European Journal of Combinatorics 33 (2012) 488–490

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Nordhaus–Gaddum for treewidth

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ARTICLE INFO

Article history:
Received 13 September 2011
Received in revised form
11 October 2011
Accepted 11 October 2011

ABSTRACT

We prove that, for every n-vertex graph G, the treewidth of G plus the treewidth of the complement of G is at least $n-2$. This bound is tight.

Nordhaus–Gaddum-type theorems establish bounds on $f(G)+f(\overline{G})$ for some graph parameter f, where $\overline{G}$ is the complement of a graph G. The literature has numerous examples; see [3,8,4,6,13,14,11] for a few. Our main result is the following Nordhaus–Gaddum-type theorem for treewidth,[1] which is a graph parameter of particular importance in structural and algorithmic graph theory. Let $\mathrm{tw}(G)$ denote the treewidth of a graph G.

Theorem 1. *For every graph G with n vertices,*

$$\mathrm{tw}(G)+\mathrm{tw}(\overline{G}) \geq n-2.$$

The following lemma is the key to the proof of Theorem 1.

Lemma 2. *For every n-vertex graph G with no induced 4-cycle and no k-clique,*

$$\mathrm{tw}(\overline{G}) \geq n-k.$$

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[1] While treewidth is normally defined in terms of tree decompositions (see [2]), it can also be defined as follows. A graph G is a *k-tree* if $G \cong K_{k+1}$ or $G-v$ is a k-tree for some vertex v whose neighbours induce a k-clique. Then the *treewidth* of a graph G is the minimum integer k such that G is a spanning subgraph of a k-tree. See [1,10] for surveys on treewidth.

Let G be a graph. Two subsets of vertices A and B in G*touch* if $A \cap B \neq \emptyset$, or some edge of G has one endpoint in A and the other endpoint in B. A *bramble* in G is a set of subsets of $V(G)$ that induce connected subgraphs and pairwise touch. A set S of vertices in G is a *hitting set* of a bramble $\mathcal{B}$ if S intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set. Seymour and Thomas [12] proved the Treewidth Duality Theorem, which says that a graph G has treewidth at least k if and only if G contains a bramble of order at least $k+1$.

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doi:10.1016/j.ejc.2011.10.005

Proof. Let $\mathcal{B} := \{\{v, w\} : vw \in E(\overline{G})\}$. If $\{v, w\}$ and $\{x, y\}$ do not touch for some $vw, xy \in E(\overline{G})$, then the four endpoints are distinct and (v, x, w, y) is an induced 4-cycle in G, which is a contradiction. Thus $\mathcal{B}$ is a bramble in $\overline{G}$. Let S be a hitting set for $\mathcal{B}$. Thus no edge in $\overline{G}$ has both endpoints in $V(\overline{G}) \setminus S$. Hence $V(G) \setminus S$ is a clique in G. Therefore $n - |S| \leq k - 1$ and $|S| \geq n - k + 1$. That is, the order of $\mathcal{B}$ is at least $n - k + 1$. By the Treewidth Duality Theorem, $\text{tw}(\overline{G}) \geq n - k$, as desired. □

Proof of Theorem 1. Let $k := \text{tw}(G)$. Let H be a k-tree that contains G as a spanning subgraph. Thus H has no induced 4-cycle (it is chordal) and has no $(k + 2)$-clique. By Lemma 2, and since $\overline{G} \supseteq \overline{H}$, we have $\text{tw}(\overline{G}) \geq \text{tw}(\overline{H}) \geq n - k - 2$. Therefore $\text{tw}(G) + \text{tw}(\overline{G}) \geq n - 2$. □

Lemma 2 immediately implies the following result of independent interest.

Theorem 3. *For every n-vertex graph G with girth at least 5,*

$$\text{tw}(\overline{G}) \geq n - 3.$$

We now show that Theorem 1 is tight.

Lemma 4. *Let G be a graph with treewidth k that contains a $(k + 1)$-clique C such that each vertex in C has a neighbour outside of C. Then*

$$\text{tw}(G) + \text{tw}(\overline{G}) = n - 2.$$

Proof. We describe an $(n-k-2)$-tree H that contains $\overline{G}$. Let $A := V(G) \setminus C$ be the starting $(n-k-1)$-clique of H. For each vertex $x \in C$, add x to H adjacent to $A \setminus \{y\}$, where y is a neighbour of x outside of C. Observe that H is an $(n - k - 2)$-tree and $\overline{G}$ is a spanning subgraph of H. Thus $\text{tw}(\overline{G}) \leq n - k - 2$ and $\text{tw}(G) + \text{tw}(\overline{G}) \leq n - 2$, with equality by Theorem 1. □

For k-trees, we have the following precise result. Let Q_n^k be the k-tree consisting of a k-clique C with $n - k$ vertices adjacent only to C.

Theorem 5. *For every n-vertex k-tree G,*

$$\text{tw}(G) + \text{tw}(\overline{G}) = \begin{cases} n - 1 & \text{if } G \cong Q_n^k \\ n - 2 & \text{otherwise.} \end{cases}$$

Proof. First, suppose that $G \cong Q_n^k$. Then $\overline{G}$ consists of K_{n-k} and k isolated vertices. Thus $\text{tw}(\overline{G}) = n - k - 1$, and $\text{tw}(G) + \text{tw}(\overline{G}) = n - 1$. Now assume that $G \ncong Q_n^k$. By the definition of a k-tree, $V(G)$ can be labelled $v_1, \ldots, v_n$ such that $\{v_1, \ldots, v_{k+1}\}$ is a clique, and, for $j \in \{k + 2, \ldots, n\}$, the neighbourhood of v_j in $G[\{v_1, \ldots, v_{j-1}\}]$ is a k-clique C_j. If $C_{k+2}, \ldots, C_n$ are all equal, then $G \cong Q_n^k$. Thus $C_j \neq C_{k+2}$ for some minimum integer j. Observe that each vertex in C_j has a neighbour outside of C_j. The result follows from Lemma 4. □

In view of Theorem 1, it is natural to also consider how large $\text{tw}(G) + \text{tw}(\overline{G})$ can be. Every n-vertex graph G satisfies $\text{tw}(G) \leq n - 1$, implying that $\text{tw}(G) + \text{tw}(\overline{G}) \leq 2n - 2$. It turns out that this trivial upper bound is tight up to lower-order terms. Indeed, Perarnau and Serra [9] proved that, if $G \in \mathcal{G}(n, p)$ is a random n-vertex graph with edge probability $p = \omega(\frac{1}{n})$ in the sense of Erdős and Rényi, then asymptotically almost surely $\text{tw}(G) = n - o(n)$; see [5,7] for related results. Setting $p = \frac{1}{2}$, it follows that, asymptotically almost surely, $\text{tw}(G) = n - o(n)$ and $\text{tw}(\overline{G}) = n - o(n)$, and hence $\text{tw}(G) + \text{tw}(\overline{G}) = 2n - o(n)$. Theorems 1 and 5 can be reinterpreted as follows, where, for graphs G_1 and G_2, the union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ (where G_1 and G_2 may intersect).

Proposition 6. *For all graphs G_1 and G_2, the union $G_1 \cup G_2$ contains no clique on $\text{tw}(G_1) + \text{tw}(G_2) + 3$ vertices. This result is sharp, since there exist graphs G_1 and G_2 such that $G_1 \cup G_2$ contains a clique on $\text{tw}(G_1) + \text{tw}(G_2) + 2$ vertices.*

Proof. For the first claim, we may assume that $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. Let S be a clique in $G_1 \cup G_2$. Thus $\overline{G_1[S]} = G_2[S]$. By Theorem 1, $\text{tw}(G_1) + \text{tw}(G_2) \geq \text{tw}(G_1[S]) + \text{tw}(G_2[S]) \geq |S| - 2$. Thus $|S| \leq \text{tw}(G_1) + \text{tw}(G_2) + 2$ as desired. The sharpness example follows from Theorem 5. □

Proposition 6 suggests studying $G_1 \cup G_2$ further. For example, what is the maximum of $\chi(G_1 \cup G_2)$ taken over all graphs G_1 and G_2 with $\text{tw}(G_1) \leq k$ and $\text{tw}(G_2) \leq k$? By Proposition 6, the answer is at least $2k + 2$. A minimum-degree greedy algorithm shows that $\chi(G_1 \cup G_2) \leq 4k$. This question is somewhat similar to Ringel's earth–moon problem, which asks for the maximum chromatic number of the union of two planar graphs.

Acknowledgements

This work was supported in part by the Actions de Recherche Concertées (ARC) fund of the Communauté française de Belgique. Gwenaël Joret is a Postdoctoral Researcher of the Fonds National de la Recherche Scientifique (F.R.S.–FNRS), and is also supported by an Endeavour Fellowship from the Australian Government. David Wood is supported by a QEII Research Fellowship from the Australian Research Council (ARC).

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