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The extremal function for Petersen minors

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ABSTRACT

We prove that every graph with n vertices and at least $5n-8$ edges contains the Petersen graph as a minor, and this bound is best possible. Moreover we characterise all Petersen-minor-free graphs with at least $5n-11$ edges. It follows that every graph containing no Petersen minor is 9-colourable and has vertex arboricity at most 5. These results are also best possible.

1. Introduction

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by the following operations: vertex deletion, edge deletion and edge contraction. The theory of graph minors, initiated in the seminal work of Robertson and Seymour, is at the forefront of research in graph theory. A fundamental question at the intersection of graph minor theory and extremal graph theory asks, for a given graph H, what is the maximum number $\mathrm{ex_m}(n,H)$ of edges in an n-vertex graph containing no H-minor? The function $\mathrm{ex_m}(n,H)$ is called the *extremal function* for H-minors.

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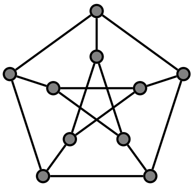

Fig. 1. The Petersen graph.

The extremal function is known for several graphs, including the complete graphs K_4 and K_5 [49,10], K_6 and K_7 [30], K_8 [19] and K_9 [44], the bipartite graphs $K_{3,3}$ [14] and $K_{2,t}$ [6], and the octahedron $K_{2,2,2}$ [8], and the complete graph on eight vertices minus an edge K_8^- [43]. Tight bounds on the extremal function are known for general complete graphs K_t [12,23,24,45,46], unbalanced complete bipartite graphs $K_{s,t}$ [25–28], disjoint unions of complete graphs [47], disjoint unions of cycles [15,7], general dense graphs [32] and general sparse graphs [4,16].

1.1. Petersen minors

We study the extremal function when the excluded minor is the Petersen graph (see Fig. 1), denoted by $\mathcal{P}$. Our primary result is the following:

Theorem 1. $\mathrm{ex_m}(n, \mathcal{P}) \leqslant 5n - 9$, *with equality if and only if* $n \equiv 2 \pmod 7$.

For $n \equiv 2 \pmod 7$, we in fact completely characterise the extremal graphs (see Theorem 2 below).

The class of $\mathcal{P}$-minor-free graphs is interesting for several reasons. As an extension of the 4-colour theorem, Tutte [48] conjectured that every bridgeless graph with no $\mathcal{P}$-minor has a nowhere zero 4-flow. Edwards, Robertson, Sanders, Seymour and Thomas [35,37, 36,40,11] have announced a proof that every bridgeless cubic $\mathcal{P}$-minor-free graph is edge 3-colourable, which is equivalent to Tutte's conjecture in the cubic case. Alspach, Goddyn and Zhang [3] showed that a graph has the circuit cover property if and only if it has no $\mathcal{P}$-minor. It is recognised that determining the structure of $\mathcal{P}$-minor-free graphs is a key open problem in graph minor theory (see [9,31] for example). Theorem 1 is a step in this direction.

1.2. Extremal graphs

We now present the lower bound in Theorem 1, and describe the class of extremal graphs. For a graph H and non-negative integer t, an (H, t)-*cockade* is defined as follows: H itself is an (H, t)-cockade, and any other graph G is an (H, t)-cockade if there are (H, t)-cockades G_1 and G_2 distinct from G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 \cong K_t$. It is well known that for every $(t + 1)$-connected graph H and every non-negative integer

$s < |V(H)|$, every (K_s, t)-cockade is H-minor-free (see Appendix A for a proof). Since $\mathcal{P}$ is 3-connected and $|V(\mathcal{P})| = 10$, every $(K_9, 2)$-cockade is $\mathcal{P}$-minor-free. Every n-vertex $(K_9, 2)$-cockade has $5n - 9$ edges. For $n \equiv 2 \pmod 7$ there is at least one n-vertex $(K_9, 2)$-cockade, hence $\mathrm{ex_m}(n, \mathcal{P}) \geqslant 5n - 9$ for $n \equiv 2 \pmod 7$.

Theorem 1 is implied by the following stronger result, which also shows that $(K_9, 2)$-cockades are the unique extremal examples of $\mathcal{P}$-minor-free graphs. Indeed, this theorem characterises $\mathcal{P}$-minor-free graphs that are within two edges of extremal.

Theorem 2. *Every graph with $n \geqslant 10$ vertices and $m \geqslant 5n - 11$ edges contains a Petersen minor or is a $(K_9, 2)$-cockade minus at most two edges.*

Since $(K_9, 2)$-cockades have connectivity 2, it is interesting to ask for the maximum number of edges in more highly connected $\mathcal{P}$-minor-free graphs. First note that Theorem 2 implies that 3-connected $\mathcal{P}$-minor-free graphs, with the exception of K_9, have at most $5n - 12$ edges. To see that this is tight, consider the class $\mathcal{C}$ of all graphs G such that there is some subset S of the vertices of G such that $|S| \leqslant 3$ and each component of $G - S$ contains at most five vertices. Then $\mathcal{C}$ is minor-closed, and it is quick to check that $\mathcal{P}$ is not in $\mathcal{C}$. If $G \in \mathcal{C}$ is such that $|S| = 3$, every vertex in S is dominant, and every component of $G - S$ is a copy of K_5, then G has $5n - 12$ edges and is 3-connected, and is $\mathcal{P}$-minor-free.

We now show that there are 5-connected $\mathcal{P}$-minor-free graphs with almost as many edges as $(K_9, 2)$-cockades. Consider the class $\mathcal{C}'$ of all graphs G with a vertex cover of size at most 5. C' is minor-closed, and $\mathcal{P}$ is not in $\mathcal{C}'$. Let $G := K_5 + \overline{K_{n-5}}$ for $n \geqslant 6$. Then G is 5-connected with $|E(G)| = 5n - 15$, and G is in $\mathcal{C}'$ and thus is $\mathcal{P}$-minor-free.

Now consider 6-connected $\mathcal{P}$-minor-free graphs. A graph G is *apex* if $G - v$ is *planar* for some vertex v. Since $K_{3,3}$ is a minor of $\mathcal{P} - v$ for each vertex v, the Petersen graph is not apex and every apex graph is $\mathcal{P}$-minor-free. A graph G obtained from a 5-connected planar triangulation by adding one dominant vertex is 6-connected, $\mathcal{P}$-minor-free, and has $4n - 10$ edges. We know of no infinite families of 6-connected $\mathcal{P}$-minor-free graphs with more edges. We also know of no infinite families of 7-connected $\mathcal{P}$-minor-free graphs. Indeed, it is possible that every sufficiently large 7-connected graph contains a $\mathcal{P}$-minor. The following conjecture is even possible.

Conjecture 3. *Every sufficiently large 6-connected $\mathcal{P}$-minor-free graph is apex.*

This is reminiscent of Jørgensen's conjecture [19], which asserts that every 6-connected K_6-minor-free graph is apex. Jørgensen's conjecture has recently been proved for sufficiently large graphs [21,22]. In this respect, K_6 and $\mathcal{P}$ possibly behave similarly. Indeed, they are both members of the so-called Petersen family [39,34,29]. Note however, that the extremal functions of K_6 and $\mathcal{P}$ are different, since $\mathrm{ex_m}(n, K_6) = 4n - 10$ [30].

1.3. Graph colouring

Graph colouring provides further motivation for studying extremal functions for graph minors. A graph is *k-colourable* if each vertex can be assigned one of k colours such that adjacent vertices get distinct colours. The *chromatic number* of a graph G is the minimum integer k such that G is k-colourable. In 1943, Hadwiger [13] conjectured that every K_t-minor-free graph is $(t-1)$-colourable. This is widely regarded as one of the most significant open problems in graph theory; see [41] for a recent survey, and see [38,2] for recent results. Extremal functions provide a natural approach for colouring graphs excluding a given minor, as summarised in the following folklore result (see Appendix A for a proof).

Lemma 4. *Let H be a graph such that $\mathrm{ex_m}(n,H) < cn$ for some positive integer c. Then every H-minor-free graph is $2c$-colourable, and if $|V(H)| \leqslant 2c$ then every H-minor-free graph is $(2c-1)$-colourable.*

Theorem 1 and Lemma 4 with $c = 5$ imply the following Hadwiger-type theorem for $\mathcal{P}$-minors, which is best possible for $\mathcal{P}$-minor-free graphs with K_9 subgraphs, for example $(K_9, 2)$-cockades.

Theorem 5. *Every $\mathcal{P}$-minor-free graph is 9-colourable.*

For a given graph G, a graph colouring can be thought of as a partition of $V(G)$ such that each part induces an edgeless subgraph, equivalently a subgraph with no K_2-minor. One way of generalising this is to instead ask for a partition of $V(G)$ such that each part induces a K_t-minor-free subgraph for some larger value of t. The minimum integer k such that there exists a partition of $V(G)$ into k sets such that each set induces a K_3-minor-free subgraph (equivalently a forest), is called the *vertex arboricity* of G. A graph is *d-degenerate* if every subgraph has minimum degree at most d. Chartrand and Kronk [5] proved that every d-degenerate graph has vertex arboricity at most $\lceil \frac{d+1}{2} \rceil$. By Theorem 1 every $\mathcal{P}$-minor-free graph is 9-degenerate. Hence, we have the following result, which again is best possible for $\mathcal{P}$-minor-free graphs with K_9 subgraphs.

Theorem 6. *Every $\mathcal{P}$-minor-free graph has vertex arboricity at most 5.*

Other classes of graphs for which the maximum vertex arboricity is known include planar graphs [5], locally planar graphs [42], triangle-free locally planar graphs [42], for each $k \in \{3, 4, 5, 6, 7\}$ the class of planar graphs with no k-cycles [33,17], planar graphs of diameter 2 [1], K_5-minor-free graphs of diameter 2 [18], and $K_{4,4}$-minor-free graphs [20].

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1.4. Notation

The following notation will be used throughout the paper. Let G be a graph, and let vw be an edge of G. The graph G/vw is the graph obtained from $G - \{v, w\}$ by adding a new vertex adjacent to all the neighbours of v except w and all the neighbours of w except v. The operation which takes G to G/e is a *contraction*. If a graph isomorphic to H can be obtained from G by performing edge deletions, vertex deletions and contractions, then H is a *minor* of G. A graph G is H*-minor-free* if H is not a minor of G.

The *components* of G are the maximal connected subgraphs of G. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. If $G[S]$ is a complete graph, S is a *clique*. We denote by $G - S$ the graph $G[V(G) \setminus S]$. Similarly, if $S \subseteq E(G)$, let $G - S$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus S$. For simplicity, we write $G - x$ for $G - \{x\}$. For any subgraph H of G, we write $G - H$ for $G - V(H)$.

For each vertex v in G, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ and $N_G[v] := \{v\} \cup N_G(v)$. Similarly, for each subgraph C of G, let $N_G(C)$ be the set of vertices in $G - C$ that are adjacent in G to some vertex of C, and let $N_G[C] := V(C) \cup N_G(C)$. When there is no ambiguity, we write $N(v)$, $N[v]$, $N(C)$ and $N[C]$ respectively for $N_G(v)$, $N_G[v]$, $N_G(C)$ and $N_G[C]$. A vertex v is *dominant* in G if $N_G[v] = V(G)$, and *isolated* if $N_G(v) = \emptyset$.

We denote by $\delta(G)$ the minimum degree of G and by $\Delta(G)$ the maximum degree of G. For $i \in \mathbb{N}$, we denote by $V_i(G)$ the set of vertices in G with degree i, and by $V_{\geqslant i}(G)$ the set of vertices of G of degree at least i.

For a tree T and $v, w \in V(T)$, let vTw be the path in T from v to w. A vertex of T is *high degree* if it is in $V_{\geqslant 3}(T)$. For a path P with endpoints x and y, $\operatorname{int}(P) := xy$ if $E(P) = \{xy\}$ and $\operatorname{int}(P) := V(P) \setminus \{x, y\}$ otherwise.

We denote by $G \dot\cup H$ the disjoint union of two graphs G and H. A subset S of $V(G)$ is a *fragment* if $G[S]$ is connected. Distinct fragments X and Y are *adjacent* if some vertex in X is adjacent to some vertex in Y.

2. Outline of proof

We now sketch the proof of Theorem 2. Assume to the contrary that there is some counterexample to Theorem 2, and select a minor-minimal counterexample G. Define $\mathcal{L}$ to be the set of vertices v of G such that $\deg(v) \leqslant 9$ and there is no vertex u with $N[u] \subsetneq N[v]$. For a vertex $v \in V(G)$, a subgraph $H \subseteq G$ is v*-suitable* if it is a component of $G - N[v]$ that contains some vertex of $\mathcal{L}$.

Section 3 shows some elementary results that are used throughout the other sections. In particular, it shows that $\delta(G) \in \{6, 7, 8, 9\}$, and hence that $\mathcal{L} \neq \emptyset$. Sections 4 and 5 respectively show that no vertex of G has degree 7 and that no vertex of G has degree 8. Sections 6 and 7 show that for every $v \in \mathcal{L}$ with degree 6 or 9 respectively there is some v-suitable subgraph, and that for each $v \in \mathcal{L}$ with degree 6 or 9 and every v-suitable subgraph C of G there is some v-suitable subgraph C' of G such that $N(C') \setminus N(C) \neq \emptyset$.

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Pick $u \in \mathcal{L}$ and a u-suitable subgraph H of G such that $|V(H)|$ is minimised. By the definition of u-suitable, there is some $v \in \mathcal{L} \cap V(H)$. Let C be a v-suitable subgraph of G containing u, and let C' be a v-suitable subgraph of G such that $N(C') \setminus N(C) \neq \emptyset$. Section 8 shows that C' selected in this way is a proper subgraph of H, contradicting our choice of H.

The basic idea of our proof is similar to proofs used for example in [44] and [2], with the major points of difference conceptually being the use of skeletons, defined in Section 3, to rule out certain configurations, and the proof in Section 3 that the minimal counterexample is 4-connected.

3. Basic results

To prove Theorem 2, suppose for contradiction that G is a minor-minimal counterexample to Theorem 2. That is, G is a graph with the following properties:

(i) $|V(G)| \geqslant 3$,
(ii) $|E(G)| \geqslant 5|V(G)| - 11$,
(iii) G is not a spanning subgraph of a $(K_9, 2)$-cockade,
(iv) $\mathcal{P}$ is not a minor of G,
(v) Every proper minor H of G with at least three vertices satisfies $|E(H)| \leqslant 5|V(H)| - 12$ or is a spanning subgraph of a $(K_9, 2)$-cockade.

If H is a $(K_9, 2)$-cockade or K_2, then $|E(H)| = 5|V(H)| - 9$. Hence, (v) immediately implies:

(vi) Every proper minor H of G with at least two vertices satisfies $|E(H)| \leqslant 5|V(H)| - 9$.

Lemma 7. *G has at least 10 vertices.*

Proof. Since $5n - 11 > \binom{n}{2}$ for $n \in \{2, 3, \ldots, 8\}$, every graph satisfying (i) and (ii) has at least 9 vertices. Every 9-vertex graph is a spanning subgraph of a $(K_9, 2)$-cockade. □

A *separation* of a graph H is a pair (A, B) of subsets of $V(H)$ such that both $A \setminus B$ and $B \setminus A$ are non-empty and $H = H[A] \cup H[B]$. The *order* of a separation (A, B) is $|A \cap B|$. A *k-separation* is a separation of order k. A *$(\leqslant k)$-separation* is a separation of order at most k. A graph is *k-connected* if it has at least $k + 1$ vertices and no separation of order less than k.

Let x, y and z be distinct vertices of a graph H. A K_3-minor *rooted at* $\{x, y, z\}$ is a set of three pairwise-disjoint, pairwise-adjacent fragments $\{X, Y, Z\}$ of H such that $x \in X$, $y \in Y$, $z \in Z$. The following lemma is well known and has been proved, for example, by Wood and Linusson [50].

Lemma 8. *Let x, y and z be distinct vertices of a graph H. There is a K_3-minor of H rooted at $\{x, y, z\}$ if and only if there is no vertex $v \in V(H)$ for which the vertices in $\{x, y, z\} \setminus \{v\}$ are in distinct components of $H - v$.*

Lemma 9. *G is 4-connected.*

Proof. By Lemma 7, $|V(G)| \geqslant 10$. Suppose for contradiction that there is a $(\leqslant 3)$-separation (A, B) of G. Note that $A \setminus B$ and $B \setminus A$ are both non-empty by definition. We separate into cases based on $|A \cap B|$ and on whether $|A \setminus B|$ is a singleton. Note that while Case 1 is redundant, it is useful to know that Case 1 does not hold when proving that Cases 2 and 4 do not hold.

Case 1. *There is a $(\leqslant 3)$-separation (A, B) of G such that $|A \setminus B| = \{v\}$:*

By Lemma 7, $|B| \geqslant 9$. Now by (vi) we have

$$|E(G)| \leqslant |E(G[B])| + \deg(v) \leqslant 5(|V(G)| - 1) - 9 + 3 = 5|V(G)| - 11.$$

By (ii), equality holds throughout. In particular $\deg(v) = 3$ and $|E(G[B])| = 5|B| - 9$ so $G[B]$ is a $(K_9, 2)$-cockade by (v). For every edge e incident to v, we have $E(G/e) = E(G[B])$ by (vi). Hence, $|A \cap B|$ is a clique, and is therefore contained in a subgraph $H \cong K_9$ of $G[B]$. Then $\mathcal{P} \subseteq H \cup G[A] \subseteq G$ contradicting (iv).

Case 2. *There is a $(\leqslant 1)$-separation (A, B) of G:*

If either $|A \setminus B| = 1$ or $|B \setminus A| = 1$ then we are in Case 1. Otherwise, $|A| \geqslant 2$ and $|B| \geqslant 2$, so by (v) we have $|E(G[A])| \leqslant 5|A| - 9$, with equality if and only if $G[A] \cong K_2$ or $G[A]$ is a $(K_9, 2)$-cockade, and the same for B. Now

$$|E(G)| = |E(G[A])| + |E(G[B])| \leqslant 5(|V(G)| + 1) - 9 - 9 = 5|V(G)| - 13,$$

contradicting (ii).

Case 3. *There is a 2-separation (A, B) of G:*

If there is a component C of $G - (A \cap B)$ such that $N(C) \neq A \cap B$, then G has a $(\leqslant 1)$-separation, and we are in Case 2. Otherwise, let C_B be a component of $G - A$ and let G_A be the graph obtained from G by contracting $G[N[C_B]]$ down to a copy of K_2 rooted at $A \cap B$ and deleting all other vertices of B. Let G_B be defined analogously. If $|E(G_A)| \leqslant 5|A| - 12$, then

$$|E(G)| \leqslant |E(G_A)| + |E(G_B)| - 1 \leqslant 5(|V(G)| + 2) - 12 - 9 - 1 = 5|V(G)| - 12,$$

contradicting (ii). Hence, $|E(G_A)| \geqslant 5|A| - 11$, and by (v), G_A is a spanning subgraph of a $(K_9, 2)$-cockade H_A. By symmetry, G_B is a spanning subgraph of a $(K_9, 2)$-cockade H_B. Then G is a spanning subgraph of the $(K_9, 2)$-cockade formed by gluing H_A and H_B together on $A \cap B$, contradicting (iii).

Case 4. *There is a 3-separation (A, B) of G:*

First, suppose that $G[A]$ does not contain a K_3 minor rooted at $A \cap B$. Then there exists a vertex v such that the vertices in $A \cap B$ are in distinct components of $G[A] - v$ by Lemma 8. Recall that $|A \setminus B| > 1$, so there is a vertex $w \neq v$ in $A \setminus B$. Let C be the component of $G[A] - v$ containing w. Then there is a $(\leqslant 2)$-separation (A', B') of G where $A' \setminus B' = V(C) \setminus (A \cap B)$, so we are in either Case 2 or Case 3. Hence, there is a K_3 minor of $G[A]$ rooted at $A \cap B$, and by the same argument a K_3 minor of $G[B]$ rooted at $A \cap B$. Let G_A be obtained from G by contracting $G[B]$ down to a triangle on $A \cap B$, and let G_B be obtained from G by contracting $G[A]$ down to a triangle on $A \cap B$. Suppose $|E(G_A)| \geqslant 5|A| - 11$. Since G satisfies (v), we have that G_A is a spanning subgraph of a $(K_9, 2)$-cockade, and so G_A is a $(K_9, 2)$-cockade minus at most two edges. Since $A \cap B$ is a clique of G_A, there is some set S of nine vertices in A, containing $A \cap B$, such that $G_A[S]$ is K_9 minus at most two edges. Let C be a component of $G - A$, and note that $N(C) = A \cap B$, or else we are in Case 2 or Case 3. Now it is quick to check that the graph obtained from $G[S \cup V(C)]$ by contracting C to a single vertex contains $\mathcal{P}$ as a subgraph, contradicting (iv). Hence, $|E(G_A)| \leqslant 5|A| - 12$, and by symmetry $|E(G_B)| \leqslant 5|B| - 12$. Now

$$|E(G)| \leqslant |E(G_A)| + |E(G_B)| - 3 \leqslant 5(|V(G)| + 3) - 12 - 12 - 3 = 5|V(G)| - 12,$$

contradicting (ii). □

Lemma 10. $\delta(G) \in \{6, 7, 8, 9\}$ *and every edge is in at least five triangles.*

Proof. Suppose for contradiction that some edge vw is in t triangles with $t \leqslant 4$. Now

$$|E(G/vw)| \geqslant |E(G)| - t - 1 \geqslant 5|V(G)| - 12 - t \geqslant 5|V(G/e)| - 11.$$

Since G satisfies (v), G/vw is a spanning subgraph of some $(K_9, 2)$-cockade H. By Lemma 9, G is 4-connected, which implies G/vw is 3-connected, so G/vw is K_9 minus at most two edges. It follows from (ii) that G is a 10-vertex graph with at most six non-edges. It is possible at this point to manually prove that $\mathcal{P} \subseteq G$. Rather than detailing this argument, we instead report that a simple random searching algorithm verifies (in six minutes) that $\mathcal{P}$ is a subgraph of every 10-vertex graph with at most six non-edges. Hence, every edge of G is in at least five triangles. By Lemma 9, G has no isolated vertex, and $\delta(G) \geqslant 6$.

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Let e be an edge of G. By (vi), $|E(G-e)| \leqslant 5|V(G)|-9$, so $|E(G)| \leqslant 5|V(G)|-8$, and hence $\delta(G) \leqslant 9$. □

Recall that $\mathcal{L}$ is the set of vertices v of G such that $\deg(v) \leqslant 9$ and there is no vertex u with $N[u] \subsetneq N[v]$. By Lemma 10, every vertex of minimum degree is in $\mathcal{L}$, and $\mathcal{L} \neq \emptyset$.

The following result is the tool we use for finding v-suitable subgraphs.

Lemma 11. *If (A,B) is a separation of G of order $k \leqslant 6$ such that there is a vertex $v \in B \setminus A$ with $A \cap B \subseteq N(v)$, then there is some vertex $u \in (A \setminus B) \cap \mathcal{L}$.*

Proof. We may assume that every vertex in $A \cap B$ has a neighbour in $A \setminus B$.

Let u be a vertex in $A \setminus B$ with minimum degree in G. Suppose for a contradiction that $\deg_G(u) \geqslant 10$. It follows that every vertex in $A \setminus B$ has degree at least 10 in $G[A]$. Hence, $G[A]$ has at most six vertices of degree less than 10, so $G[A]$ is not a spanning subgraph of a $(K_9,2)$-cockade. Now $|A| \geqslant |N[u]| \geqslant 11$, so by (v),

$$\begin{aligned}\sum_{w\in A\cap B} \deg_{G[A]}(w) &= 2|E(G[A])| - \sum_{w\in A\setminus B} \deg_{G[A]}(w) \leqslant 2(5|A|-12) - 10|A\setminus B| \\ &= 10k-24. \qquad (1)\end{aligned}$$

Let X be the set of edges of G with one endpoint in $A \cap B$ and the other endpoint in $A \setminus B$. It follows from Lemma 9 that there are a pair of disjoint edges e_1 and e_2 in X, since deleting the endpoints of an edge $e_1 \in X$ from G does not leave a disconnected graph and $|A \setminus B| \geqslant |N[u]| - k \geqslant 5$. By Lemma 10, e_1 is in at least five triangles. Each of these triangles contains some edge in $X \setminus \{e_1, e_2\}$, so $|X| \geqslant 7$. By (1),

$$\begin{aligned}\delta(G[A\cap B]) &\leqslant \frac{1}{k}\sum_{w\in A\cap B} \deg_{G[A\cap B]}(w) = \frac{1}{k}\left(\left(\sum_{w\in A\cap B}\deg_{G[A]}(w)\right) - |X|\right) \\ &\leqslant \frac{1}{k}(10k-31).\end{aligned}$$

Since $k \leqslant 6$, some vertex $x \in A \cap B$ has degree at most 4 in $G[A \cap B]$. Let $G' := G[A \cup \{v\}]/vx$. Then $|E(G')| \geqslant |E(G[A])| + (k-5)$. Recall that every vertex in $A \setminus B$ has degree at least 10 in $G[A]$. Further, every vertex in $A \cap B$ is incident with some edge in X, and hence has at least six neighbours in A by Lemma 10. Hence $|E(G')| \geqslant \frac{1}{2}(10|A \setminus B| + 6k) + (k-5) \geqslant \frac{1}{2}(10|A| - 4k) + k - 5 \geqslant 5|A| - 11$. Then G' is a spanning subgraph of a $(K_9,2)$-cockade by (v), and so $G[A]$ is a spanning subgraph of a $(K_9,2)$-cockade, a contradiction.

Hence, $\deg_G(u) \leqslant 9$. Suppose for contradiction that $N[w] \subsetneq N[u]$ for some vertex w. Then $w \in N(u)$ and $\deg_G(w) < \deg_G(u)$, so $w \in A \cap B$. But $N[w] \subseteq N[u]$, so $w \notin N(v)$, which contradicts the assumption that $A \cap B \subseteq N(v)$. Therefore $u \in \mathcal{L}$, as required. □

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For an induced subgraph H of G, a subtree T of $G[N[H]]$ is a *skeleton* of H if $V_1(T) = N(H)$.

Lemma 12. *Let S be a fragment of G, let T be a skeleton of $G[S]$, and let v and w be distinct vertices of T. If $vw \notin E(T)$ and $T \neq vTw$, then there is a path P of $G[N[S]] - \{v, w\}$ from vTw to $T - vTw$ with no internal vertex in T.*

Proof. $G - \{v, w\}$ is connected by Lemma 9, so there is a path in $G - \{v, w\}$ from vTw to $T - vTw$. Let P be a vertex-minimal example of such a path with endpoints x in vTw and y in $T - vTw$.

Suppose to the contrary that there is some internal vertex z of P in T. Then either z is in vTw and the subpath of P from z to y contradicts the minimality of P, or z is in $T - vTw$ and the subpath of P from x to z contradicts the minimality of P.

Suppose to the contrary that there is some vertex z in $P - N[S]$. The subpath P' of P from x to z has one end in S and one end in $G - N[S]$, so there is some internal vertex z' of P' in $N(S)$. But $N(S) \subseteq V(T)$, so z' is an internal vertex of P in T, a contradiction. □

Lemma 13. *If (A, B) is a separation of G such that $N(A \setminus B) = A \cap B$, $|A \setminus B| \geqslant 2$ and $G[A \setminus B]$ is connected, then there is a skeleton of $G[A \setminus B]$ with at least two high degree vertices.*

Proof. There is at least one subtree of $G[A]$ in which every vertex of $A \cap B$ is a leaf, since we can obtain such a tree by taking a spanning subtree of $G[A \setminus B]$ and adding the vertices in $A \cap B$ and, for each vertex in $A \cap B$, exactly one edge $e \in E(G)$ between that vertex and some vertex of $A \setminus B$. We can therefore select T a subtree of $G[A]$ such that $A \cap B \subseteq V_1(T)$ and such that there is no proper subtree T' of T such that $A \cap B \subseteq V_1(T')$. There is no vertex v in $V_1(T) \setminus B$, since for any such vertex $T - v$ is a proper subtree of T and $A \cap B \subseteq V_1(T - v)$, a contradiction. Hence, $V_1(T) = A \cap B$. If $|V_{\geqslant 3}(T)| \geqslant 2$ then we are done, so we may assume there is a unique vertex w in $V_{\geqslant 3}(T)$.

Suppose that for some $x \in A \cap B$ there is some vertex in $\mathrm{int}(xTw)$. By Lemma 12, there is a path P of $G[A] - \{x, w\}$ from xTw to $T - xTw$ with no internal vertex in T. Let y be the endpoint of P in xTw and let z be the other endpoint. Then $T' := (T \cup P) - \mathrm{int}(zTw)$ is a skeleton of $G[A \setminus B]$ that has a vertex of degree exactly 3. Since $|V_1(T')| = |A \cap B| \geqslant 4$ by Lemma 9, T' has at least two high degree vertices (namely y and w).

Suppose instead that $V(T) = \{w\} \cup (A \cap B)$. By Lemma 9 G is 4-connected, so $(A \setminus B, B \cup \{w\})$ is not a separation of G, so there is some vertex y in $A \setminus (B \cup \{w\})$ adjacent to some vertex x in $A \cap B$. Let P_1 be a minimal length path from y to $A \cap B$ in $G - \{x, w\}$ (and hence in $G[A] - \{x, w\}$), and let z be the endpoint of P_1 in $A \cap B$. Let P_1' be the path formed by adding the vertex x and the edge xy to P_1. Since $G[A \setminus B]$ is connected, we can select a minimal length path P_2 of $G[A \setminus B]$ from P_1 to w. Then $(T \cup P_1' \cup P_2) - \{xw, zw\}$ is a skeleton of $G[A \setminus B]$ that has a degree 3 vertex, and therefore at least two high degree vertices (namely the endpoints of P_2). □

For any graph H a *table* of H is an ordered 6-tuple $\mathcal{X} := (X_1, \dots, X_6)$ of pairwise disjoint fragments of H such that X_5 is adjacent to X_1, X_2 and X_6, and X_6 adjacent to X_3 and X_4. For any subset S of $V(H)$, $\mathcal{X}$ is *rooted* at S if $|X_i \cap S| = 1$ for $i \in \{1, 2, 3, 4\}$ and $X_5 \cap S = X_6 \cap S = \emptyset$.

Lemma 14. *If (A, B) is a separation of G such that $N(A \setminus B) = A \cap B$, $|A \cap B| \geqslant 4$, $|A \setminus B| \geqslant 2$ and $G[A \setminus B]$ is connected, then there is a table of $G[A]$ rooted at $A \cap B$.*

Proof. By Lemma 13, there is some skeleton T of $G[A \setminus B]$ such that $|V_{\geqslant 3}(T)| \geqslant 2$. Let w and x be distinct vertices in $V_{\geqslant 3}(T)$. Let w_1, w_2 and w' be three neighbours of w in T, and let x', x_3 and x_4 be three neighbours of x in T, labelled so that w' and x' are both in $V(xTw)$. For $i \in \{1, 2\}$ let X_i be the vertex set of a path from w_i to a leaf of T in the component subtree of $T - w$ that contains w_i, and for $i \in \{3, 4\}$ let X_i be the vertex set of a path from x_i to a leaf of T in the component subtree of $T - x$ that contains x_i. Since $V_1(T) = A \cap B$, $|X_i \cap B| = 1$ for $i \in \{1, 2, 3, 4\}$. Let $X_5 := V(wTx')$ and let $X_6 := \{x\}$. Then $\mathcal{X} := (X_1, \dots, X_6)$ satisfies our claim. $\square$

4. Degree 7 vertices

In this section we show that $V_7(G) = \emptyset$.

Claim 15. *If $v \in V_7(G)$, then there is no isolated vertex in $G - N[v]$.*

Proof. Suppose for contradiction that there is some isolated vertex u in $G - N[v]$. By Lemma 10, $|N(u)| \geqslant 6$. By Lemma 7, there is some component C of $G - N[v]$ not containing u. Since $|N(C)| \geqslant 4$ by Lemma 9 and $|N(u) \cup N(C)| \leqslant |N(v)| = 7$, there is some vertex v_1 in $N(u) \cap N(C)$. Let v_1, v_2 and v_3 be distinct vertices in $N(C)$, and let v_4 and v_5 be distinct vertices in $N(u) \setminus \{v_1, v_2, v_3\}$. Let v_6 and v_7 be the remaining vertices of $N(v)$. By Lemma 10, for $i \in \{1, 2, \dots, 7\}$, $N(v_i) \cap N(v) \geqslant 5$. If some vertex in $\{v_2, v_3\}$, say v_2, is not adjacent to some vertex in $\{v_4, v_5\}$, say v_5, then v_2 and v_5 are both adjacent to every other vertex in $N(v)$, and in particular v_2v_4 and v_3v_5 are edges in G. Hence, there are two disjoint edges between $\{v_2, v_3\}$ and $\{v_4, v_5\}$. Without loss of generality, $\{v_2v_4, v_3v_5\} \subseteq E(G)$. We now consider two cases depending on whether $v_6v_7 \in E(G)$.

Case 1. $v_6v_7 \in E(G)$:

Since v_1 is adjacent to all but at most one of the other neighbours of v, either $v_1v_6 \in E(G)$ or $v_1v_7 \in E(G)$, so without loss of generality $v_1v_6 \in E(G)$. Since v_7 is adjacent to all but at most one of the other neighbours of v, either $\{v_7v_2, v_7v_5\} \subseteq E(G)$ or $\{v_7v_3, v_7v_4\} \subseteq E(G)$, so without loss of generality $\{v_7v_2, v_7v_5\} \subseteq E(G)$. Let G' be obtained from G by contracting C to a single vertex. Then $\mathcal{P} \subseteq G'$ (see Fig. 2a), contradicting (iv).

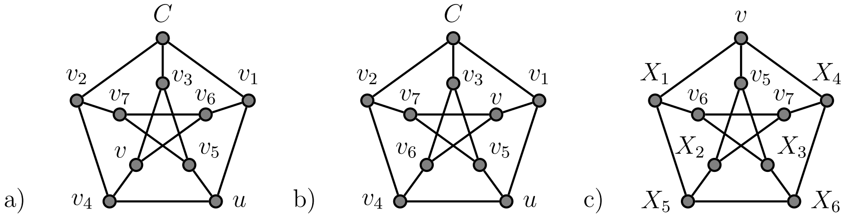


Fig. 2. Petersen subgraphs in Claim 15 and Lemma 16.

Case 2. $v_6v_7 \notin E(G)$:

Then v_6 and v_7 are both adjacent to every other neighbour of v. Let G' be obtained from G by contracting C to a single vertex. Then $\mathcal{P} \subseteq G'$ (see Fig. 2b), contradicting (iv). □

The following is the main result of this section.

Lemma 16. *$V_7(G) = \emptyset$.*

Proof. Suppose for contradiction that there is some vertex $v \in V_7(G)$. By Lemma 7, there is a non-empty component C of $G - N[v]$. By Lemma 9, $|N(C)| \geqslant 4$ and by Claim 15, $|V(C)| \geqslant 2$. Hence, by Lemma 14 with $A := N[C]$ and $B := V(G - C)$, there is a table $\mathcal{X} := (X_1, \dots, X_6)$ of $G[N[C]]$ rooted at $N(C)$.

Let $\{v_1, \dots, v_7\} := N(v)$, with $v_i \in X_i$ for $i \in \{1, 2, 3, 4\}$. By Lemma 10, $|N(v_i) \cap N(v)| \geqslant 5$ for $i \in \{1, 2, \dots, 7\}$. We consider two cases depending on whether $v_5v_6v_7$ is a triangle of G.

Case 1. $v_5v_6v_7$ *is a triangle of* G:

Let Q be the bipartite graph with bipartition $V := \{v_1, v_2, v_3, v_4\}$, $W := \{v_5, v_6, v_7\}$ and $E(Q) := \{xy : xy \notin E(G), x \in V, y \in W\}$. Then $\Delta(Q) \leqslant 1$, so without loss of generality $E(Q) \subseteq \{v_1v_5, v_2v_6, v_3v_7\}$. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1, 2, \dots, 6\}$. Then $\mathcal{P} \subseteq G'$ (see Fig. 2c), contradicting (iv).

Case 2. $v_5v_6v_7$ *is not a triangle of* G:

We may assume without loss of generality that $v_5v_6 \notin E(G)$. Then v_5 and v_6 are both adjacent to every other neighbour of v. At most one neighbour of v is not adjacent to v_7, so v_7 has some neighbour in $\{v_1, v_2\}$, say v_2, and some neighbour in $\{v_3, v_4\}$, say v_4. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1, 2, \dots, 7\}$. Then $\mathcal{P} \subseteq G'$ (see Fig. 2c), contradicting (iv). □

5. Degree 8 vertices

We now prove that $V_8(G) = \emptyset$. Note that the following lemma applies to any graph, not just G. This means we can apply it to minors of G, which we do in Claims 26 and 28.

Claim 17. *If H is a graph that contains a vertex v such that* $\deg(v) = 8$, $|N(v') \cap N(v)| \geqslant 5$ *for all $v' \in N(v)$, and C is a component of $H \setminus N[v]$ with $|N_H(C)| \geqslant 3$, then $\mathcal{P}$ is a minor of H unless all of the following conditions hold:*

1. $\overline{K_3}$ *is an induced subgraph of* $H[N(v) \setminus N(C)]$,
2. $\overline{C_4}$ *is an induced subgraph of* $H[N(v)]$,
3. $H[N(C)] \cong K_3$.

Proof. By assumption, $\delta(H[N(v)]) \geqslant 5$. Let H' be an edge-minimal spanning subgraph of $H[N(v)]$ such that $\delta(H') \geqslant 5$. Every edge e in H' is incident to some vertex of degree 5, since otherwise $\delta(H' - e) \geqslant 5$, contradicting the minimality of H'. Hence, the vertices of degree at most 1 in $\overline{H'}$ form a clique in $\overline{H'}$. Now $\Delta(\overline{H}) \leqslant 2$, since $|V(H')| = \deg(v) = 8$ and $\delta(H') \geqslant 5$. It follows that $\overline{H'}$ is the disjoint union of some number of cycles, all on at least three vertices, and a complete graph on at most two vertices. Let x, y and z be three vertices in $N(C)$, and let $1, 2, \ldots, 5$ be the remaining vertices of $N(v)$. Colour x, y, and z white and colour $1, 2, \ldots, 5$ black. In Table 1 we examine every possible graph $\overline{H'}$, up to colour preserving isomorphism. We use cycle notation to label the graphs, with an ordered pair representing an edge and a singleton representing an isolated vertex. In each case we find $\mathcal{P}$ as a subgraph of the graph G' obtained from G by contracting C to a single vertex, except in the unique case where K_3 is an induced subgraph of $\overline{H'} - \{x, y, z\}$, C_4 is an induced subgraph of $\overline{H'}$ and $\{x, y, z\}$ is an independent set of vertices in $\overline{H'}$.

It follows that if $N(C) = \{x, y, z\}$, then the claim holds. Suppose to the contrary that $\mathcal{P}$ is not a minor of H and $|N(C)| \geqslant 4$. As Table 1 shows, $\overline{H'}$ contains both K_3 and C_4 as induced subgraphs. Since $\Delta(\overline{H'}) \leqslant 2$, no vertex of $\overline{H'}$ is in more than one cycle,

Table 1
Petersen subgraphs in Claim 17.

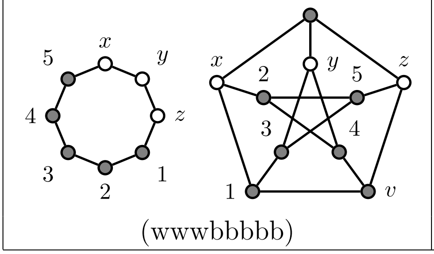
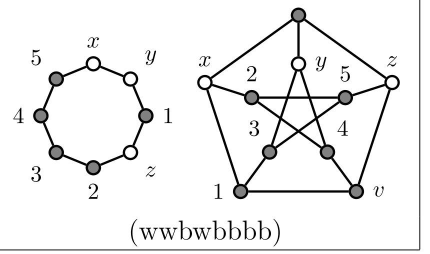
 (wwwbbbbb)	 (wwbwbbbb)

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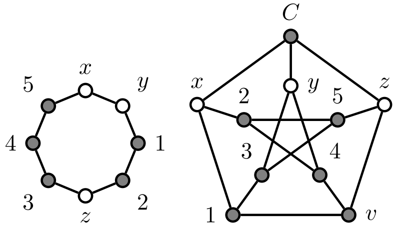
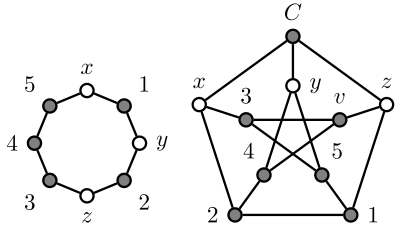
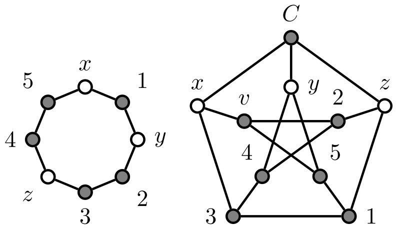
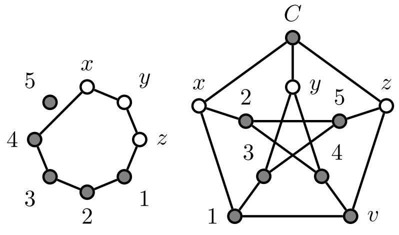
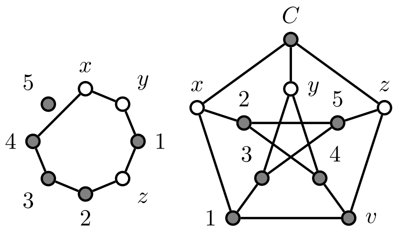
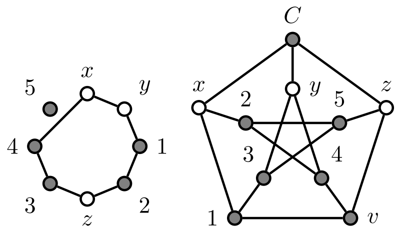
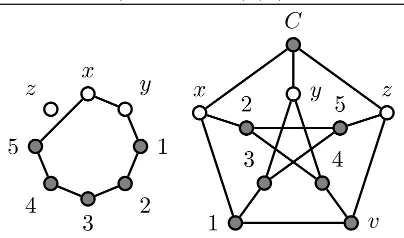
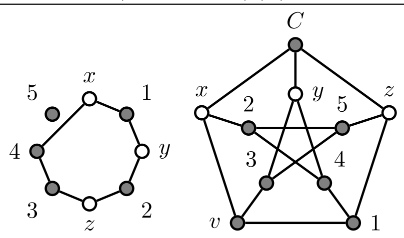
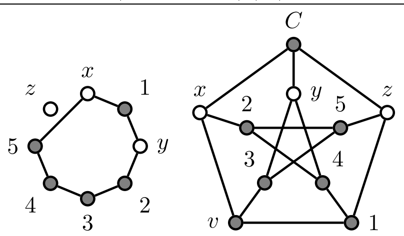
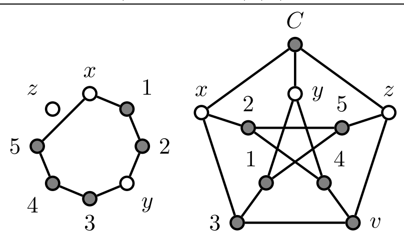
Table 1 (*continued*)

 (wwbbwbbb)	 (wbwbwbbb)
 (wbwbbwbb)	 (wwwbbbb)(b)
 (wwbwbbb)(b)	 (wwbbwbb)(b)
 (wwbbbbb)(w)	 (wbwbwbb)(b)
 (wbwbbbb)(w)	 (wbbwbbb)(w)

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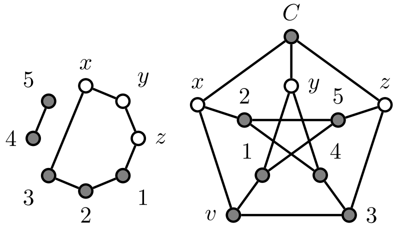
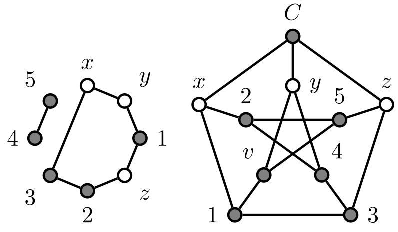
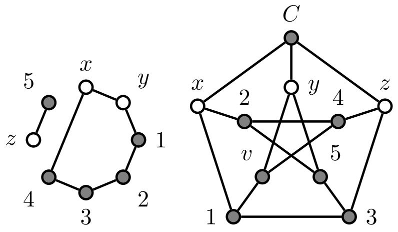
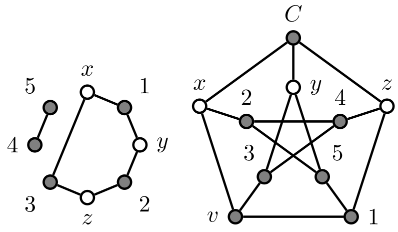
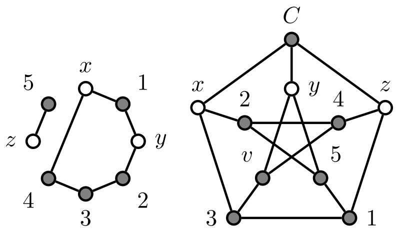
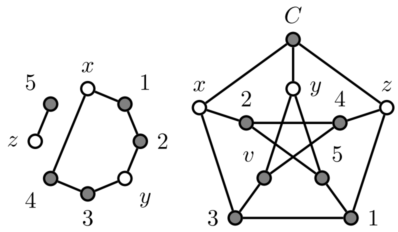
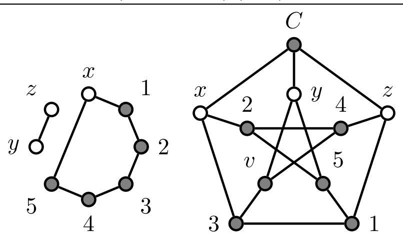
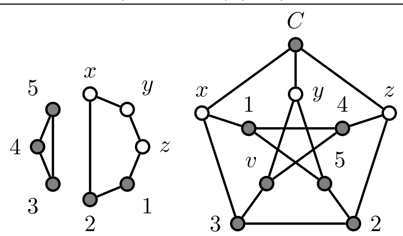
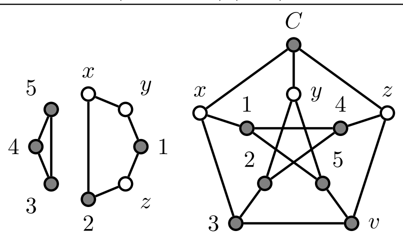
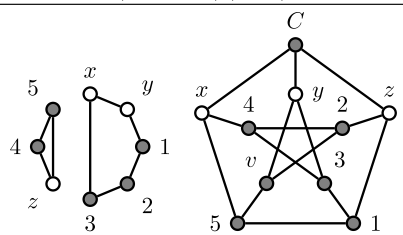
 (wwwbbb)(bb)	 (wwbwbb)(bb)
 (wwbbbb)(wb)	 (wbwbwb)(bb)
 (wbwbbb)(wb)	 (wbbwbb)(wb)
 (wbbbbb)(ww)	 (wwwbb)(bbb)
 (wwbwb)(bbb)	 (wwbbb)(wbb)

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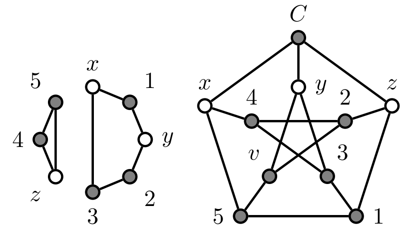
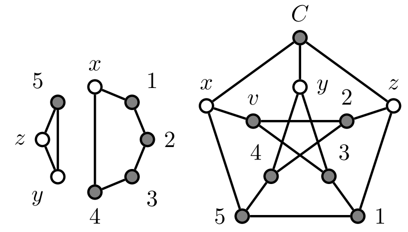
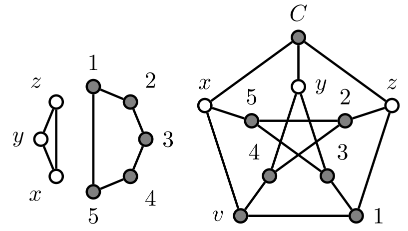
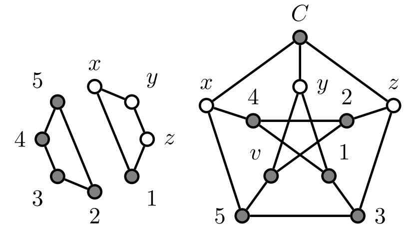
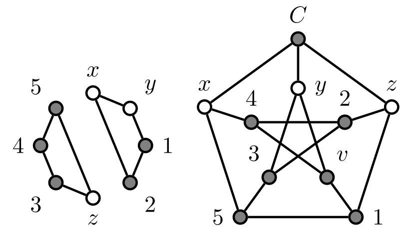
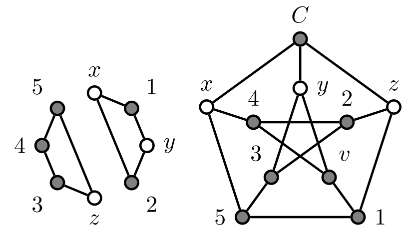
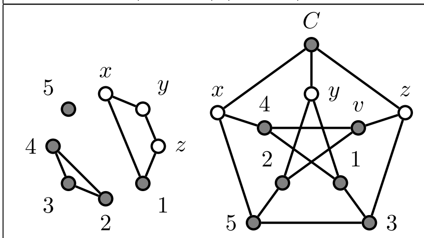
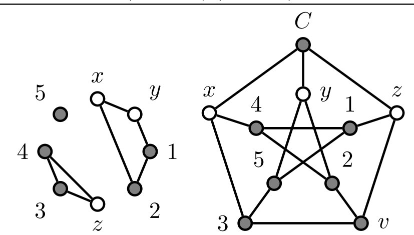
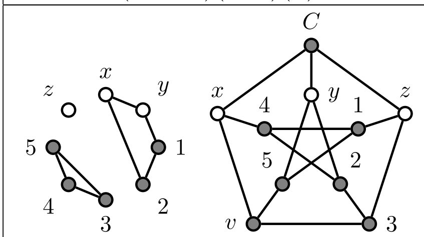
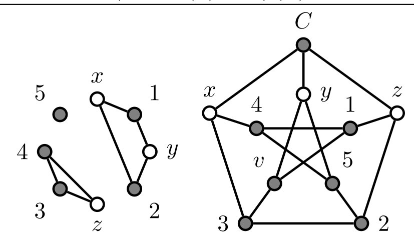
 (wbwbb)(wbb)	 (wbbbb)(wwb)
 (bbbbb)(www)	 (wwwb)(bbbb)
 (wwbb)(wbbb)	 (wbwb)(wbbb)
 (wwwb)(bbb)(b)	 (wwbb)(wbb)(b)
 (wwbb)(bbb)(w)	 (wbwb)(wbb)(b)

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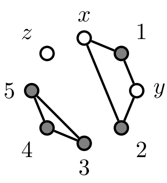
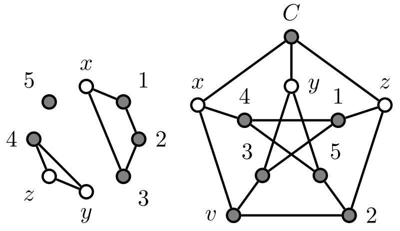
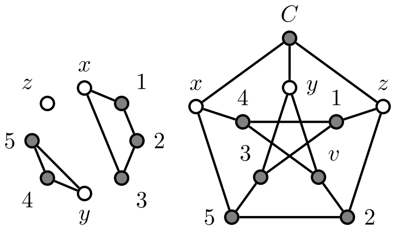
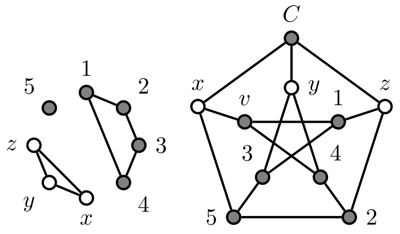
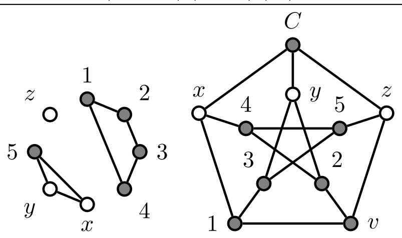
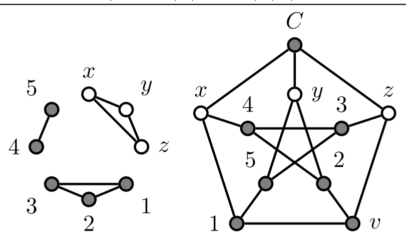
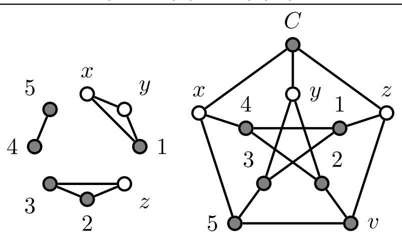
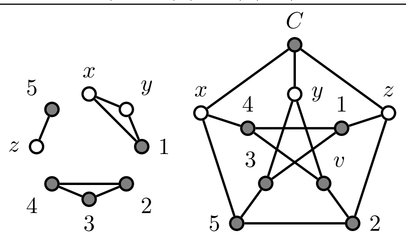
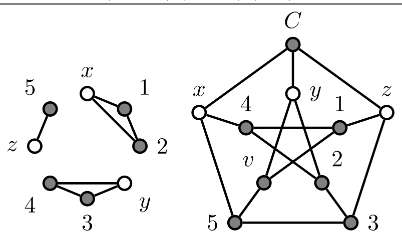
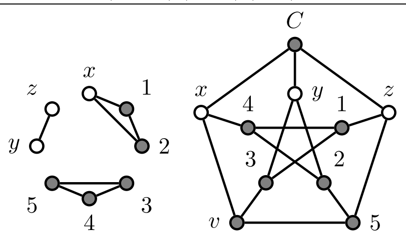
 1., 2. and 3. hold. (wbwb)(bbb)(w)	 (wbbb)(wwb)(b)
 (wbbb)(wbb)(w)	 (bbbb)(www)(b)
 (bbbb)(wwb)(w)	 (www)(bbb)(bb)
 (wwb)(wbb)(bb)	 (wwb)(bbb)(wb)
 (wbb)(wbb)(wb)	 (wbb)(bbb)(ww)

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so there is a unique triangle in $\overline{H'}$. For any subset $S \subseteq N(C)$ of size 3, S is a set of independent vertices in $\overline{H'}$, disjoint from the unique triangle of $\overline{H'}$ by the case analysis in Table 1. Hence, $N(C)$ is an independent set of at least four vertices in $\overline{H'}$, disjoint from the unique triangle of $\overline{H'}$. However, given the structure of H, there is no such set, a contradiction. □

The following is the main result of this section.

Lemma 18. *$V_8(G) = \emptyset$.*

Proof. Suppose to the contrary that $v \in V(G)$ has degree 8. By Lemma 10, $|N(v') \cap N(v)| \geqslant 5$ for all $v' \in N(v)$. By Lemma 7, $G - N[v]$ has some non-empty component C. By Lemma 9, $|N(C)| \geqslant 4$, so $G[N(C)] \ncong K_3$. Hence, by Claim 17, $\mathcal{P}$ is a minor of G, contradicting (iv). □

6. Degree 6 vertices

In this section we focus on vertices of degree 6 in G. Recall that for a given vertex v of our minimal counterexample G, a subgraph H of G is *v-suitable* if it is a component of $G - N[v]$ that contains some vertex of $\mathcal{L}$. The main result of this section is that if $v \in V_6(G)$, then for any v-suitable subgraph H there is a v-suitable subgraph H' such that $N(H') \setminus N(H) \neq \emptyset$ (see Lemma 23).

Claim 19. *If $v \in V_6(G)$, then $N[v]$ is a clique.*

Proof. By definition, v is dominant in $G[N[v]]$. Let w be a vertex in $N(v)$. Then w is adjacent to each of the five other vertices in $N(v)$, by Lemma 10 applied to the edge vw. □

This result is useful because it means that for an induced subgraph H of $\mathcal{P}$ on seven or fewer vertices, $H \subseteq G[N[v]]$. Throughout this section we show that certain statements about the structure of G imply $\mathcal{P}$ is a minor of G, and are therefore false. When illustrating this, the vertices of $N[v]$ will be coloured white, for ease of checking.

Claim 20. *If $v \in V_6(G)$ and C is a component of $G - N[v]$ with $|N(C)| \geqslant 5$, then $|V(C)| = 1$.*

Proof. Suppose for contradiction that $|V(C)| > 1$. By Lemma 13 with $A := N[C]$ and $B := V(G - C)$, there is a skeleton T of C with at least two high degree vertices. The handshaking lemma implies

$$\sum_{i=3}^{\infty}(i-2)\cdot|V_i(T)| = |V_1(T)| - 2. \tag{2}$$

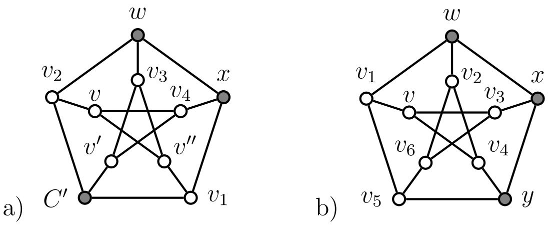

Fig. 3. Cases 1 and 2 in Claim 20.

Note that $|V_1(T)| = |N(C)|$ and $|N(C)| \in \{5, 6\}$, so $|V_1(T)| - 2 \in \{3, 4\}$. Hence either $|V_{\geqslant 3}(T)| \in \{3, 4\}$ (Case 2 below), $V_3(T) = \emptyset$ and $|V_4(T)| = 2$ (Cases 3 and 4 below), or $|V_3(T)| = 1$ and $|V_{\geqslant 4}(T)| = 1$ (Case 5).

Case 1. $|V(C)| = 2$:

Since C is connected, the two vertices w and x of C are adjacent. By Lemma 10 applied to wx, w and x have at least five common neighbours, $v_1, \dots, v_5$. By Lemma 7, $|V(G - N[v] - C)| \geqslant 1$, so there is some component $C' \neq C$ of $G - N[v]$. By Lemma 9, $|N(C')| \geqslant 4$. Both $N(C')$ and $\{v_1, \dots, v_5\}$ are subsets of $N(v)$ and $|N(v)| = 6$, so $|N(C') \cap \{v_1, \dots, v_5\}| \geqslant 3$. Assume without loss of generality that $N(C') \supseteq \{v_1, v_2, v'\}$, where v' is neither v_3 nor v_4. Let v'' be the unique vertex in $N(v) \setminus \{v_1, v_2, v_3, v_4, v'\}$. Let G' be obtained from G by contracting C' to a single vertex. Then $\mathcal{P} \subseteq G'$ by Claim 19 (see Fig. 3a), contradicting (iv).

Case 2. *C has a skeleton T with at least three high degree vertices*:

By repeatedly contracting edges of $T \cap C$, we can obtain a minor T' of T such that T' is a tree, $V_1(T') = N(C)$, there are at least three vertices in $V_{\geqslant 3}(T')$ and $|V_{\geqslant 3}(T'/e)| \leqslant 2$ for every edge $e \in E(T' - V_1(T'))$. Contracting an edge of $T' - V_1(T')$ can only reduce $|V_{\geqslant 3}(T')|$ by 1, and only if both endpoints of the edge are in $|V_{\geqslant 3}(T')|$. Hence, there are exactly three vertices of $T' - V_1(T)$, and each has degree at least 3 in T'. Now $T' - V_1(T)$ is a tree on three vertices, and hence is a path wxy. Since w, x and y all have degree at least 3 in T', there are distinct vertices $v_1, \dots, v_5$ such that w is adjacent to v_1 and v_2 in T', y is adjacent to v_4 and v_5 in T', and x is adjacent to v_3 in T'. Let v_6 be the remaining vertex in $N(v)$, and recall that $G[N[v]]$ is a complete subgraph of G by Claim 19. Let E be the set of edges that were contracted to obtain T', and let $G' := G/E$. Then $\mathcal{P} \subseteq G'$ (see Fig. 3b), contradicting (iv).

Case 3. *There is a skeleton T of C with $|V_4(T)| = 2$ and with some $y \in V_2(T)$*:

Let w and x be the vertices in $V_4(T)$.

First, suppose that y is in xTw. Then by Lemma 12, there is a path P of $G[N[C]]$ from xTw to $T - xTw$ with no internal vertex in T. Let a be the endpoint of P in

xTw and let b be the other endpoint. Without loss of generality, $w \notin V(xTb)$. Let $R := (T \cup P) - \text{int}(xTb)$. Then R is a skeleton of C and $V_{\geqslant 3}(R) = \{x, w, a\}$, so we are in Case 2.

Suppose instead that y is not in xTw. Without loss of generality, y is in the component of $T - \text{int}(xTw)$ containing x. Let z be the leaf of T such that y is in xTz. By Lemma 12, there is a path P of $G[N(C)] - \{x, z\}$ from xTz to $T - xTz$ with no internal vertex in T. Let a be the endpoint of P in xTz and let b be the other endpoint. If $w \notin V(xTb)$ or $w = b$, then let $R := (T \cup P) - \text{int}(xTb)$. Otherwise, let $R := (T \cup P) - \text{int}(wTb)$. In either case, R is a skeleton of C and $V_{\geqslant 3}(R) = \{x, w, a\}$, so we are in Case 2.

Case 4. *There is a skeleton T of C with $|V_4(T)| = 2$ and $V_2(T) = \emptyset$:*

Since T is a skeleton of C, $|V_1(T)| = |N(C)| \leqslant 6$. It then follows from (2) that $V(T) \setminus V_1(T) = V_4(T)$, and $|V_1(T)| = 6$. We may assume that we are not in Case 1, so there is some vertex in $C - V_4(T)$. Since C is connected, there is some vertex y in $C - V_4(T)$ adjacent to some vertex x in $V_4(T)$. Let w be the other vertex of $V_4(T)$. By Lemma 9, there is a path of $G - x$ from y to T. Let P be a vertex-minimal example of such a path, and note that $\text{int}(P)$ is disjoint from T. Also, since $N(C) \subseteq V(T)$, every vertex of P is in $N[C]$. Let P' be the path formed from P by adding x and the edge xy, and let b be the other endpoint of P'.

Suppose that either $b = w$ or $w \notin V(bTx)$. Let $R := (T \cup P') - \text{int}(bTx)$. Then R is a skeleton of C with $|V_4(T)| = 2$ and $y \in V_2(T)$, so we are in Case 3.

Suppose instead that $w \in \text{int}(bTx)$. Note that $V(T) = \{x, w\} \cup V_1(T)$, and hence $xTw = xw$. Hence, by Lemma 10, x and w have at least five common neighbours. If some common neighbour z of x and w is in C, then $R := (T \cup wzx) - \text{int}(xTw)$ is a skeleton of C with $|V_4(R)| = 2$ and $z \in V_2(R)$ and we are in Case 3. We may therefore assume that $N(x) \cap N(w) \subseteq N(C)$. Let $v_1, \ldots, v_5$ be distinct vertices in $N(x) \cap N(w)$, and let v_6 be the remaining vertex of $N(C)$. Let w_1, w_2 and w_3 be distinct neighbours of w in $\{v_1, \ldots, v_6\} \setminus \{b\}$, with $w_1 = v_6$ if possible. Since $\{v_1, \ldots, v_5\} \subseteq N(x)$ and at least one of w and x is adjacent to v_6, x has two neighbours x_1 and x_2 in $\{v_1, \ldots, v_6\} \setminus \{b, w_1, w_2, w_3\}$. Let $V(R) := \{x, w, v_1, \ldots, v_6\} \cup V(P)$ and $E(R) := \{ww_1, ww_2, ww_3, xx_1, xx_2, xw\} \cup E(P')$. Then R is a skeleton of C with $V_4(R) = \{x, w\}$ and $y \in V_2(R)$, and we are in Case 3.

Case 5. *There is a skeleton T of C with exactly one vertex $x \in V_3(T)$ and exactly one vertex $w \in V_{\geqslant 4}(T)$:*

Since $\deg_T(x) = 3$ there are distinct leaves v_1 and v_2 such that $w \notin V(v_1Tv_2)$. Let $v_3, v_4, \ldots, v_k$ be the remaining leaves of T, where $k = |N(C)|$. Let C' be the component of $C - w$ containing x, and note that $N(C') \subseteq N(C) \cup \{w\}$. Since G is 4-connected by Lemma 9, there is some vertex in $N(C') \cap (N(C) \setminus \{v_1, v_2\})$, and hence some path P of $G[N[C] \setminus \{w, v_1, v_2\}]$ from x to $N(C) \setminus \{v_1, v_2\}$. Let P' be a subpath of P of shortest possible length while having an endpoint a in the component $T - w$ containing x and an endpoint b in some other component of $T - w$. Note that $P' \subseteq G[N[C] - \{w, v_1, v_2\}]$ and no internal vertex of P' is in T. Let $R := (T \cup P') - \mathrm{int}(bTw)$, and note that R is a skeleton of C. If $a \neq x$, then $V_{\geqslant 3}(R) = \{a, x, w\}$, and we are in Case 2. If $a = x$ and $w \in V_5(T)$, then $V_4(R) = \{x, w\}$, and we are in Case 3 or Case 4. Hence, we may assume $x = a$ and $w \in V_4(T)$, meaning $|N(C)| = 5$. We now consider two subcases, depending on whether $xw \in E(T)$.

Case 5a. $wx \notin E(T)$:

By Lemma 12, there is a path Q of $G[N[C]] - \{x, w\}$ from xTw to $T - xTw$ with no internal vertex in T. Let c be the endpoint of Q in xTw, and let d be the other endpoint.

Suppose first that Q intersects P'. Let Q' be the subpath of Q from c to P' that is internally disjoint from P', and let d' be the endpoint of Q' in P'. Let $S := (R \cup Q') - \mathrm{int}(d'Rx)$. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, c, w\}$, and we are in Case 2.

Suppose instead that Q is disjoint from P'. If $x \notin V(dTw)$, then let $S := (T \cup Q) - \mathrm{int}(dTw)$. Otherwise, let $S := (R \cup Q) - \mathrm{int}(dRx)$. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, c, w\}$, and we are in Case 2.

Case 5b. $xTw = xw$:

By Lemma 10 applied to the edge xw, $|N(x) \cap N(w)| \geqslant 5$.

Suppose there is some vertex $y \in (N(x) \cap N(w)) \setminus N(C)$. If $y \in (N(x) \cap N(w)) \setminus V(T)$, then let $S := (T \cup xyw) - xw$. Then S is a skeleton of C with exactly one vertex $x \in V_3(S)$ and exactly one vertex $w \in V_{\geqslant 4}(S)$ and $xw \notin E(S)$, so we are in Case 5a. If $y \in N(x) \cap N(w) \cap V(xTv_i - v_i)$ for some $i \in \{1, 2\}$, then let S be the graph obtained from R by adding the edge wy and deleting the edge wx. If $y \in N(x) \cap N(w) \cap V(xTv_i - v_i)$ for some $i \in \{3, 4, 5\}$, then let S be the graph obtained from T by adding the edge xy and deleting the edge wx. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, y, w\}$, and we are in Case 2.

Suppose instead that $N(x) \cap N(w) \subseteq N(C)$. Since $|N(C)| = 5$, we have $N(x) \cap N(w) = N(C)$. We may assume we are not in Case 1, so by Lemma 9, there is some vertex y in $C - \{x, w\}$ adjacent to some vertex in $N(C)$. Since $\{x, w\}$ is complete to $N(C)$, assume without loss of generality that $v_5 \in N(y)$. Since C is connected, there is a path Q of C from y to $\{w, x\}$. Choose Q to be of shortest possible length, so that $\mathrm{int}(Q)$ is disjoint from $\{x, w\}$, and without loss of generality assume x is an endpoint of Q (since $\{x, w\}$ is complete to $N(C)$). Let S be the skeleton with $V(S) := \{w, v_1, \dots, v_5\} \cup V(Q)$ and $E(S) := \{wv_1, wv_2, wv_3, wx, xv_4, yv_5\} \cup E(Q)$. By Lemma 12, there is a path Q' of $G[N[C]] - \{x, v_5\}$ from xSv_5 to $S - xSv_5$, internally disjoint from S. Let c be the endpoint of Q' in xSv_5 and let d be the other endpoint. If $d \in \{v_1, v_2, v_3\}$, then let $S' := (S \cup Q') - dw$. Then S' is a skeleton of C with $V_{\geqslant 3}(S') = \{w, x, c\}$, and we are in Case 2. If either $d = w$ and there is some vertex in $\mathrm{int}(Q')$, or $d = v_4$, then let $S' := (S \cup Q') - dx$. Then S' is a skeleton of C with exactly one vertex $c \in V_3(S')$ and exactly one vertex $w \in V_{\geqslant 4}(S')$, and $cw \notin E(S)$, so we are in Case 5a. If $d = w$ and there

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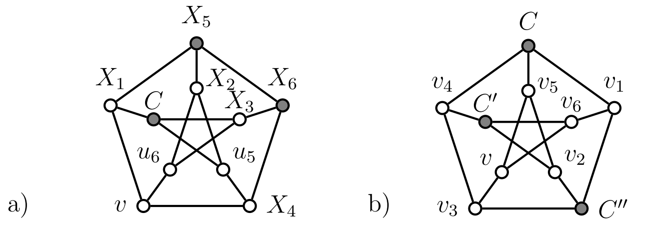

Fig. 4. Petersen subgraphs in Claim 21.

is no vertex in $\mathrm{int}(Q')$, then either $c \in N(x) \cap N(w)$, contradicting the assumption that $N(x) \cap N(w) \subseteq N(C)$, or $|V(ySc \cup Q')| < |V(Q)|$, contradicting our choice of Q. $\square$

Claim 21. *If $v \in V_6(G)$ and C is a component of $G - N[v]$, then $V(C) \neq \emptyset$ and $|N(C)| = 4$.*

Proof. By Lemma 7, $V(G) \setminus N[v]$ is non-empty, so $V(C) \neq \emptyset$. Hence $|N(C)| \geqslant 4$ by Lemma 9. Suppose for contradiction that $|N(C)| \geqslant 5$. Then $|V(C)| = 1$ by Claim 20. Hence, by Lemma 10, $|N(C)| \geqslant 6$, so $N(C) = N(v)$.

Suppose that there is some component C' of $G - N[v]$ with $|N(C')| = 4$. By Lemma 10, $|V(C')| \geqslant 3$. Hence, by Lemma 14 with $A := N[C']$ and $B := V(G - C')$, there is a table $\mathcal{X} := (X_1, \ldots, X_6)$ of $G[N[C']]$ rooted at $N(C')$. For $i \in \{1, 2, 3, 4\}$, let v_i be the unique vertex in $X_i \cap N(C')$. Let v_5 and v_6 be the remaining vertices of $N(v)$. By Claim 19, $G[N[v]] \cong K_7$. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1, 2, \ldots, 6\}$. Then $\mathcal{P} \subseteq G'$ (see Fig. 4a), contradicting (iv).

Suppose instead that every component C' of $G - N[v]$ satisfies $|N(C')| \geqslant 5$. Then by Claim 20 every component of $G - N[v]$ is an isolated vertex and by Lemma 10 each component C' of $G - N[v]$ satisfies $N(C') = N(v)$. Now by Lemma 7 there are at least three distinct components C, C' and C'' of $G - N[v]$. Hence, by Claim 19, $\mathcal{P} \subseteq G$ (see Fig. 4b), contradicting (iv). $\square$

Claim 21 and Lemma 11 immediately imply the following corollary, which we use in the final step of the proof (in Section 8).

Corollary 22. *For every vertex $v \in V_6(G)$, there is at least one v-suitable subgraph.*

We now prove the main result of this section.

Lemma 23. *If $v \in V_6(G)$ and H is a v-suitable subgraph of G, then there is some v-suitable subgraph H' of G such that $N(H') \setminus N(H) \neq \emptyset$.*

Proof. By Claim 21, $|N(H)| = 4$. Suppose for contradiction that there exist distinct vertices $w, x \in N(v)$ such that $N[x] \subseteq N[v]$ and $N[w] \subseteq N[v]$. Let $G' := G - \{v, w, x\}$. By (ii),

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$$|E(G')| \geqslant |E(G)| - 3 - 3(4) \geqslant (5|V(G)| - 11) - 15 = 5|V(G')| - 11.$$

By (v), G' is a $(K_9, 2)$-cockade minus at most two edges. Every $(K_9, 2)$-cockade has at least nine vertices of degree exactly 8, so $|V_8(G')| \geqslant 5$. Then some vertex in $V(G') \setminus N[v]$ has degree exactly 8 in G, contradicting Lemma 18.

Hence there is at most one vertex w in $N(v)$ such that $N[w] \subseteq N[v]$, so there is some vertex x in $N(v) \setminus N(H)$ with some neighbour y in $G - N[v]$. Let H' be the component of $G - N[v]$ that contains y. The vertex x is in $N(H')$, so $N(H') \setminus N(H) \neq \emptyset$. By Claim 21 and Lemma 11, H' is v-suitable, as required. $\square$

7. Degree 9 vertices

In this section, we focus on vertices in $V_9(G) \cap \mathcal{L}$. For each such vertex v, the minimum degree of $G[N(v)]$ is at least 5, by Lemma 10 applied to each edge incident to v. Let H_v be the complement of an edge-minimal spanning subgraph of $G[N(v)]$ with minimum degree 5.

The main result of this section, Lemma 29, states that for each component C of $G - N[v]$, there is some v-suitable subgraph C' with a neighbour not in the neighbourhood of C. We argue for this claim directly when each component C' of $G - N[v]$ has $|N(C')| = 4$. Otherwise, we first look at the case where the maximum distance between two vertices of degree 3 in H_v is at most 2. Then we consider the case where there are two vertices of degree 3 at distance at least 3 in H_v. A useful technique is that a graph obtained by contracting some edge in $G[N(v)]$ must violate some condition of Claim 17.

Claim 24. *If $v \in V_9(G) \cap \mathcal{L}$, then $\Delta(H_v) = 3$ and the vertices of H_v with degree at most 2 form a clique.*

Proof. Since $|V(H_v)| = |N(v)| = 9$, if a vertex u has degree greater than 3 in H_v, then u has degree less than 5 in $\overline{H_v}$, a contradiction. If two non-adjacent vertices x and y in H_v both have degree at most 2 in H_v, then $\overline{H_v} - xy$ is a spanning subgraph of $G[N(v)]$ with minimum degree at least 5, contradicting the definition of H_v. Thus the vertices of degree at most 2 form a clique of size at most 3, so there is indeed a vertex of degree 3 in H_v. $\square$

The following claim guarantees that $|V(G)| \geqslant 11$ if we find a vertex $v \in V_9(G) \cap \mathcal{L}$, and hence that the components of $G - N[v]$ are non-empty.

Claim 25. *If $v \in V_9(G) \cap \mathcal{L}$, then $V(G - N[v]) \neq \emptyset$.*

Proof. By (iv), $\mathcal{P} \not\subseteq G[N[v]]$, so $G[v] \ncong K_{10}$. Hence, there is some vertex $w \in N(v)$ such that $N[w] \neq N[v]$. By the definition of $\mathcal{L}$, there is some vertex $x \in N[w] \setminus N[v]$ and $x \in V(G - N[v])$. $\square$

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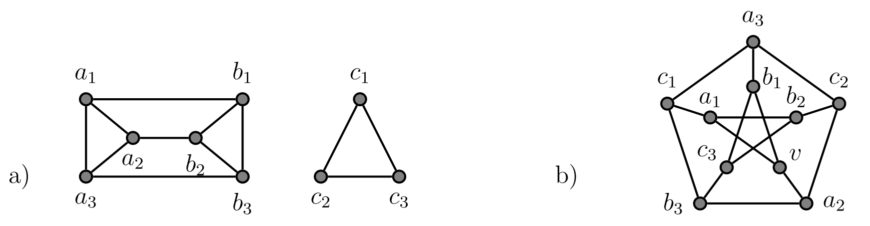

Fig. 5. Illustration for the proof of Claim 26.

A graph is *cubic* if every vertex has degree exactly 3.

Claim 26. *If $v \in V_9(G) \cap \mathcal{L}$, then there are vertices x and y in $V_3(H_v)$ such that $\operatorname{dist}_{H_v}(x, y) \geqslant 3$, unless either $|N(C)| = 4$ for every component C of $G - N[v]$ or $H_v \cong K_{3,3} \dot\cup K_3$.*

Proof. Suppose for contradiction that $\operatorname{dist}_{H_v}(x, y) \leqslant 2$ whenever $\{x, y\} \subseteq V_3(H_v)$, there is some component C of $G - N[v]$ such that $|N(C)| \neq 4$ and $H_v \ncong K_{3,3} \dot\cup K_3$. By Claim 25, $V(C) \neq \emptyset$, so by Lemma 9, $|N(C)| \geqslant 5$. Let $S := V_0(H_v) \cup V_1(H_v) \cup V_2(H_v)$. By Claim 24, S is a clique, so $|S| \leqslant 3$. Since $|V(H_v)| = 9$, the number of vertices of odd degree in H_v is even and $V(H_v) \setminus S = V_3(H_v)$, we have $S \neq \emptyset$. We consider five cases depending on S and whether there is any triangle in H_v.

Case 1. $|S| = 3$:

In this case, $S = V_2(H_v)$ and $H_v[S] \cong K_3$, and there is no edge in H_v from a vertex in S to a vertex not in S. Hence, $H_v - S$ is a 6-vertex cubic graph. By assumption, $H_v \ncong K_{3,3}$. There is only one other 6-vertex cubic graph, so H_v is the graph depicted in Fig. 5. Then $\mathcal{P} \subseteq G[N[v]]$ (see Fig. 5b), contradicting (iv).

Case 2. $|S| = 2$:

Since $|V(H_v)|$ is odd, there are an odd number of vertices of even degree in H_v. Since S is a clique, $\delta(H_v) \geqslant 1$. Hence, by Claim 24, there is a unique vertex $x \in V_2(H_v)$, and since $|S| = 2$, there is some vertex $v_1 \in V_3(H_v)$ adjacent to x in H_v. Let v_2 and v_3 be the other neighbours of v_1 in H_v, and note that $\{v_2, v_3\} \subseteq V_3(H_v)$. Since $\operatorname{dist}_{H_v}(v_1, y) \leqslant 2$ for every vertex y in $V_3(H_v)$, each of the four remaining vertices of $H_v - S$ is adjacent to $\{v_2, v_3\}$. Since v_2 and v_3 each have only three neighbours in H_v, $v_2v_3 \notin E(H_v)$. Let G' be obtained from G by deleting every edge in $G \cap H_v$ and then contracting v_2v_3. Now $v \in V_8(G')$. Let v' be a vertex in $N_{G'}(v)$. If $v' \in S$, then $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 8 - \deg_{H_v}(v') - 1 \geqslant 5$. If v' is in $H_v - (S \cup \{v_2, v_3\})$, then $|N_{G'}(v') \cap N_{G'}(v)| = 8 - \deg_{H_v}(v') = 5$. If v' is the new vertex of G', then $|N_{G'}(v') \cap N_{G'}(v)| = 8 - |N_{H_v}(v_2) \cap N_{H_v}(v_3)| - 1 = 6$. Hence, $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 5$ for any vertex $v' \in N_{G'}(v)$. Finally, $|N_{G'}(C)| \geqslant |N_G(C)| - 1 \geqslant 4$, so $G'[N_{G'}(C)] \ncong K_3$. Hence $\mathcal{P}$ is a minor of G by Claim 17, contradicting (iv).

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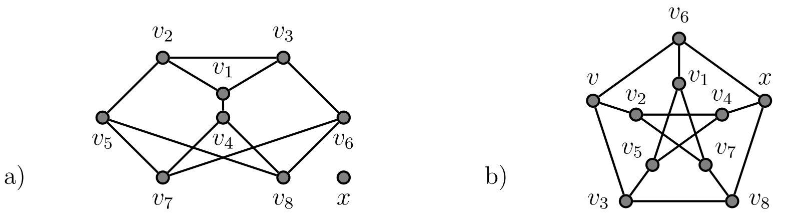


Fig. 6. Illustration for the proof of Claim 26.

Case 3. *There is some triangle* $v_1v_2v_3$ *of* H_v *and* $S = V_0(H_v) = \{x\}$:

Let $\{v_4, v_5, \dots, v_8\}$ be the other vertices of H_v, where $v_4v_1 \in E(H_v)$. For every vertex y in $H_v - S$ we have $\mathrm{dist}_{H_v}(v_1, y) \leqslant 2$ by assumption, so y is either adjacent to v_1 or adjacent to a neighbour of v_1. Since $\{v_2, v_3, v_4\} \subseteq V_3(H_v)$, we may assume without loss of generality that $\{v_2v_5, v_3v_6, v_4v_7, v_4v_8\} \subseteq E(H_v)$. Since $\Delta(H_v) = 3$ and $\mathrm{dist}_{H_v}(v_i, v_j) \leqslant 2$ for $i \in \{2, 3\}$ and $j \in \{7, 8\}$, H_v is the graph depicted in Fig. 6a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Fig. 6b), contradicting (iv).

Case 4. *There is no triangle of* H_v *and* $S = V_0(H_v) = \{x\}$:

So $H_v - x$ is a cubic, triangle-free graph, with diameter 2 and exactly eight vertices. We now show that there is exactly one such graph, namely the Wagner graph. Let v_1 be a vertex of $H_v - S$, and let v_2, v_3 and v_4 be its neighbours in H_v. Since H_v contains no triangle, $\{v_2, v_3, v_4\}$ is an independent set in H_v. Let $\{v_5, v_6, v_7, v_8\}$ be the remaining vertices of $H_v - S$. If v_2, v_3 and v_4 all share some common neighbour, say v_5, in H_v, then there are six edges in $H_v[\{v_1, \dots, v_5\}]$, and at most three other edges in H_v incident to some vertex in $\{v_1, \dots, v_5\}$. By the handshaking lemma, $E(H_v - S) = E(H_v) = 12$, since $S = V_0(H_v)$ and $V(H_v - S) = V_3(H_v)$. Hence $v_6v_7v_8$ is a triangle of H_v, a contradiction. If for every pair $i, j \in \{2, 3, 4\}$ v_i and v_j share a neighbour in H_v distinct from v_1, then $|N_{H_v}[v_2] \cup N_{H_v}[v_3] \cup N_{H_v}[v_4]| \leqslant 3(4) - 3(2) + 1 = 7$ by inclusion-exclusion, contradicting the assumption that $\mathrm{dist}_{H_v}(v_1, y)$ for each of the 8 vertices y in $V_3(H_v)$. Hence, without loss of generality, v_2 and v_3 have no common neighbour in H_v, and $\{v_2v_5, v_2v_6, v_3v_7, v_3v_8\} \subseteq E(H_v)$. Without loss of generality $v_8 \in N_{H_v}(v_4)$, since $\{v_5, v_6, v_7, v_8\} \cap N_{H_v}(v_4) \neq \emptyset$. Since $v_7v_3v_8$ is a path in H_v and H_v contains no triangle, the other vertex adjacent to v_8 is either v_5 or v_6, so without loss of generality $v_8v_6 \in E(H_v)$. Since $v_5v_2v_6$ and $v_4v_8v_6$ are paths in H_v, the remaining vertex adjacent to v_6 is v_7. Since $V_3(H_v) = V(H_v) \setminus \{x\}$ and $x \in V_0(H_v)$, the remaining two vertices adjacent to v_5 are v_7 and v_4. Hence H_v is the Wagner Graph, plus a single isolated vertex, as illustrated in Fig. 7a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Fig. 7b), contradicting (iv).

Case 5. $S = \{x\}$ *and* $x \notin V_0(H_v)$:

The number of vertices of odd degree in H_v is even, so $x \in V_2(H_v)$. By contracting an edge of H_v incident to x, we obtain a cubic graph on eight vertices with diameter at

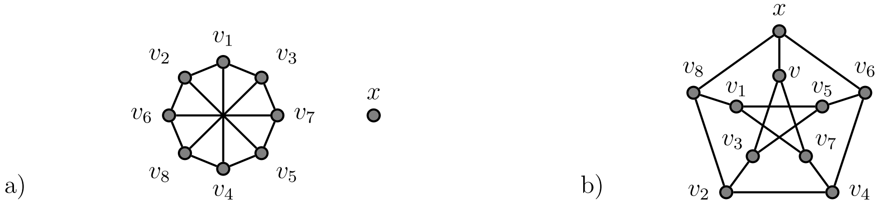


Fig. 7. Case 4 in Claim 26.

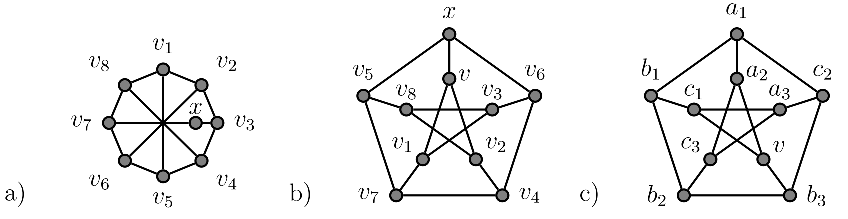


Fig. 8. Illustrations for Claims 26 and 27.

most 2. In Cases 3 and 4 we showed that there are only two such graphs (one with and one without a triangle), so H_v is a copy of one of these in which exactly one edge is subdivided exactly once. It is quick to check that the only such graph in which $\text{dist}(x', y') \leqslant 2$ whenever x' and y' both have degree 3 is the graph depicted in Fig. 8a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Fig. 8b), contradicting (iv). □

Claim 27. *If $v \in V_9(G) \cap \mathcal{L}$ and $H_v \cong K_{3,3} \dot\cup K_3$, then for each component C of $G - N[v]$, there is some v-suitable subgraph C' with $N(C') \setminus N(C) \neq \emptyset$.*

Proof. Let $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\} := V(H_v)$, with $a_i b_j \in E(H_v)$ for $i, j \in \{1, 2, 3\}$, and $c_i c_j \in E(H_v)$ for distinct $i, j \in \{1, 2, 3\}$. Suppose for contradiction that there is a path P of G from a_i to b_j with no internal vertex in $N[v]$ for some $i, j \in \{1, 2, 3\}$. Without loss of generality, $i = j = 1$. Let G' be obtained from G by contracting all but one edge of P. Then $\mathcal{P} \subseteq G'$ (see Fig. 8c), contradicting (iv). Hence, there is no such path P. In particular, no vertex v' in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is adjacent to every vertex of $N(v) \setminus \{v'\}$. Hence, since $v \in \mathcal{L}$, for each $v' \in \{a_1, a_2, a_3, b_1, b_2, b_3\}$ there is some component C of $G - N[v]$ such that $v' \in N(C)$. However, there is no component C such that $N(C)$ contains some vertex in $\{a_1, a_2, a_3\}$ and some vertex in $\{b_1, b_2, b_3\}$. Hence, for each component C of $G - N[v]$ there is a component C' of $G - N[v]$ with $N(C') \setminus N(C) \neq \emptyset$. Suppose for contradiction that C' is not v-suitable. By Lemma 11 $|N(C')| \geqslant 7$. Since $\overline{G[N(C')]} \subseteq H_v$, there is some vertex in $\{a_1, a_2, a_3\} \cap N(C')$ and some vertex in $\{b_1, b_2, b_3\} \cap N(C')$, a contradiction. Hence C' satisfies our claim. □

Claim 28. *If $v \in V_9(G) \cap \mathcal{L}$ and there are two vertices x and y in $V_3(H_v)$ such that $\text{dist}_{H_v}(x, y) \geqslant 3$ and there is some component C of $G - N[v]$ with $|N(C)| \geqslant 5$, then for each component C' of $G - N[v]$ there is a v-suitable subgraph C'' with $N(C'') \setminus N(C') \neq \emptyset$.*

Proof. By Claim 25, $V(C) \neq \emptyset$. Choose x and y, if possible, so that

$$N_{H_v}(x) \cup N_{H_v}(y) \subseteq V_3(H_v). \tag{3}$$

Let $G' := G/xy$, let x' be the new vertex of G', and let $H' = H_v - \{x, y\}$. Note that $\deg_{G'}(v) = 8$. Since $|N_G(C)| \geqslant 5$, we have $|N_{G'}(C)| \geqslant 4$, and hence $G'[N_{G'}(C)] \ncong K_3$. By Claim 17 and (iv), G' does not satisfy $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 5$ for all $v' \in N_{G'}(v)$.

Now $\{x, y\} \subseteq V_3(H_v)$ and $\text{dist}_{H_v}(x, y) \geqslant 3$, so $|N_{H_v}(x) \cap N_{H_v}(y)| = 6$. Also, since $\overline{G[N(v)]} \subseteq H_v$, there is no common neighbour of x and y in $\overline{G[N(v)]}$, so x' is dominant in $G'[N_{G'}[v]]$, and $|N_{G'}(x') \cap N_{G'}(v)| = 7 > 5$.

By Claim 24, $\Delta(H_v) = 3$. If $v' \in N_{H_v}(x) \cup N_{H_v}(y)$, then since v' is not adjacent to both x and y in $\overline{H_v}$, we have $|N_{G'}(v') \cap N_{G'}(v)| \geqslant |N_{\overline{H_v}}(v')| \geqslant 8 - 3 = 5$.

Hence, the unique vertex z in $H' - (N_{H_v}(x) \cup N_{H_v}(y))$ satisfies $|N_{G'}(z) \cap N_{G'}(v)| \leqslant 4$. Thus z has at most three neighbours in $G'[N(v) \setminus \{x, y\}]$ and hence $|N_{H'}(z)| \geqslant 7 - 1 - 3 = 3$. Since $\Delta(H') \leqslant \Delta(H_v) = 3$, we have $\deg_{H_v}(z) = 3$.

There are an even number of vertices, including z, with odd degree in H'. We have $\deg_{H'}(v') \leqslant \Delta(H_v) - 1 = 2$ for the six vertices v' in $N_{H_v}(x) \cup N_{H_v}(y) = V(H' - z)$, so there are an odd number of vertices in $V_1(H')$. Each vertex in $V_1(H')$ has degree at most 2 in H_v since x and y have no common neighbour in H_v. So $V_1(H')$ is a clique of H_v by Claim 24, and hence a clique of H'. Since $|V_1(H')|$ is odd, there is a unique vertex w in $V_1(H')$. By the same argument, the vertices of $V_0(H') \cup V_1(H')$ form a clique of H'. No vertex in $V_0(H')$ is adjacent in H' to w, so $V_0(H') = \emptyset$. Hence, $V_1(H') = \{w\}$, $V_3(H') = \{z\}$ and $V_2(H') = V(H' - \{w, z\})$.

Now w is one of the six vertices of $N_{H_v}(x) \cup N_{H_v}(y)$, and $\deg_{H_v}(w) \leqslant \deg_{H'}(w) + 1 \leqslant 2$. In particular x and y do not satisfy (3), so no such pair satisfy (3). This means, there are no two vertices x' and y' in $V_3(H_v)$ that satisfy (3) such that $\text{dist}_{H_v}(x', y') \geqslant 3$.

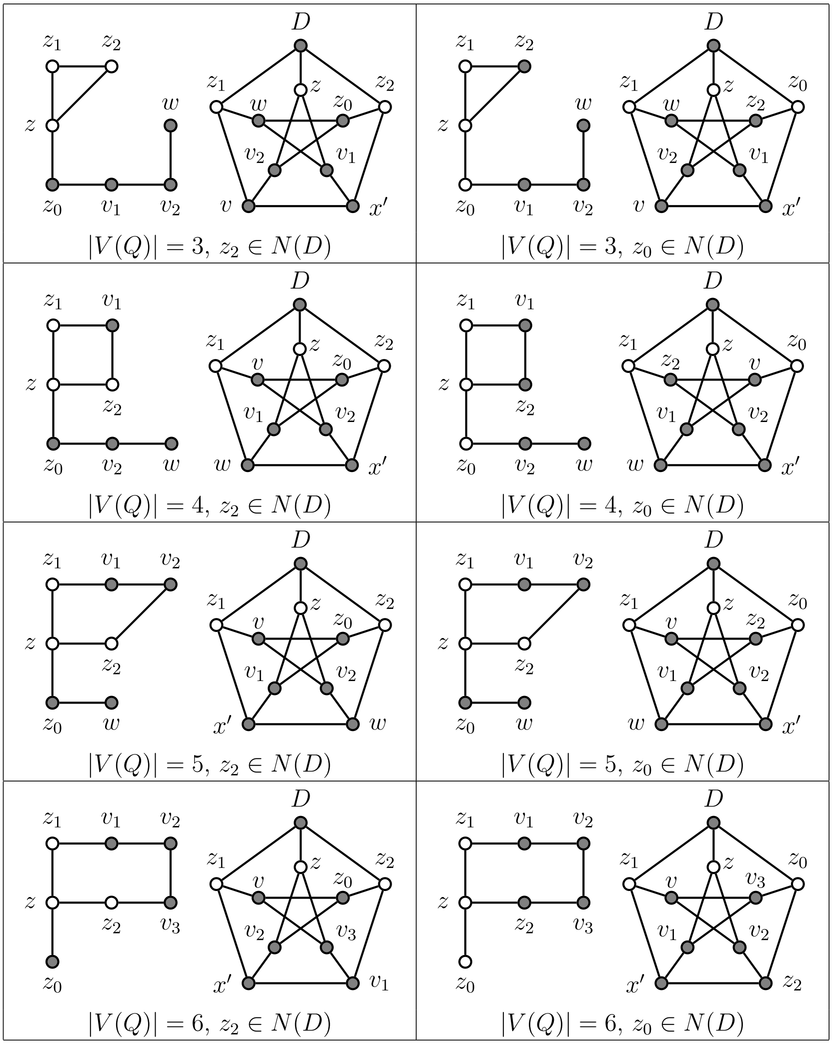
We consider four cases depending on whether H' is connected and on the components of $G - N[v]$.

Case 1. *H' is not connected*:

Since each connected component of H' has an even number of vertices of odd degree, z and w are in the same component, and each other component is a cycle. Since $|V(H') \setminus N_{H'}[z]| = 3$, there is a unique component D of H' not containing z and D is a triangle. Since $|V_2(H')| = 5$, there is some vertex x_0 of degree 2 not in D and not adjacent to w. Assume without loss of generality that x_0 is adjacent to x in H_v. Since $x \in V_3(H_v)$, there is some vertex y_0 in D such that $y_0 x \notin E(H_v)$. Now y_0 is adjacent to no neighbour of x_0 in H_v, so $\text{dist}_{H_v}(x, y) \geqslant 3$. But the vertices adjacent to $\{x_0, y_0\}$ in H_v are all in $V_3(H_v)$ since w is adjacent to neither x_0 nor y_0 in H_v. Therefore x_0 and y_0 satisfy (3), a contradiction.

For the remaining cases, H' is a connected graph such that $|V_1(H')| = |V_3(H')| = 1$ and every other vertex has degree 2. Hence, H' is composed of a path P from z to w and

Table 2
Petersen subgraphs in Case 2 of Claim 28.

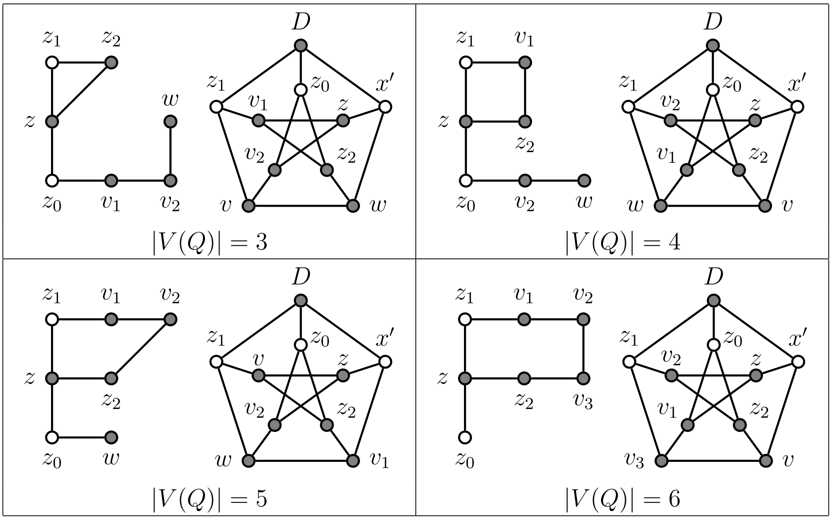


a cycle Q of size at least 3 containing z, with $V(P \cap Q) = \{z\}$. Let z_0 be the neighbour of z in the path from z to w, and let z_1 and z_2 be the other neighbours of z in H'.

Case 2. *H' is connected and there is some component D of $G - N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 2$:*

At least one vertex is in $\{z_1, z_2\} \cap N(D)$, so without loss of generality $z_1 \in N(D)$. Either z_0 or z_2 is also in $N(D)$. Since $V(Q) \subseteq V(H') \setminus \{w\}$, we have $3 \leqslant |V(Q)| \leqslant 6$. Let $G'' := G'/E(D)$. The diagrams in Table 2 demonstrate that $\mathcal{P} \subseteq G''$, contradicting (iv).

Table 3
Petersen subgraphs in Case 3 of Claim 28.



Case 3. *H' is connected and there is some component D of $G - N[v]$ such that $z_0 \in N(D)$, $N(D) \cap \{z_1, z_2\} \neq \emptyset$ and $N(D) \cap \{x, y\} \neq \emptyset$:*

Without loss of generality, $z_1 \in N(D)$. Note that $\{z_1, z_0, x'\} \subseteq N_{G'}(D)$, and let $G'' := G'/E(D)$. The diagrams in Table 3 demonstrate that $\mathcal{P} \subseteq G''$, contradicting (iv).

Case 4. *H' is connected and there is no component D of $G - N[v]$ such that either $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 2$ or $z_0 \in N(D)$, $N(D) \cap \{z_1, z_2\} \neq \emptyset$ and $N(D) \cap \{x, y\} \neq \emptyset$:*

Recall that $|N_{G'}(z) \cap N_{G'}(v)| \leqslant 4$. Hence z has at least three non-neighbours in $G'[N(v)]$. Since $\overline{G'[N(v) \setminus \{x, y\}]} \subseteq H'$ and x' is dominant in $G'[N_{G'}(v)]$, z is non-adjacent in G' to each vertex in $N_{H'}(z)$. Hence, for every vertex $z' \in N_{H'}[z]$ there is a component $C_{z'}$ of $G - N[v]$ such that $z' \in N(C_{z'})$, since $v \in \mathcal{L}$.

By Lemma 11, each component $C_{z'}$ of $G - N[v]$ satisfying $|N(C_{z'})| \leqslant 6$ is v-suitable.

Recall that C' is an arbitrary component of $G - N[v]$. We now show that, for some $z' \in N_{H_v}[z]$, $C_{z'}$ is v-suitable and $N(C_{z'}) \setminus N(C') \neq \emptyset$, as required.

Suppose first that there is no component D of $G - N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 1$. Then $|N(C_z)| \leqslant 6$. Furthermore, $z \notin N(C_{z_0})$ and either $N(C_{z_0}) \cap \{z_1, z_2\} = \emptyset$ or $N(C_{z_0}) \cap \{x, y\} = \emptyset$ since Case 3 does not apply, so $|N(C_{z_0})| \leqslant 6$. Hence, C_z and C_{z_0} are both v-suitable. By assumption, $N(C)$ does not contain both z and z_0, so $z' \notin N(C)$ for some vertex $z' \in \{z, z_0\}$. Hence, $N(C_{z'}) \setminus N(C') \neq \emptyset$, and the claim holds.

Now assume that there is some component D of $G - N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 1$. Since Case 2 does not apply, $|N(D) \cap N_{H'}(z)| = 1$. Let $\{z', z''\} := N_{H'}(z) \setminus N(D)$. If $|N(C_{z'})| \leqslant 6$ and $|N(C_{z''})| \leqslant 6$ (in which case $C_{z'}$ and $C_{z''}$ are both v-suitable), and $\{z', z''\} \nsubseteq N(C')$, then the claim holds. So we may assume that either $D' := C'$ satisfies $\{z', z''\} \subseteq N(D')$ or some $D' \in \{C_{z'}, C_{z''}\}$ satisfies $|N(D')| \geqslant 7$. Now D' is distinct from D since $N(D') \cap \{z', z''\} \neq \emptyset$, and $|N_{G'}(D')| \geqslant 3$ since $|N(D')| \geqslant 4$ by Lemma 9. Let G'' be obtained from G' by contracting D onto z. Then $v \in V_8(G'')$, $|N_{G''}(v) \cap N_{G''}(v')| \geqslant 5$ for every vertex $v' \in N_{G''}(v)$, and $|N_{G''}(D')| = |N_{G'}(D')| \geqslant 3$. Furthermore, there is at most one cycle in $\overline{G''[N(v)]}$, namely Q, so $\overline{K_3}$ and $\overline{C_4}$ are not both induced subgraphs of $G''[N(v)]$. Hence by Claim 17, $\mathcal{P} \subseteq G''$, contradicting (iv). □

We finally reach the main result of this section.

Lemma 29. *If $v \in V_9(G) \cap \mathcal{L}$ and C is a component of $G - N[v]$, then there is some v-suitable subgraph C' such that $N(C') \setminus N(C) \neq \emptyset$.*

Proof. Suppose first that each component C' of $G - N[v]$ has $|N(C')| = 4$. Then every component of $G - N[v]$ is v-suitable by Lemma 11. Suppose for contradiction that there is no v-suitable subgraph C' such that $N(C') \setminus N(C) \neq \emptyset$. Then $N(C') \subseteq N(C)$ for every component C' of $G - N[v]$, so there are at least five vertices in $N(v)$ with no neighbour outside of $N[v]$. Since $v \in \mathcal{L}$, each of these vertices is dominant in $G[N[v]]$. Let G' be obtained from G by contracting C onto some vertex x of $N(C)$ and then deleting all other components of $G - N[v]$. There are at most three non-dominant vertices in G', so $|E(G')| \geqslant \binom{10}{2} - 3 = 42 = 5|V(G')| - 8$, contradicting (vi).

Suppose instead that there is some component C' of $G - N[v]$ with $|N(C')| \geqslant 5$. By Claims 26 and 27, we may assume that there are two vertices x and y in $V_3(H_v)$ such that $\mathrm{dist}_{H_v}(x, y) \geqslant 3$. The result then follows directly from Claim 28. □

Lemma 29 immediately implies the following corollary, which we use in Section 8.

Corollary 30. *For every vertex $v \in V_9(G)$ there is at least one v-suitable subgraph.*

8. Final step

We now complete the proof sketched in Section 2.

Proof of Theorem 2. Let G be the minimum counterexample defined at the start of Section 3. By Lemmas 10, 16 and 18, $\mathcal{L} \subseteq V_6(G) \cup V_9(G)$, so for every vertex $v \in \mathcal{L}$ there is some v-suitable subgraph of G by Corollaries 22 and 30. Choose $v \in \mathcal{L}$ and H a v-suitable subgraph of G so that $|V(H)|$ is minimised. Let u be a vertex of $\mathcal{L}$ in H. Since $u \in V(H)$ and H is a component of $G - N[v]$, u is not adjacent to v, so v is in some component C of $G - N[u]$. Since $v \in \mathcal{L}$, C is u-suitable. By Lemmas 23 and 29, there is some u-suitable subgraph C' of G with $N(C') \setminus N(C) \neq \emptyset$.

Now $N(C') \subseteq N(u)$, so $v \notin N(C')$. Since $N(C') \setminus N(C) \neq \emptyset$, we have that C and C' are distinct (and thus disjoint), so $v \notin N[C']$ and C' is disjoint from $N[v]$. Hence $G[V(C') \cup (N(C') \setminus N(C)) \cup \{u\}]$ is a connected subgraph of $G - N[v]$, and thus a subgraph of H. But $u \in V(H) \setminus V(C')$, so $|V(C')| < |V(H)|$, contradicting our choice of v and H. This contradiction shows that in fact there are no counterexamples to Theorem 2. $\square$

Acknowledgments

The authors are grateful to the referees for their helpful comments, and especially for an insightful suggestion that lead to a significant shortening of Section 7.

Appendix A

We now prove the two well known lemmas used in Section 1.

Lemma 31. *For every $(t+1)$-connected graph H and every non-negative integer $s < |V(H)|$, every (K_s, t)-cockade is H-minor-free.*

Proof. Let G be a (K_s, t)-cockade. We proceed by induction on $|V(G)| + |E(G)|$. The claim is trivial if $G = K_s$, since $s < |V(H)|$. Assume that there are (K_s, t)-cockades G_1 and G_2 distinct from G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 \cong K_t$. Note that G_1 and G_2 are proper subgraphs of G, and hence by induction are H-minor-free. Suppose for contradiction that G contains an H-minor. Then there is a set of pairwise disjoint connected subgraphs of G such that if every edge inside one of these subgraphs is contracted and every vertex not in one of these subgraphs is deleted, then the graph obtained is a supergraph H' of H such that $|V(H')| = |V(H)|$. Each of these subgraphs will contract down to a separate vertex, so we call these subgraphs *prevertices*. There are exactly t vertices in $G_1 \cap G_2$, so the set S of prevertices that intersect $G_1 \cap G_2$ has size at most t. Since H is $(t+1)$-connected, each prevertex not in S is in the same connected component of $G - S$. Without loss of generality, each prevertex not in S is a subgraph of G_1. Now, there is no path of G between two non-adjacent vertices of G_1 that is internally disjoint from G_1. Hence, by deleting every vertex of $G_2 \setminus G_1$ and then contracting the remaining edges of the prevertices and deleting the remaining vertices that are not in any prevertex, we obtain H', contradicting the assumption the G_1 contains no H-minor. $\square$

Proof of Lemma 4. Let G be an n-vertex H-minor-free graph. We proceed by induction on n. The base case with $n \leqslant 2c - 1$ is trivial. For $n \geqslant 2c$, $|E(G)| < c|V(G)|$, implying G has average degree less than $2c$. Thus G has a vertex v of degree at most $2c - 1$. By induction, $G - v$ is $2c$-colourable. Some colour is not used on the neighbours of v, which can be assigned to v. Hence G is $2c$-colourable. It remains to prove that G is $(2c-1)$-colourable under the assumption that $|V(H)| \leqslant 2c$. First suppose that $\deg(v) \leqslant 2c - 2$. By induction, $G - v$ is $(2c - 1)$-colourable. Some colour is not used on the neighbours of v, which can be assigned to v. Hence G is $(2c-1)$-colourable. Now assume that $\deg(v) = 2c - 1$. There is some pair of non-adjacent vertices x and y in $N(v)$, as otherwise G contains K_{2c} and hence H (since $|V(H)| \leqslant 2c$). Let G' be the graph obtained from G by contracting the edges vx and vy into a new vertex z. By induction, G' is $(2c-1)$-colourable. Colour each vertex of $G - \{v, x, y\}$ by the colour assigned to the corresponding vertex in G'. Colour x and y by the colour assigned to z. Since every vertex adjacent to x or y in $G - v$ is adjacent to z in G', this defines a $(2c-1)$-colouring of $G - v$. Now v has $2c - 1$ neighbours, two of which have the same colour. Thus there is an unused colour on the neighbours of v, which can be assigned to v. Therefore G is $(2c-1)$-colourable. □

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