Three dimensional graph drawing with fixed vertices and one bend per edge

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A *three-dimensional grid-drawing* of a graph represents the vertices by distinct points in $\mathbb{Z}^3$ (called *grid-points*), and represents each edge as a polyline between its endpoints with bends (if any) also at gridpoints, such that distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. This topic has been previously studied in [1–5]. We focus on the problem of producing such a drawing, where the vertices are fixed at given grid-points. This variant has been previously studied in [5, 6]. Meijer and Wismath [5] recently proved the following theorem:

Theorem 1. *For every graph G with n vertices, given fixed locations for the vertices of G in $\mathbb{Z}^3$, there is a three-dimensional grid-drawing of G with at most three bends per edge.*

We prove the same result with one bend per edge.

Theorem 2. *For every graph G with n vertices and m edges, given fixed locations for the vertices of G in $\mathbb{Z}^3$, there is a three-dimensional grid-drawing of G with one bend per edge.*

Proof. Consider each edge vw of G in turn. Say $v = (a, b, c)$ and $w = (p, q, r)$ in $\mathbb{Z}^3$. Choose $x \in \{a - 1, a + 1\} \setminus \{p\}$ and $y \in \{q - 1, q + 1\} \setminus \{b\}$. Let $L(v, w) := \{(x, y, z) : z \in \mathbb{Z}\}$. Observe that $L(v, w)$ is contained in a vertical line, and every point in $L(v, w)$ is visible from both v and w. That is, a segment from v or w to any point in $L(v, w)$ passes through no other point in $\mathbb{Z}^3$. Choose a point $(x, y, z) \in L(v, w)$ such that (1) no vertex of G is positioned at (x, y, z), (2) the segment between v and (x, y, z) does not intersect any already drawn edge segment, and (3) the segment between w and (x, y, z) does not intersect any already drawn edge segment. Rule (1) forbids less than n points in $L(v, w)$. Note that no edge-segment is drawn as a vertical line by this algorithm. Thus each edge-segment that is already drawn intersects the vertical line containing $L(v, w)$ in at most one point. Hence rule (2) forbids at most one point in $L(v, w)$ for each edge-segment that is already drawn. In total, rule (2) forbids less than $2m$ points in $L(v, w)$. Similarly, rule (3) forbids less than $2m$ points in $L(v, w)$. Since $L(v, w)$ has infinitely many points, there is a point $(x, y, z) \in L(v, w)$ satisfying (1), (2) and (3). Draw vw with one bend at (x, y, z). Then vw passes through no vertex and intersects no other edge (except of course at v or w). □

The *volume* of a three-dimensional grid-drawing is the number of grid points in a minimum axis-aligned box that contains the drawing. Meijer and Wismath [5] considered the volume of the drawing

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produced by Theorem 1 to be "unconstrained", although they did provide volume bounds for a different result with the vertices in the plane. Meijer and Wismath [5] state that "the general 3D point-set embeddability problem in which the specified point-set is not constrained to a plane remains as an interesting open problem if the volume must be constrained." We now show that the drawings produced by Theorem 2 have constrained volume. In fact, in a certain sense the volume is optimal.

Say the initial vertex set is contained in an $X \times Y \times Z$ bounding box, without loss of generality, $[1, X] \times [1, Y] \times [1, Z]$. Then for each edge, the algorithm may choose the bend point (x, y, z) with $x \in [0, X+1]$ and $y \in [0, Y+1]$ and $z \in [1, \max\{Z, n+4m\}]$. Thus the drawing is contained in an $(X+2) \times (Y+2) \times \max\{Z, n+4m\}$ bounding box.

We now show that in a special case, this volume bound is best possible. Say $G = K_n$ with the vertices at $(1, 0, 0), \ldots, (n, 0, 0)$. Using the above notation, $X = n$ and $Y = 1$ and $Z = 1$. The above volume upper bound is $(X+2)(Y+2)\max\{Z, n+4m\} \leqslant O(n^3)$. Morin and Wood [6] proved that every 1-bend drawing of an n-vertex graph G with vertices fixed on a line has volume at least $kn/2$ where k is the cutwidth of G. The cutwidth of K_n equals $\lfloor n^2/4 \rfloor$. Thus the volume of any 1-bend drawing of K_n, with these vertex locations, is at least $n^3/8$, which is within a constant factor of the above volume upper bound.

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