. Graph classes that are not 2-colourable with bounded clustering

These three examples all need three colours in a colouring with bounded clustering. The main result of this section is the following converse result.

Theorem 14. *Let $\mathcal{G}$ be a minor-closed graph class. Then $\chi_\star(\mathcal{G}) \leqslant 2$ if and only if for some integer $k \geqslant 2$, the k-fan, the k-fat path, and the k-fat star are not in $\mathcal{G}$.*

Lemma 24 below shows that every graph containing no k-fan minor, no k-fat path minor, and no k-fat star minor is 2-colourable with clustering $f(k)$ for some explicit function f. Along with the above discussion, this implies Theorem 14. We assume $k \geqslant 2$ for the remainder of this section.

The following definition is a key to the proof. For an h-vertex graph H with vertex set $\{v_1, \dots, v_h\}$, a *k-strong H-model* in a graph G consists of h pairwise disjoint connected subgraphs $X_1, \dots, X_h$ in G, such that for each edge $v_i v_j$ of H there are at least k vertices in $V(G) \setminus \bigcup_{i=1}^{h} V(X_i)$ adjacent to both X_i and X_j. Note that a vertex in $V(G) \setminus \bigcup_{i=1}^{h} V(X_i)$ might count towards this set of k vertices for distinct edges of H. This definition leads to the following sufficient condition for a graph to contain a k-fat star or k-fat path

Lemma 15. *If a graph G contains a $k(k+1)$-strong H-model for some connected graph H with k^k edges, then G contains a k-fat star or a k-fat path as a minor.*

Proof. Use the notation introduced in the definition of k-strong H-model. Since H is connected with k^k edges, H contains a k-vertex path or a k-leaf star as a subgraph. Suppose that $(v_1, \dots, v_k)$ is a k-vertex path in H. For

$i=1,2,\ldots,k-1$, let N_i be a set of $k+1$ vertices in

$$\Big(V(G)\setminus\bigcup_{j=1}^{h}V(X_j)\Big)\setminus\bigcup_{j=1}^{i-1}N_j,$$

each of which is adjacent to both X_i and X_{i+1}. Such a set exists since X_i and X_{i+1} have at least $k(k+1)$ common neighbours in $V(G)\setminus\bigcup_{j=1}^{h}V(X_j)$. For $i\in[1,k-1]$, contract one vertex of N_i into X_i. Then contract each of $X_1,\ldots,X_h$ into a single vertex. We obtain the k-fat path as a minor in G. The case of a k-leaf star is analogous. ∎

Lemma 16. *If a connected graph G contains a $(k+2c-2)$-strong H-model, for some graph H with c connected components, then G contains a k-strong H'-model for some connected graph H' with $|E(H')|=|E(H)|$.*

Proof. We proceed by induction on $c\geqslant 1$. The case $c=1$ is vacuous. Assume $c\geqslant 2$, and the result holds for $c-1$. Let $H_1,\ldots,H_c$ be the components of H. We may assume that H has no isolated vertices. Say $X_1,\ldots,X_h$ is a $(k+2c-2)$-strong H-model in G. For each edge v_iv_j in H, let N_{ij} be a set of $k+2c-2$ common neighbours of X_i and X_j. For each component H_a of H, note that $(\bigcup_{v_i\in V(H_a)}V(X_i))\cup(\bigcup_{v_iv_j\in E(H_a)}N_{ij})$ induces a connected subgraph in G, which we denote by G_a. Since G is connected, there is a path P between G_a and G_b, for some distinct $a,b\in[1,c]$, such that no internal vertex of P is in $G_1\cup\cdots\cup G_c$. Note that P might be a single vertex. For some edge $v_iv_{i'}$ in H_a and some edge $v_jv_{j'}$ in H_b, without loss of generality, P joins some vertex x in $V(X_i)\cup N_{ii'}$ and some vertex y in $V(X_j)\cup N_{jj'}$. Let H' be the graph obtained from H by identifying v_i and v_j into a new vertex v_0. Now H' has $c-1$ components and $|E(H')|=|E(H)|$. Define $X_0:=X_i\cup X_j\cup P$. If $x\notin V(X_i)$, then add the edge between x and X_i to X_0. Similarly, if $y\notin V(X_j)$, then add the edge between y and X_j to X_0. Remove x and/or y from $N_{\alpha\beta}$ for each edge $v_\alpha v_\beta$ of H'. Now $|N_{\alpha\beta}|\geqslant k+2(c-1)-2$. We obtain a $(k+2(c-1)-2)$-strong H'-model in G. By induction, G contains a k-strong H''-model for some connected graph H'' with $|E(H'')|=|E(H)|$. ∎

Lemma 17. *If a connected graph G contains a $3k^k$-strong H-model for some graph H with at least k^k edges, then G contains a k-fat star or a k-fat path as a minor.*

Proof. We may assume that H has exactly k^k edges and has no isolated vertices. Say H has c connected components. Then $c\leqslant k^k$ and $3k^k\geqslant k^2+k+2c-2$. Hence G contains a $(k^2+k+2c-2)$-strong H-model. The result then follows from Lemmas 15 and 16. ∎

Lemma 18. *Let G be a connected graph such that $\deg_G(v) \geqslant 2\ell k$ for some non-cut-vertex v and integers $k, \ell \geqslant 1$. Then G contains a k-fan as a minor, or G contains a connected subgraph X and v has ℓ neighbours not in X and all adjacent to X (thus contracting X gives a $K_{2,\ell}$ minor).*

Proof. Let r be a vertex of $G-v$. For each $w \in N_G(v)$, let P_w be a wr-path in $G-v$. If $|P_w \cap N_G(v)| \geqslant k$ for some $w \in N_G(v)$, then G contains a k-fan minor. Now assume that $|P_w \cap N_G(v)| \leqslant k-1$ for each $w \in N_G(v)$. Let H be the digraph with vertex set $N_G(v)$, where $N_H^+(w) := V(P_w) \cap N_G(v)$ for each vertex w. Thus H has maximum outdegree at most $k-1$, and the underlying undirected graph of H has average degree at most $2k-2$. Since $|V(H)| \geqslant 2\ell k$, by Turán's Theorem, H contains a stable set S of size ℓ. Let $X := \bigcup\{P_w : w \in S\} - S$, which is connected since S is stable. Each vertex in S is adjacent to v and to X, as desired. ∎

Lemma 19. *Let G be a graph with distinct vertices $v_1, \dots, v_k$, such that $C := G - \{v_1, \dots, v_k\}$ is connected and $\deg_C(v_i) \geqslant k^3$ for each $i \in [1,k]$. Then G contains a k-fan or k-fat star as a minor.*

Proof. The idea of the proof is to attempt to build a k-fan model by constructing a subtree X such that each v_i is adjacent to a subset S_i of k leaves of X (where the S_i are disjoint). We construct X and the S_i by adding, one at a time, paths to some neighbour w of some v_i to increase the size of S_i. We always choose a neighbour at maximal distance from some root vertex, among all neighbours of all v_i for which S_i is not yet large enough: this ensures that later paths will not pass through the sets S_i that have been previously constructed.

We now formalise this idea. Let r be a vertex in C. Let $V_0, V_1, \dots, V_n$ be a BFS layering of C starting at r. Initialise $t := n$ and $X := \{r\}$ and $S_i := \emptyset$ for $i \in [1,k]$ and $S := \emptyset$. The following properties trivially hold:

(0) $S = \bigcup_{i \in [1,k]} S_i$ and $S \subseteq V_t \cup V_{t+1} \cup \dots \cup V_n$.

(1) X is a (connected) subtree of C rooted at r with (non-root) leaf set S.

(2) $S_i \cap S_j = \emptyset$ for distinct $i, j \in [1,k]$.

(3) S_i is a set of at most $k+1$ neighbours of v_i for $i \in [1,k]$ (and so $|S| \leqslant k(k+1)$).

(4) $|N_{C-V(X)}(v_i)| \geqslant k^3 - 1 - (k-1)|S| > 0$ for $i \in [1,k]$.

Now execute the following algorithm, which maintains properties (0)–(4). Think of V_t as the 'current' layer.

While $|S_i| \leqslant k$ for some $i \in [1,k]$ repeat the following: If $V_t \cap N_{C-V(X)}(v_i) = \emptyset$ for all $i \in [1,k]$ with $|S_i| \leqslant k$, then let $t := t-1$. Properties (0)–(4) are trivially maintained. Otherwise, let w be a vertex in $V_t \cap N_{C-V(X)}(v_i)$ for some $i \in [1,k]$ with $|S_i| \leqslant k$. Since $V_0, V_1, \dots, V_n$ is a BFS layering of C rooted at r and r is in X, there is a path P from w to X consisting of at most one vertex from each of $V_0, \dots, V_t$, and with no internal vertices in X. By (0) and since $w \notin S$, P avoids S. By (1), the endpoint of P in X is not a leaf of X. If P contains at least k vertices in $N_C(v_j)$ for some $j \in [1,k]$, then G contains a k-fan minor and we are done. Now assume that P contains at most $k-1$ vertices in $N_C(v_j)$ for each $j \in [1,k]$. Let $S_i := S_i \cup \{w\}$ and $S := S \cup \{w\}$ and $X := X \cup P$. Now w is a leaf of X, and property (1) is maintained. Properties (0), (2) and (3) are maintained by construction. Property (4) is maintained since $|S|$ increases by 1 and P contains at most $k-1$ vertices in $N_C(v_j)$ for each $j \in [1,k]$.

The algorithm terminates when $|S_i| = k+1$ for each $i \in [1,k]$. Delete $C - V(X)$. Contract $X - S$ (which is connected by (1)) to a single vertex z. Since S is the set of leaves of X, each vertex in S_i is adjacent to both v_i and z. Contract one edge between v_i and S_i for each $i \in [1,k]$. We obtain the k-fat star as a minor. ∎

Lemma 20. *Let G be a bipartite graph with bipartition A, B, such that at least p vertices in A have degree at least $k|A|$, and every vertex in B has degree at least 2. Then G contains a k-strong H-model for some graph H with at least $p/2$ edges.*

Proof. Let H be the graph with $V(H) := A$ where $vw \in E(H)$ whenever $|N_G(v) \cap N_G(w)| \geqslant k$. Since every vertex in B has degree at least 2, every vertex in A with degree at least $k|A|$ is incident with some edge in H. Thus H has at least $p/2$ edges. By construction, G contains a k-strong H-model. ∎

For the remainder of this section, let $d := (k+2)k^k(18k^{2k+1}+1)$. A vertex v is *high-degree* if $\deg(v) \geqslant d$, otherwise v is *low-degree.*

Lemma 21. *If a 2-connected graph G has at least $(k+2)k^k$ high-degree vertices, then G contains a k-fat path, a k-fat star, or a k-fan as a minor.*

Proof. Let A be a set of exactly $(k+2)k^k$ high-degree vertices in G. Let $C_1, \dots, C_p$ be the components of $G - A$. Say (v, C_j) is a *heavy pair* if $v \in A$ and v has at least $6k^{k+1}$ neighbours in C_j. Since $6k^{k+1} \geqslant k^3$, by Lemma 19, if some C_j is in at least k heavy pairs, then G contains a k-fan or k-fat star as a minor, and we are done. Now assume that each C_j is in fewer than k heavy pairs. Let h be the total number of heavy pairs. Then there is a set P of at least h/k heavy pairs containing at most one heavy pair for each component C_j. For each such heavy pair (v, C_j), by Lemma 18 with $\ell = 3k^k$, $G[V(C_j) \cup \{v\}]$ contains a k-fan as a minor (and we are done) or a $K_{2,3k^k}$

minor, where $G[\{v\}]$ is the subgraph corresponding to one of the vertices in the colour class of size 2 in $K_{2,3k^k}$. We obtain a $3k^k$-strong H-model for some graph H, where $|E(H)|=|P|\geqslant h/k$. If $h/k\geqslant k^k$, then we are done by Lemma 17. Now assume that $h<k^{k+1}$. In particular, the number of vertices in A that are in a heavy pair is less than k^{k+1}. Let A' be the set of vertices in A in no heavy pair; thus $|A'|\geqslant 2k^k$. Let H be the bipartite graph with bipartition A,B, where there is one vertex w_j in B for each component C_j, and $v\in A$ is adjacent to $w_j\in B$ if and only if v is adjacent to some vertex in C_j. In H, every vertex in A' has degree at least $(d-|A|)/6k^{k+1}$, which is at least $3k^k|A|$. (Note that d is defined so that this property holds.) Since G is 2-connected, each C_j is adjacent to at least two vertices in A. Thus, every vertex in B has degree at least 2 in H. By Lemma 20, H contains a $3k^k$-strong model of a graph with at least $|A'|/2\geqslant k^k$ edges. By Lemma 17 we are done. ∎

Lemma 22. *Let $V_0,V_1,\dots$ be a BFS layering in a connected graph G. If $G[V_i\cup V_{i+1}\cup\cdots\cup V_{i+c}]$ contains a path on at least k^{c+1} vertices for some $i,c\geqslant 0$, then G contains a k-fan minor.*

Proof. We proceed by induction on c. Let P be a path in

$$G[V_i\cup V_{i+1}\cup\ldots\cup V_{i+c}]$$

on k^{c+1} vertices. First suppose that P contains k vertices $v_1,\dots,v_k$ in V_i (which must happen in the base case $c=0$). Each vertex v_i has a neighbour in V_{i-1}. Thus, contracting $G[V_0\cup\cdots\cup V_{i-1}]$ into a single vertex and contracting P between v_i and v_{i+1} to an edge (for $i\in[1,k-1]$) gives a k-fan minor. Now assume that P contains at most $k-1$ vertices in V_i and $c\geqslant 1$. Thus $P-V_i$ has at least $k^{c+1}-(k-1)$ vertices and at most k components. Thus, some component of $P-V_i$ has at least $\lceil (k^{c+1}-k+1)/k\rceil=k^c$ vertices and is contained in $G[V_{i+1}\cup V_{i+2}\cup\cdots\cup V_{i+c}]$. By induction, G contains a k-fan minor. ∎

Say a vertex v in a coloured graph is *properly* coloured if no neighbour of v gets the same colour as v.

Lemma 23. *Let G be a 2-connected graph containing no k-fan, k-fat star or k-fat path as a minor. Let h be the number of high-degree vertices in G. Let r be a vertex in G. Then G is 2-colourable with clustering at most $d^{k^{3(k+2)k^k}}$. Moreover, if $h=0$, then we can additionally demand that r is properly coloured.*

Proof. Let $V_0, V_1, \dots$ be the BFS layering of G starting at r.

First suppose that $h=0$. Colour each vertex $v \in V_i$ by $i \bmod 2$. Then r is properly coloured. Every monochromatic component is contained in some V_i. Suppose that some component X of $G[V_i]$ has at least d^k vertices. Thus $i \geqslant 1$. Since G and thus X has maximum degree at most d, X contains a path of k vertices. Contracting $G[V_0 \cup \dots \cup V_{i-1}]$ into a single vertex gives a k-fan minor. This contradiction shows that the 2-colouring has clustering at most d^k.

Now assume that $h \geqslant 1$. By Lemma 21, $h \leqslant (k+2)k^k$. Colour all the high-degree vertices black. Let I be the set of integers $i \geqslant 0$ such that V_i contains a high-degree vertex. Colour all the low-degree vertices in $\bigcup\{V_i : i \in I\}$ white.

Let $V_i, V_{i+1}, \dots, V_{i+c}$ be a maximal sequence of layers with no high-degree vertices, where $c \geqslant 0$. Thus V_{i-1} is empty or contains a high-degree vertex. Similarly, V_{i+c+1} is empty or contains a high-degree vertex. If c is even, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c}$ white and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-1}$ black. If c is odd, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c-1}$ and V_{i+c} white, and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-2}$ black. Note that if $c \geqslant 2$, then at least one of $V_{i+1}, \dots, V_{i+c-1}$ is black.

We now show that each black component X has bounded size. If X contains some high-degree vertex, then every vertex in X is high-degree and $|X| \leqslant h \leqslant (k+2)k^k$. Now assume that X contains no high-degree vertices. Say X intersects V_j. Since each black layer is preceded by and followed by a white layer, X is contained in V_j. Every vertex in X has degree at most d in G. Thus if X has at least d^k vertices, then X contains a path of length k, and contracting $V_0 \cup \dots \cup V_{j-1}$ to a single vertex gives a k-fan. Hence X has at most d^k vertices.

Finally, let X be a white component. Then X is contained within at most $3h \leqslant 3(k+2)k^k$ consecutive layers (since in the notation above, if all of $V_i, V_{i+1}, \dots, V_{i+c}$ are white, then $c \leqslant 1$). Suppose that $|X| \geqslant d^{k^{3(k+2)k^k}}$. Since X has maximum degree at most d, X contains a path of length $k^{3(k+2)k^k}$. Thus, Lemma 22 with $c+1 = 3(k+2)k^k$ implies that G contains a k-fan minor. Hence $|X| \leqslant d^{k^{3(k+2)k^k}}$. ∎

We now complete the proof of Theorem 14.

Lemma 24. *Let G be a graph containing no k-fan, no k-fat path, and no k-fat star as a minor. Then G is 2-colourable with clustering $kd^{k^{3(k+2)k^k}}$.*

Proof. We may assume that G is connected. Let r be a vertex of G. If B is a block of G containing r, then consider B to be rooted at r. If B is a block of G not containing r, then consider B to be rooted at the unique vertex in

B that separates B from r. Say (B,v) is a *high-degree pair* if B is a block of G and v has high-degree in B. Note that one vertex might be in several high-degree pairs.

Suppose that some vertex v is in at least k high-degree pairs with blocks $B_1,\dots,B_k$. Since $d \geqslant 2k(k+1)$, by Lemma 18 with $\ell = k+1$, for $i \in [k]$, there is a connected subgraph X_i in $B_i - v$ and there is a set $N_i \subseteq N_{B_i}(v) \setminus V(X_i)$ of size $k+1$, such that each vertex in N_i is adjacent to X_i. For $i \in [1,k]$, contract X_i into a single vertex, and contract one edge between v and N_i. We obtain a k-fat star as a minor. Now assume that each vertex is in fewer than k high-degree pairs.

Colour each block B in non-decreasing order of the distance in G from r to the root of B. Let B be a block of G rooted at v (possibly equal to r). Then v is already coloured in the parent block of B. Let h_B be the number of high-degree pairs involving B. By Lemma 23, B is 2-colourable with clustering at most $d^{k^{3(k+2)k^k}}$, such that if $h_B = 0$, then v is properly coloured. Permute the colours in B so that the colour assigned to v matches the colour assigned to v by the parent block. Then the monochromatic component containing v is contained within the parent block of B along with those blocks rooted at v that form a high-degree pair with v. As shown above, there are at most k such blocks. Thus, each monochromatic component has at most $kd^{k^{3(k+2)k^k}}$ vertices. ∎

4. Excluding a Fat Star

This section considers colourings of graphs excluding a fat star. We need the following more general lemma.

Lemma 25. *For every planar graph H,*

$$\chi_\star(\mathcal{M}_H) \leqslant 2\,\chi_\Delta(\mathcal{M}_H).$$

Proof. The grid minor theorem of Robertson and Seymour [18] says that every graph in $\mathcal{M}_H$ has tree-width at most some function $w(H)$. (Chekuri and Chuzhoy [2] recently showed that w can be taken to be polynomial in $|V(H)|$.) Alon, Ding, Oporowski, and Vertigan [1] observed that every graph with tree-width w and maximum degree Δ is 2-colourable with clustering $24w\Delta$. Let $k := \chi_\Delta(\mathcal{M}_H)$. That is, every H-minor-free graph G is k-colourable with monochromatic components of maximum degree at most some function $d(H)$. Apply the above result of Alon et al. [1] to each monochromatic component. Thus G is $2k$-colourable with clustering $24\,w(H)\,d(H)$. Hence $\chi_\star(\mathcal{M}_H) \leqslant 2k$. ∎

A variant of Lemma 25 holds for arbitrary graphs H with "2" replaced by "3". The proof uses a result of Liu and Oum [13] in place of the result of Alon et al. [1]; see [5,6].

Theorem 26. *For $k \geqslant 3$, the clustered chromatic number of the class of graphs containing no k-fat star minor equals 4.*

Proof. As illustrated in Figure 2, the k-fat star is planar. Ossona de Mendez et al. [16] proved that graphs containing no k-fat star minor are 2-colourable with defect $O(k^{13})$. Thus, Lemma 25 implies that the clustered chromatic number of the class of graphs containing no k-fat star is at most 4. To obtain a bound on the clustering, note that a result of Leaf and Seymour [12] implies that every graph containing no k-fat star minor has tree-width $O(k^2)$. It follows from the proof of Lemma 25 that every graph containing no k-fat star minor is 4-colourable with clustering $O(k^{15})$. Since the 3-fat star is $C\langle 3,3\rangle$, Lemma 13 implies that for $k\geqslant 3$, the clustered chromatic number of the class of graphs containing no k-fat star minor is at least 4. ∎

Every graph H with $\overline{\text{td}}(H) \leqslant 3$ is a subgraph of the k-fat star for some $k \leqslant |V(H)|$. Thus Theorem 26 implies Conjecture 4 in the case of connected tree-depth 3.

Corollary 27. *For every graph H with $\overline{\text{td}}(H)\leqslant 3$,*

$$\chi_\star(\mathcal{M}_H) \leqslant 4.$$

We can push this result further.

Theorem 28. *For every graph H with $\text{td}(H)\leqslant 3$,*

$$\chi_\star(\mathcal{M}_H) \leqslant 5.$$

Proof. Say H has p components. Each component of H is a subgraph of the k-fat star for some $k \leqslant |V(H)|$. Let H' consist of p pairwise disjoint copies of the k-fat star. Let G be an H-minor-free graph. Thus G is also H'-minor-free. By the Grid Minor Theorem of Robertson and Seymour [18] and since H' is planar, G has treewidth at most $w = w(H')$. By Theorem 5, there is a set X of at most $(p-1)(w-1)$ vertices in G, such that $G-X$ contains no k-fat star as a minor. By Theorem 26, $G-X$ is 4-colourable with clustering at most some function of H. Assign vertices in X a fifth colour. Thus G is 5-colourable with clustering at most some function of H. ∎

5. A Conjecture about Clustered Colouring

We now formulate a conjecture about the clustered chromatic number of an arbitrary minor-closed class of graphs. Consider the following recursively defined class of graphs. Let $\mathcal{X}_{1,c} := \{P_{c+1}, K_{1,c}\}$. Here P_{c+1} is the path with $c+1$ vertices, and $K_{1,c}$ is the star with c leaves. As illustrated in Figure 3, for $k \geqslant 2$, let $\mathcal{X}_{k,c}$ be the set of graphs obtained by the following three operations. For the first two operations, consider an arbitrary graph $G \in \mathcal{X}_{k-1,c}$.

- Let G' be the graph obtained from c disjoint copies of G by adding one dominant vertex. Then G' is in $\mathcal{X}_{k,c}$.
- Let G^+ be the graph obtained from G as follows: for each k-clique D in G, add a stable set of $k(c-1)+1$ vertices complete to D. Then G^+ is in $\mathcal{X}_{k,c}$.
- If $k \geqslant 3$ and $G \in \mathcal{X}_{k-2,c}$, then let G^{++} be the graph obtained from G as follows: for each $(k-1)$-clique D in G, add a path of $(c^2-1)(k-1)+(c+1)$ vertices complete to D. Then G^{++} is in $\mathcal{X}_{k,c}$.

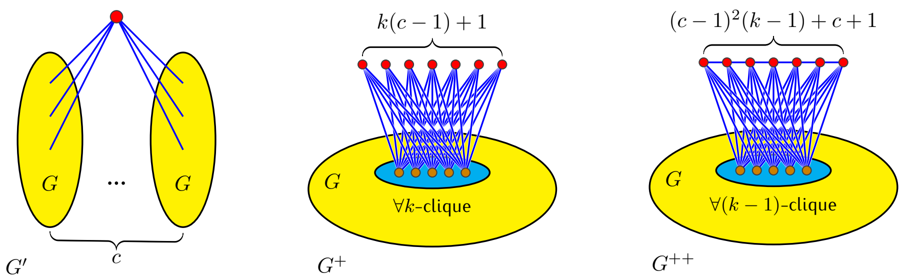

Figure 3. Construction of $\mathcal{X}_{k,c}$

A vertex-coloured graph is *rainbow* if every vertex receives a distinct colour.

Lemma 29. *For every $c \geqslant 1$ and $k \geqslant 2$, for every graph $G \in \mathcal{X}_{k,c}$, every colouring of G with clustering c contains a rainbow K_{k+1}. In particular, no graph in $\mathcal{X}_{k,c}$ is k-colourable with clustering c.*

Proof. We proceed by induction on $k \geqslant 1$. In the case $k = 1$, every colouring of P_{c+1} or $K_{1,c}$ with clustering c contains an edge whose endpoints receive distinct colours, and we are done. Now assume the claim for $k-1$ and for $k-2$ (if $k \geqslant 3$).

Let $G \in \mathcal{X}_{k-1,c}$. Consider a colouring of G' with clustering c. Say the dominant vertex v is blue. At most $c-1$ copies of G contain a blue vertex. Thus, some copy of G has no blue vertex. By induction, this copy of G contains a rainbow K_k. With v we obtain a rainbow K_{k+1}.

Now consider a colouring of G^+ with clustering c. By induction, the copy of G in G^+ contains a clique $w_1,\dots,w_k$ receiving distinct colours. Let S be the set of $k(c-1)+1$ vertices adjacent to $w_1,\dots,w_k$ in G^+. At most $c-1$ vertices in S receive the same colour as w_i. Thus some vertex in S receives a colour distinct from the colours assigned to $w_1,\dots,w_k$. Hence G^+ contains a rainbow K_{k+1}.

Now suppose $k \geqslant 3$ and $G \in \mathcal{X}_{k-2,c}$. Consider a colouring of G^{++} with clustering c. By induction, the copy of G in G^{++} contains a clique $w_1,\dots,w_{k-1}$ receiving distinct colours. Let P be the path of $(c^2-1)(k-1)+(c+1)$ vertices in G^{++} complete to $w_1,\dots,w_{k-1}$. Let X_i be the set of vertices in P assigned the same colour as w_i, and let $X := \bigcup_i X_i$. Thus $|X_i| \leqslant c-1$ and $|X| \leqslant (c-1)(k-1)$. Hence $P-X$ has at most $(c-1)(k-1)+1$ components, and $|V(P-X)| \geqslant (c^2-1)(k-1)+(c+1)-(c-1)(k-1) = c\big((c-1)(k-1)+1\big)+1$. Some component of $P-X$ has at least $c+1$ vertices, and therefore contains a bichromatic edge xy. Then $\{w_1,\dots,w_{k-1}\} \cup \{x,y\}$ induces a rainbow K_{k+1} in G^{++}. ∎

We conjecture that a minor-closed class that excludes every graph in $\mathcal{X}_{k,c}$ for some c is k-colourable with bounded clustering. More precisely:

Conjecture 30. *For every minor-closed class $\mathcal{M}$ of graphs,*

$$\chi_\star(\mathcal{M}) = \min\{k\colon \exists c\; \mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset\}.$$

Several comments about Conjecture 30 are in order:

- To prove the lower bound in Conjecture 30, let k be the minimum integer such that $\mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset$ for some integer c. Thus, for every integer c some graph $G \in \mathcal{X}_{k-1,c}$ is in $\mathcal{M}$. By Lemma 29, G has no $(k-1)$-colouring with clustering c. Thus $\chi_\star(\mathcal{M}) \geqslant k$.
- Note that the $k=1$ case of Conjecture 30 is trivial: a graph is 1-colourable with bounded clustering if and only if each component has bounded size, which holds if and only if every path has bounded length and every vertex has bounded degree.
- We note that Theorem 14 implies Conjecture 30 with $k=2$. If $G = P_{c+1}$, then G' is contained in the $c(c+1)$-fan and G^+ is contained in the $(2c-1)$-fat path. If $G = K_{1,c}$, then G' is the c-fat star and G^+ is contained in the $(2c-1)$-fat star. It follows that if a minor-closed class $\mathcal{M}$ excludes every

graph in $\mathcal{X}_{2,c}$ for some c , then $\mathcal{M}$ excludes the $c(c+1)$-fan, the $(2c-1)$-fat path, and the $(2c-1)$-fat star. Then $\chi_\star(\mathcal{M})\leqslant 2$ by Theorem 14.

- We now relate Conjectures 4 and 30. Fix a graph H. Conjecture 30 says that the clustered chromatic number of $\mathcal{M}_H$ equals the minimum integer k such that for some integer c, every graph in $\mathcal{X}_{k,c}$ contains H as a minor. Let $k:=\overline{\mathrm{td}}(H)\geqslant 2$. An easy inductive argument shows that every graph in $\mathcal{X}_{2k-2,c}$ contains a $C\langle k,c\rangle$ minor. Thus, for a suitable value of c, every graph in $\mathcal{X}_{2k-2,c}$ contains H as a minor. Hence, Conjecture 30 implies Conjecture 4.
- Consider the case of excluding the complete bipartite graph $K_{s,t}$ as a minor for $s\leqslant t$. Van den Heuvel and Wood [6] proved the lower bound, $\chi_\star(\mathcal{M}_{K_{s,t}})\geqslant s+1$ for $t\geqslant \max\{s,3\}$. Their construction is a special case of the construction above. We claim that Conjecture 30 asserts that $\chi_\star(\mathcal{M}_{K_{s,t}})=s+1$ for $t\geqslant\max\{s,3\}$. To see this, first note that an easy inductive argument shows that every graph in $\mathcal{X}_{s+1,t}$ contains a $K_{s,t}$ subgraph; thus $\mathcal{M}_{K_{s,t}}\cap\mathcal{X}_{s+1,t}=\emptyset$. Furthermore, another easy inductive argument shows that for all $s,c\geqslant 1$, there is a graph in $\mathcal{X}_{s,c}$ containing no $K_{s,\max\{s,3\}}$ minor. This implies that $\mathcal{M}_{K_{s,t}}\cap\mathcal{X}_{s,c}\neq\emptyset$ for all $t\geqslant\max\{s,3\}$. Together these observations show that $\min\{k\colon \exists c\,\mathcal{M}_{s,t}\cap\mathcal{X}_{k,c}=\emptyset\}=s+1$ for $t\geqslant\max\{s,3\}$. That is, Conjecture 30 asserts that $\chi_\star(\mathcal{M}_{K_{s,t}})=s+1$ for $t\geqslant\max\{s,3\}$. Van den Heuvel and Wood [6] proved the upper bound, $\chi_\star(\mathcal{M}_{K_{s,t}})\leqslant 3s$ for $t\geqslant s$, which was improved to $2s+2$ by Dvořák and Norin [4].

6. An Alternative View

We conclude the paper with alternative versions of Conjectures 2 and 30 that shift the focus to characterising minimal minor-closed classes of given defective and clustered chromatic number.

We start with some definitions. Let H and G be two vertex-disjoint graphs, and let $S\subseteq V(G)$. Let G' be obtained from $G\cup H$ by joining every vertex of S to every vertex of H by an edge. Then we say that G' is obtained from G by *taking a join with H along S*. Let $\mathcal{H}$ be a class of graphs. We say that a graph G' is an *$\mathcal{H}$-decoration* of a graph G, if G' is obtained from G by repeatedly taking joins with graphs in $\mathcal{H}$ along cliques of G. For a class of graphs $\mathcal{G}$, let $\mathcal{G}\wedge\mathcal{H}$ denote the class of all minors of $\mathcal{H}$-decorations of graphs in $\mathcal{G}$. One can routinely verify that the $\wedge$ operation is associative. The examples below show that it is not always commutative.

First, we introduce notation for some minor-closed classes that will serve as the basis for our constructions. Let $\mathcal{I}$ denote the class of graphs on at most one vertex, let $\mathcal{O}$ denote the class of edgeless graphs, and let $\mathcal{P}$ denote the class of linear forests (that is, subgraphs of paths). Let $\mathcal{T}_d$ denote the class of all graphs of tree-depth at most d. Then $\mathcal{T}_1$ is a class of all edgeless graphs. It follows from the alternative definition of tree-depth given in [14, Section 6.1] that $\mathcal{T}_{d+1} = \mathcal{O} \wedge \mathcal{T}_d$.

The operations used in Conjecture 30 can be described as follows.

- Adding a vertex adjacent to several copies of graphs in the class $\mathcal{G}$ (and taking all possible minors) produces the class $\mathcal{I} \wedge \mathcal{G}$.
- Adding stable sets complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{I}$.
- Adding paths complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{P}$.

Note that by definition $\mathcal{G} \wedge \mathcal{H}$ is a minor-closed class for any pair of minor-closed classes $\mathcal{G}$ and $\mathcal{H}$.

We next present an analogue of Lemma 29 using the notions introduced above. A class of graphs $\mathcal{G}$ is *k-cluster rainbow* (respectively, *k-defect rainbow*) if for every c there exists $G \in \mathcal{G}$ such that every colouring of G with clustering (respectively, defect) at most c contains a rainbow clique of size k. For example, $\mathcal{I}$ is 1-cluster rainbow and 1-defect rainbow, $\mathcal{P}$ is 2-cluster rainbow, but not 2-defect rainbow. Note that if a class of graphs $\mathcal{G}$ is k-cluster rainbow, then clearly $\chi_\star(\mathcal{G}) \geqslant k$. Similarly, if $\mathcal{G}$ is k-defect rainbow, then $\chi_\Delta(\mathcal{G}) \geqslant k$.

The proof of the following lemma parallels the proof of Lemma 29; we present it for completeness.

Lemma 31. *Let $\mathcal{G}, \mathcal{H}$ be graph classes, such that $\mathcal{G}$ is k-cluster rainbow and $\mathcal{H}$ is ℓ-cluster rainbow. Then $\mathcal{G} \wedge \mathcal{H}$ is $(k+\ell)$-cluster rainbow.*

Proof. Fix c, and let $G \in \mathcal{G}$ and $H \in \mathcal{H}$ be such that every colouring of G with clustering at most c contains a rainbow clique of size k, and every colouring of H with clustering at most c contains a rainbow clique of size ℓ. Let J be obtained from G by taking a join of G with H, $(c-1)k+1$ times along every clique S of G. Then $J \in \mathcal{G} \wedge \mathcal{H}$ by definition. It remains to show that every colouring $\phi \colon V(J) \to C$ of J for some set of colours C with clustering at most c contains a rainbow clique of size $k+\ell$. By the choice of J there exists a clique S in G of size k, which is rainbow in ϕ. Let $H_1, H_2, \dots, H_r$ be copies of H glued along S to G. By the choice of H, for every i there exists a clique S_i of size ℓ in H_i that is rainbow in ϕ. Suppose for a contradiction that $S \cup S_i$ is not rainbow for any i. Then there exists $s \in S$ with a neighbour of the same colour in S_i for at least c choices of i. Thus s belongs to a monochromatic component of size at least $c+1$ in ϕ, a contradiction. ∎

Note that an analogue of Lemma 31 also holds for defective colourings. The proof is identical.

Let $\mathcal{G}$ be a graph class obtained by taking a wedge-product of v copies of $\mathcal{I}$ and p copies of $\mathcal{P}$ in some order such that $v+2p=k+1$. Then we say that $\mathcal{G}$ is *k-cluster critical*. By Lemma 31 the clustered chromatic number of a k-cluster critical class is at least $k+1$. (In fact, it is not difficult to see that equality holds.) For example, the class $\mathcal{I}\wedge\mathcal{P}$ of minors of fans, the class $\mathcal{I}\wedge\mathcal{I}\wedge\mathcal{I}$ of minors of fat stars, and the class $\mathcal{P}\wedge\mathcal{I}$ of minors of fat paths are all possible 2-cluster critical classes. Thus, Theorem 14 is equivalent to the statement that $\chi_\star(\mathcal{G})\leqslant 2$ if and only if $\mathcal{G}$ contains no 2-cluster critical class.

The discussion above implies that for all k and c every graph in $\mathcal{X}_{k,c}$ is a member of some k-cluster critical class. Conversely, for all n, k there exists c such that for every graph $G\in\mathcal{X}_{k,c}$ there exists a k-cluster critical class $\mathcal{G}$ such that $\mathcal{X}_{k,c}$ contains as minors all graphs in $\mathcal{G}$ on at most n vertices. Thus Conjecture 30 can be reformulated as follows.

Conjecture 32. *Let $\mathcal{M}$ be a minor-closed class of graphs and $k\geqslant 0$ an integer. Then $\chi_\star(\mathcal{G})\geqslant k+1$ if and only if $\mathcal{G}\nsubseteq\mathcal{M}$ for some k-cluster critical class $\mathcal{G}$.*

Similarly, note that the k-term $\wedge$-product $\wedge^k\mathcal{I}=\mathcal{I}\wedge\mathcal{I}\wedge\ldots\wedge\mathcal{I}$ is the class of minors of connected graphs of tree-depth k and therefore the following conjecture is equivalent to Conjecture 2.

Conjecture 33. *Let $\mathcal{M}$ be a minor-closed class of graphs and $k\geqslant 0$ an integer. Then $\chi_\Delta(\mathcal{G})\geqslant k+1$ if and only if $\wedge^{k+1}\mathcal{I}\nsubseteq\mathcal{M}$.*

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