Proximity graphs: E, δ, Δ, χ and ω

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Abstract

Graph-theoretic properties of certain proximity graphs defined on planar point sets are investigated. We first consider some of the most common proximity graphs of the family of the Delaunay graph, and study their number of edges, minimum and maximum degree, clique number, and chromatic number. In the second part of the paper we focus on the higher order versions of some of these graphs and give bounds on the same properties.

Keywords Geometric graphs; proximity graphs; graph-theoretic properties.

1 Introduction

Loosely speaking, a proximity graph has as its vertex set a set of points in the plane, and adjacency in the graph attempts to describe some of the proximity relations of the point set. Examples of proximity graphs include relative neighborhood graphs, sphere-of-influence graphs, Yao graphs, and Gabriel graphs (see [23] for a survey). Proximity graphs extract the relevant structure or shape of point sets, and thus find applications in areas where this structure is important, which include pattern recognition, computer vision, and cluster analysis. Additionally, some proximity graphs have other desirable properties (for example, planarity) which, combined with the correlation between adjacency and proximity, makes them a useful tool in disciplines such as wireless networks, graph drawing, and terrain representation.

This paper considers a family of proximity graphs comprising several graphs —and some of their variations— that are subgraphs of the Delaunay triangulation. We study some of

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the classical graph-theoretic properties of these graphs; namely, number of edges, minimum and maximum degree, chromatic number, and clique number (see the definitions below). These parameters provide relevant information of the graphs and have in fact been considered before in the literature. The existing results, though, leave some gaps, have been developed under different degrees of non-degeneracy assumptions for the point sets, and have never been gathered to allow comparisons between distinct classes of proximity graphs. In this paper we try to address these issues by fixing the same assumptions on the point sets for all graphs, reviewing previous work on this setting and developing new bounds in order to close or narrow the existing gaps.

The first part of the paper, Section 2, is devoted to order-0 proximity graphs. We consider seven graphs of the family of the Delaunay graph, namely, the minimum spanning tree, the union of the minimum spanning trees, the relatively closest graph, the relative neighborhood graph, the Gabriel graph, the modified Gabriel graph, and the Delaunay graph (the definitions of these graphs are given below). These graphs were quite popular in the eighties and early nineties, and many of their properties can be found in [13, 23, 27, 29, 31, 32]. Specifically, a variety of properties have been investigated for the Gabriel graph and the relative neighborhood graph in [27] and [32], respectively, and also for two variations of these graphs in [13]. This group of properties comprises the size of the minimum cycle/wheel that might be contained as a subgraph, the size of the maximum complete/complete bipartite graph that might appear in the graph, constraints on the structure of the trees that can be represented as a proximity graph, maximum number of edges, expected vertex degree... In relation to expected case analysis, the size of several proximity graphs defined on points drawn at random has been determined in [14], and the expected maximum degree of Gabriel graphs has been given in [15]. Additionally, some properties of the Delaunay triangulation of random points have been considered, such as the average and maximum edge length, the minimum and maximum angles, and the expected weight of the triangulation [6, 12, 28]. Other graph-theoretic properties of the Delaunay triangulation that have been investigated are hamiltonicity and toughness [17, 18]. Finally, there exists an ample body of literature on characterizations of which combinatorial graphs can be drawn as proximity graphs of some set of points; the interested reader is referred to [25].

In this paper we focus on the basic properties mentioned earlier. First, we study the seven order-0 proximity graphs indicated, and we do not make any non-degeneracy assumption on the set of points on which the graphs are defined, since we believe that the analysis is more interesting in this case. Nevertheless, in some occasions we make remarks on the differences between the non-degenerate and the degenerate situations.

In the second part of the paper, Section 3, we look at higher order proximity graphs. Except for minimum spanning trees, all graphs in Section 2 can be generalized to order-k graphs. We focus on some of the most common, i.e., the k-relative neighborhood graph, the k-Gabriel graph, and the k-Delaunay graph. We also consider the shared k-nearest neighbor graph and the k-nearest neighbor graph, which are not analyzed in Section 2 because all bounds given in Section 3 for these graphs are tight for all values of k, and thus also for $k = 1$. Here we assume that points are in certain general position, which is partially enforced by the definitions of the graphs.

Order-k graphs have not been so extensively studied as their order-0 counterparts. The authors of [30] obtained asymptotic bounds for the size of the k-Gabriel graph and the k-Delaunay graph, and the results for the latter were refined in [1]. There are also some results on the number of edges of the k-relative neighborhood graph [9, 10, 11]. The chromatic number, diameter, and connectivity of the k-Gabriel and k-Delaunay graphs have been studied in [7], and the number of crossings of several order-k proximity graphs have been considered in [2]. As for nearest neighbor graphs, some interesting properties are given in [20], including the expected number of components of the graph and the relationship between the size of a component and its diameter.

Definitions and results All graphs considered are undirected, finite and simple, unless stated otherwise. Let G be such a graph. We denote by $V(G)$ (respectively, $E(G)$) the set of vertices (respectively, edges) of G, and by $|V(G)|$ (respectively, $|E(G)|$) the cardinality of this set. If v is a vertex in $V(G)$, we denote by $d_G(v)$ the degree of v in G. The *minimum degree* of G is $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. The *maximum degree* of G is $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$. A *clique* of G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the maximum number of vertices in a clique of G. A *k-coloring* of G is a mapping $f : V(G) \to \{1, 2, \dots, k\}$ such that $f(v) \neq f(w)$ for every edge vw of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G is k-colorable.

We denote by S a generic set of n points in the plane. When we describe a concrete example of a point set satisfying a specific property we use $\mathcal{S}$. We next list the definitions of the graphs we consider in this paper. All of them are geometric graphs on S, that is, their set of vertices is S and their edges consist of straight-line segments with endpoints in S. They are all undirected except for k-$\mathsf{NNG}(S)$, which is directed. Points in S are usually denoted by $p_1, \dots, p_i, \dots, p_n$.

Definition 1.1. Let T be a spanning tree of S. The *weight* of T is the sum of the lengths of the edges of T. A spanning tree of S with minimum weight is a *minimum spanning tree* of S, and the set of minimum spanning trees of S is denoted by $\mathsf{MST}(S)$. The graph with vertex set S consisting of the union of all $T \in \mathsf{MST}(S)$ is denoted by $\mathsf{U\text{-}MST}(S)$.

We associate two lenses[1] to any pair p_i, p_j:

$$\begin{aligned}\mathsf{O\text{-}LENS}(p_i,p_j) =& \{x \in \mathbb{R}^2 : |p_i x| < |p_i p_j| \text{ and } |p_j x| < |p_i p_j|\}\\ \mathsf{C\text{-}LENS}(p_i,p_j) =& \{x \in \mathbb{R}^2 : |p_i x| \leq |p_i p_j| \text{ and } |p_j x| \leq |p_i p_j|\}\ .\end{aligned}$$

Definition 1.2. The *relative neighborhood graph*, denoted by $\mathsf{RNG}(S)$, is the graph in which p_i, p_j are adjacent if $\mathsf{O\text{-}LENS}(p_i,p_j) \cap S = \emptyset$. The *relatively closest graph*, denoted by $\mathsf{RCG}(S)$, is the graph in which p_i, p_j are adjacent if $\mathsf{C\text{-}LENS}(p_i,p_j) \cap S = \{p_i, p_j\}$.

We also associate two discs to p_i, p_j, namely the open and the closed disc centered at the midpoint of $\overline{p_i p_j}$ with both p_i and p_j on their boundary, denoted by $\mathsf{O\text{-}DISC}(p_i,p_j)$ and $\mathsf{C\text{-}DISC}(p_i,p_j)$ respectively.

[1]It is standard in the computational geometry literature that a lens is incorrectly called a lune.

Definition 1.3. The *modified Gabriel graph*, denoted by $\mathsf{MGG}(S)$, is the graph in which p_i, p_j are adjacent if $\mathsf{O\text{-}DISC}(p_i, p_j) \cap S = \emptyset$. The *Gabriel graph*, denoted by $\mathsf{GG}(S)$, is the graph in which p_i, p_j are adjacent if $\mathsf{C\text{-}DISC}(p_i, p_j) \cap S = \{p_i, p_j\}$.

Definition 1.4. The *Delaunay graph*, denoted by $\mathsf{DG}(S)$, is the graph in which p_i, p_j are adjacent if there exists a closed disk containing p_i and p_j, and no other point from S.

If S does not contain three collinear or four concyclic points, $\mathsf{DG}(S)$ is a triangulation. In that case this graph is also denoted by $\mathsf{DT}(S)$.

The graphs defined so far are the order-0 graphs considered in Section 2. They satisfy some hierarchical relations. In particular, for every point set S, it holds that

$$\mathsf{U\text{-}MST}(S) \subseteq \mathsf{RNG}(S) \subseteq \mathsf{GG}(S) \subseteq \mathsf{DG}(S),$$
$$\mathsf{RCG}(S) \subseteq \mathsf{RNG}(S),$$
$$\mathsf{GG}(S) \subseteq \mathsf{MGG}(S).$$

(See [13, 27, 31, 32].)

We next define the higher order proximity graphs we study in Section 3.

Definition 1.5. The *k-nearest neighbor graph*, denoted by k-$\mathsf{NNG}(S)$, is the graph in which every point is connected with a directed segment to its k closest neighbors. The undirected graph consisting of the bidirectional edges of k-$\mathsf{NNG}(S)$ is called the *shared k-nearest neighbor graph*, k-$\mathsf{SNNG}(S)$.

Definition 1.6. The *k-relative neighborhood graph*, denoted by k-$\mathsf{RNG}(S)$, is the graph in which p_i, p_j are adjacent if $|\mathsf{O\text{-}LENS}(p_i, p_j) \cap S| \leq k$.

Definition 1.7. The *k-Gabriel graph*, denoted by k-$\mathsf{GG}(S)$, is the graph in which p_i, p_j are adjacent if $|\mathsf{C\text{-}DISC}(p_i, p_j) \cap S| \leq k + 2$.

Definition 1.8. The *k-Delaunay graph*, denoted by k-$\mathsf{DG}(S)$, is the graph in which p_i, p_j are adjacent if there exists a circle through p_i and p_j that contains at most k points from S in its interior.

It is well known that, for every point set S,

$$(k+1)\text{-}\mathsf{SNNG}(S) \subseteq (k+1)\text{-}\mathsf{NNG}(S) \subseteq k\text{-}\mathsf{RNG}(S) \subseteq k\text{-}\mathsf{GG}(S) \subseteq k\text{-}\mathsf{DG}(S)^2.$$

The results we review in this paper and the ones we prove are summarized in Tables 1 and 2. Subscripts containing a reference mean that the corresponding bound is proved in that reference. The asterisks indicate results that are well-known or trivial.

[2] Notice that $(k+1)$-$\mathsf{NNG}(S)$ is a directed graph, while the other graphs in the expression are undirected. When we write $(k+1)$-$\mathsf{SNNG}(S) \subseteq (k+1)$-$\mathsf{NNG}(S) \subseteq k$-$\mathsf{RNG}(S)$ we actually mean that the undirected graph resulting from suppressing the directions of the edges of $(k+1)$-$\mathsf{NNG}(S)$ satisfies these relations.

Table 1: Bounds on graph-theoretic properties for proximity graphs defined on a set of n points. No non-degeneracy assumptions are made.

	T ∈ MST	U-MST	RCG	RNG	GG	MGG	DG
$\min \lvert E\rvert$	$n-1_*$	$n-1$	$0_{[13]}$	$n-1$	$n-1$	$n-1$	$n-1_*$
$\max \lvert E\rvert \geq$	$n-1_*$	$3n-8$	$2n-6$	$3n-8$	$3n-8_{[27]}$	$4n-6\sqrt{n}+2$	$3n-6_*$
$\max \lvert E\rvert \leq$	$n-1_*$	$3n-8$	$2n-5_{[13]}$	$3n-8$	$3n-8_{[27]}$	$4n-\frac{2\sqrt{n}}{3}$	$3n-6_*$
$\max \delta$	1_*	5	$3_{[13]}$	5	$5_{[27]}$	$\in \{6,7\}$	$5_{[27]}$
$\max \Delta$	$6_{[29]}$	$n-1$	5	$n-1$	$n-1_{[27]}$	$n-1$	$n-1_{[27]}$
$\max \chi$	2_*	4	$3_{[13]}$	4	4	$\in \{4,\ldots,8\}$	4_*
$\max \omega$	2_*	3	2	$3_{[32]}$	$3_{[27]}$	4	4_*

Table 2: Bounds on properties for higher order proximity graphs. Several non-degeneracy assumptions are made. Some results only hold for specific ranges of k; see the complete statements throughout the paper.

	k-SNNG	k-NNG	k-RNG	k-GG	k-DG
$\min \lvert E\rvert \geq$	$\binom{k+1}{2}$	$\frac{kn}{2}_*$	$\frac{(k+1)n}{2}$	$\frac{(k+1)n}{2}$	$(k+1)n_{[1]}$
$\min \lvert E\rvert \leq$	$\binom{k+1}{2}$	$\frac{kn}{2}_*$	$kn+o(kn)$	$kn+o(kn)$	$\frac{3kn}{2}+o(kn)$
$\max \lvert E\rvert \geq$	$\frac{kn}{2}_*$	$kn-\binom{k+1}{2}$	$\frac{3\pi}{4\pi-3\sqrt{3}}nk$	$2kn+o(nk)$	$3kn+o(nk)$
$\max \lvert E\rvert \leq$	$\frac{kn}{2}_*$	$kn-\binom{k+1}{2}$	$\frac{5(k+1)n}{2}$	$3kn+o(nk)_{[1]}$	$3kn+o(nk)_{[1]}$
$\max \delta \geq$	k	k	$2k+2$	$3k+2$	$4k+3$
$\max \delta \leq$	k	k	$3k+3$	$6k+5$	$6k+5$
$\max \Delta$	k	$5k$	$5(k+1)$	$n-1_*$	$n-1_*$
$\max \chi \geq$	$k+1$	$k+1$	$k+2$	$3k+3_{[7]}$	$4k+4_{[7]}$
$\max \chi \leq$	$k+1$	$k+1$	$3k+4$	$6k+6_{[7]}$	$6k+6_{[7]}$
$\max \omega \geq$	$k+1$	$k+1$	$k+2$	$3k+3$	$4k+4$
$\max \omega \leq$	$k+1$	$k+1$	$k+2$	$3k+3$	$4.74k+14$

2 Order-0 graphs

In this section we look at the classical versions of proximity graphs, i.e., we consider the specific case $k=0$. These graphs were introduced earlier than their higher order counterparts, and thus much more is known about them. We review previous work on their graph-theoretic properties, and we also give some new bounds.

Throughout this section we do not make any non-degeneracy assumption. However, we make the effort whenever possible that our worst-case constructions are not degenerate.

2.1 Number of edges

We start by looking at the minimum number of edges of the graphs.

Let $T \in \mathsf{MST}(S)$. Since T is a tree, $|E(T)| = n-1$. Consequently, for every point set S, we have that $|E(\mathsf{U\text{-}MST}(S))| \geq n-1$, $|E(\mathsf{RNG}(S))| \geq n-1$, $|E(\mathsf{GG}(S))| \geq n-1$, $|E(\mathsf{MGG}(S))| \geq n-1$, and $|E(\mathsf{DG}(S))| \geq n-1$. In the next proposition we show that these bounds can be attained:

Proposition 2.1. *There exists a point set $\mathcal{S}$ such that* $|E(\mathsf{U\text{-}MST}(\mathcal{S}))| = |E(\mathsf{RNG}(\mathcal{S}))| = |E(\mathsf{GG}(\mathcal{S}))| = |E(\mathsf{MGG}(\mathcal{S}))| = |E(\mathsf{DG}(\mathcal{S}))| = n-1$.

Proof. Let $\mathcal{S}$ be a set of points lying on a line l. Then all these graphs are a path, namely the one connecting consecutive points in l. The configuration can even be perturbed so that no three points lie in a common line and the structure of all graphs except for $\mathsf{DG}(\mathcal{S})$ is maintained. In fact, since for any S the graph $\mathsf{DG}(S)$ contains the edges of the convex hull of S, $\mathsf{DG}(S)$ has at least $2n-3$ edges when this convex hull encloses a region with positive area. ⊠

As shown in [13], $\mathsf{RCG}(S)$ might be empty; this is the case when the points are placed in a triangular grid.

We next try to determine the maximum number of edges of these graphs. This question is more complicated, and in some cases the number of edges of our worst-case construction does not match our upper bound.

It is well known that the Delaunay graph is a plane graph. As a consequence, $|E(\mathsf{DG}(S))| \leq 3n-6$, and also $|E(\mathsf{U\text{-}MST}(S))| \leq 3n-6$, $|E(\mathsf{RNG}(S))| \leq 3n-6$, and $|E(\mathsf{GG}(S))| \leq 3n-6$. For some of the graphs this upper bound can be strengthened and for others it cannot.

The Delaunay graph of S is a triangulation provided that S is in general position (no three points are collinear and no four points are concyclic). Therefore, if S is in general position and the convex hull of S is a triangle, $\mathsf{DG}(S)$ contains exactly $3n-6$ edges. In contrast, Gabriel graphs cannot have so many edges: it has been shown (see [27]) that every Gabriel graph on n points has at most $3n-8$ edges, and that this bound is tight for an infinite number of values of n.

The bound for the Gabriel graph implies that $|E(\mathsf{U\text{-}MST}(S))| \leq 3n-8$. We next show that this can be attained:

Proposition 2.2. *There exist point sets $\mathcal{S}$ such that $|E(\mathsf{U\text{-}MST}(\mathcal{S}))| = 3n-8$, where $|\mathcal{S}| = n$.*

Proof. We first prove the claim for RNG and then we show that, for the particular point set $\mathcal{S}$ that we describe, $\mathsf{U\text{-}MST}(\mathcal{S}) = \mathsf{RNG}(\mathcal{S})$.

Let p_i, p_j, p_k be three points of a set S. If the three edges p_ip_j, p_ip_k, and p_jp_k belong to $\mathsf{RNG}(S)$, then these points form either an equilateral triangle or an isosceles triangles in which the unequal side is shorter. Therefore, if there exist point configurations $\mathcal{S}$ such that $|E(\mathsf{RNG}(\mathcal{S}))| = 3n-8$, then $\mathsf{RNG}(\mathcal{S})$ must look like a maximal plane graph (except for two edges) and all its interior triangles must be either equilateral or isosceles in which the unequal side is shorter. We have been able to prove that such configurations, quite twisted, exist for some values of n. We next describe an example with fourteen points.

We start with an equilateral triangle $\triangle q_2q_3q_4$. Then we place two points q_5 and q_6 that are symmetric with respect to the perpendicular bisector of q_2 and q_3, and such that $|q_2q_4| = |q_2q_6|$ and $|q_3q_4| = |q_3q_5|$ (see Figure 1). We next place a point q_1 on the perpendicular bisector of q_5 and q_6, and such that q_1 is on the convex hull of the point set. In order for the edges in Figure 1 to be in $\mathsf{RNG}(\mathcal{S})$, the remaining points must lie in the region delimited by the circular arcs in dotted lines q_4q_6, q_6q_5, and q_5q_4. Furthermore, they must be added so that the point set can be triangulated in such a way that all triangles are either equilateral or isosceles in which the unequal side is shorter. A possible way to do this is illustrated in Figure 2.

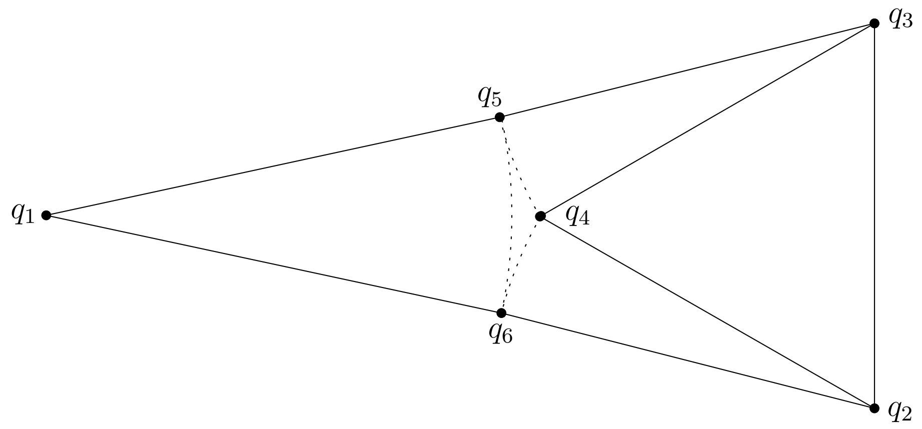

Figure 1: First steps of the construction of the point set of Proposition 2.2.

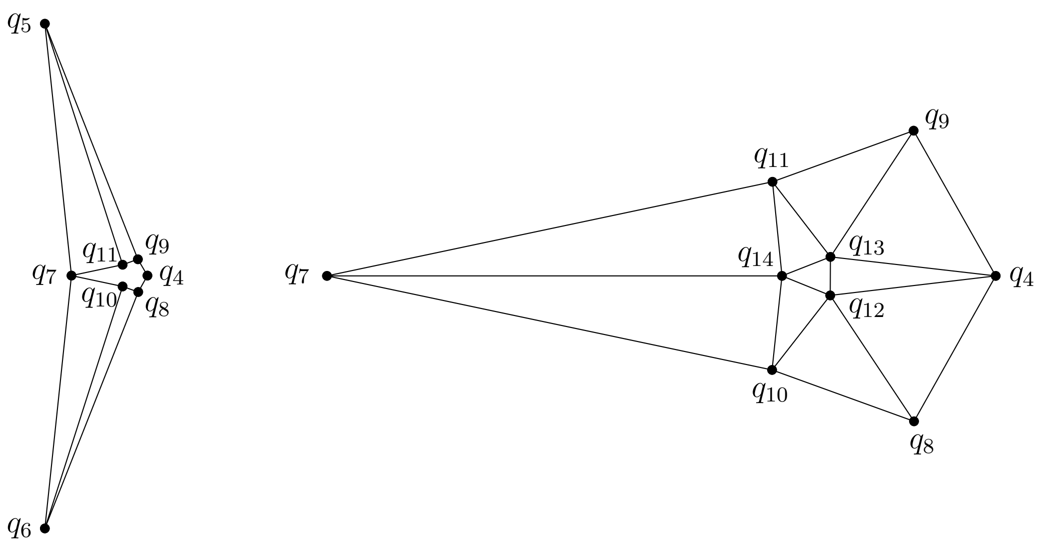

Figure 2: The region delimited by q_4, q_5, and q_6 in Figure 1 is filled with points so that we can triangulate it with isosceles triangles in which the unequal side is shorter. In the left figure, some points have been suppressed for the sake of clarity; they are shown in the right figure. Apart from the edges in the figure, RNG of the resulting point set also contains q_1q_7, q_2q_8, and q_3q_9.

It remains to argue that, for this set $\mathcal{S}$, $\mathsf{U\text{-}MST}(\mathcal{S}) = \mathsf{RNG}(\mathcal{S})$. Indeed let us suppose that we use Kruskal's algorithm to compute a particular minimum spanning tree T of $\mathcal{S}$. First we would add the edge $q_{12}q_{13}$ to T, and then either $q_{14}q_{12}$ or $q_{14}q_{13}$, so both edges belong to $\mathsf{U\text{-}MST}(\mathcal{S})$.

Next we would connect q_{10} and q_{11} to T using $q_{10}q_{12}$ or $q_{10}q_{14}$, and $q_{11}q_{13}$ or $q_{11}q_{14}$, respectively. Using similar arguments we conclude that $\mathsf{U\text{-}MST}(\mathcal{S}) = \mathsf{RNG}(\mathcal{S})$. ⊠

By adding points in the interior of the inner-most triangle of the previous example, we have constructed point sets $\mathcal{S}$ of larger size such that $|E(\mathsf{U\text{-}MST}(\mathcal{S}))| = |E(\mathsf{RNG}(\mathcal{S}))| = 3n - 8$. However, we do not have a proof that point sets with this property can be effectively constructed for arbitrarily big values of n. Using a different construction, we can prove that there exist point sets $\mathcal{S}$ of arbitrarily large size such that $|\mathcal{S}| = n$ and $|E(\mathsf{U\text{-}MST}(\mathcal{S}))| = |E(\mathsf{RNG}(\mathcal{S}))| = 3n - 9$.

Let us next look at the relative neighborhood graph. The following theorem was formulated in [32]:

Theorem 2.3 (Urquhart 1983)**.** *A relative neighborhood graph on n vertices has at most*

(i) $3n - 8$ *edges for* $n \geq 5$,
(ii) $3n - 9$ *edges for* $n \geq 6$,
(iii) *exactly* $3n - 10$ *edges for* $n = 7m + 1$ *for any* $m \geq 1$, *and*
(iv) *either* $3n - 10$ *or* $3n - 11$ *edges for all* $n \geq 8$.

The claims (ii), (iii), and (iv) are clearly not true, as shown by the example in Figures 1 and 2. In fact, the author does not attempt to prove (iii) and (iv), but gives examples of graphs achieving these bounds. As for (ii), the proof is incorrect, because it is partially based on Lemma 4.2 of [32], which is not true. This lemma states that the wheel graph W_n (the graph formed by connecting a single vertex to all vertices of an $(n-1)$-cycle) may be a relative neighborhood graph if and only if $n \geq 7$. However, the example in Figure 3 shows that W_6 may indeed be a relative neighborhood graph.

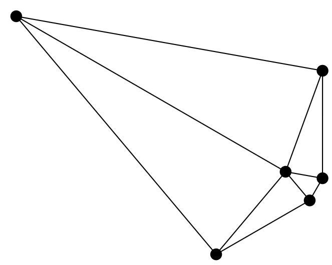

Figure 3: W_6 may be a relative neighborhood graph.

The above upper bounds on the number of edges of $\mathsf{RNG}(S)$ and the result on the drawability of RNG as a wheel graph have been cited several times in the literature (see the survey [23], and also [8, 24, 26]). We take this opportunity to stress that, after the previous considerations, the best upper bound on the maximum number of edges of $\mathsf{RNG}(S)$ is $3n - 8$ (consequence of $|E(\mathsf{GG}(S))| \leq 3n - 8$), and this can be attained, as shown in Proposition 2.2.

The question of determining the maximum number of edges of the relatively closest graph has been considered in [13], where the following lemma is proved:

Lemma 2.4 (Cimikowski 1992)**.** *Every relatively closest graph* $\mathsf{RCG}(S)$ *is a triangle-free plane graph.*

This lemma is used to prove that $|E(\mathsf{RCG}(S))| \leq 2n-5$ (see [13]). Next we give an example that almost achieves this bound. For all $n \equiv 0 \pmod 7$, consider a set of regular heptagons with the same center and orientation. Choose the size of the heptagons so that their edges are in the relatively closest graph of their vertices. Then this graph has $2n-7$ edges (see Figure 4, left). The example can be further improved by slightly modifying the inner-most heptagon so that one of the chords of the heptagon belongs to RCG (see Figure 4, right). The relatively closest graph of the new point set contains $2n-6$ edges.

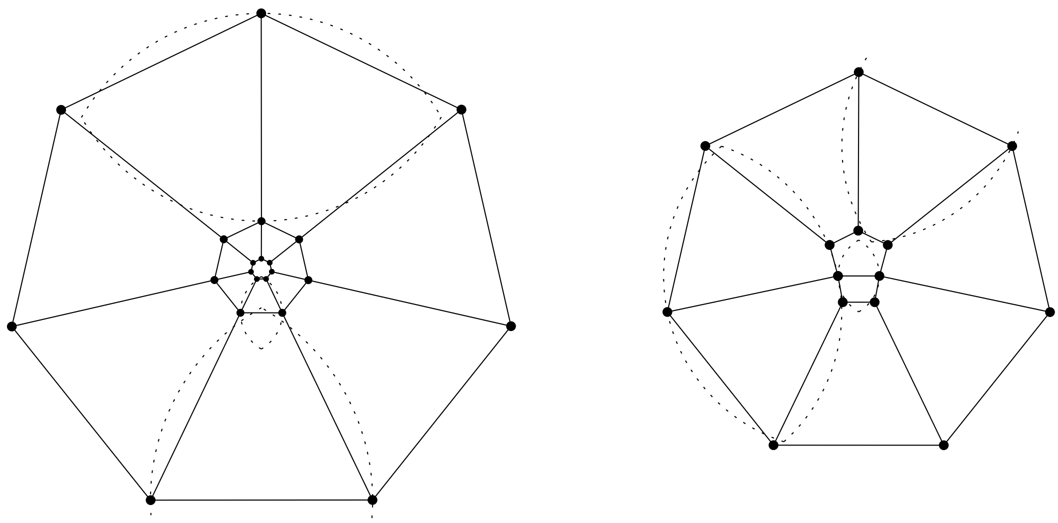

Figure 4: Left: a RCG with $2n-7$ edges. Right: modification of the inner-most heptagon to add an extra edge to RCG.

Let us finally focus on the modified Gabriel graph. This graph was studied in [13] under the assumption that no four points of S are concyclic. Since we do not make this requirement, our results are significantly different.

Lemma 2.5. *Let $H=(S,E)$ be a plane geometric graph formed by 4-cycles $p_ip_jp_lp_mp_i \in F_4$ such that $p_i,p_j,p_l,p_m \in S$ are the vertices of a closed rectangle which is empty of points from S, except for p_i,p_j,p_l,p_m. Let $|F_4|$ be the number of such 4-cycles. We say that an edge $e \in E$ is* red *if it belongs to exactly one of these 4-cycles. The number of red edges of H is at least $4\sqrt{|F_4|}$.*

Proof. We partition the 4-cycles of H into groups as follows. Let c_i, c_j be two 4-cycles of H. We say that c_i and c_j belong to the same *group* if there exists a sequence of cycles $c_i, c_{i+1}, \ldots, c_{j-1}, c_j$ such that any pair of consecutive cycles of the sequence shares one edge (see Figure 5, left for an example of a group of 4-cycles). Suppose that the 4-cycles of H are subdivided into l groups $F_4^1, F_4^2, \ldots, F_4^l$, and that, for each group F_4^i, we can prove that the number of red edges of the group is at least $4\sqrt{|F_4^i|}$. By definition of the groups, any red edge of some group is also a red edge of H. Therefore, the number of red edges of H is at least $4\sqrt{|F_4^1|}+4\sqrt{|F_4^2|}+\cdots+4\sqrt{|F_4^l|}$, which is greater than $4\sqrt{|F_4^1|+|F_4^2|+\cdots+|F_4^l|}=4\sqrt{|F_4|}$. Thus it suffices to prove the result for one group of 4-cycles.

Let F_4^1 be a group of 4-cycles. Observe that all the rectangles associated to the cycles

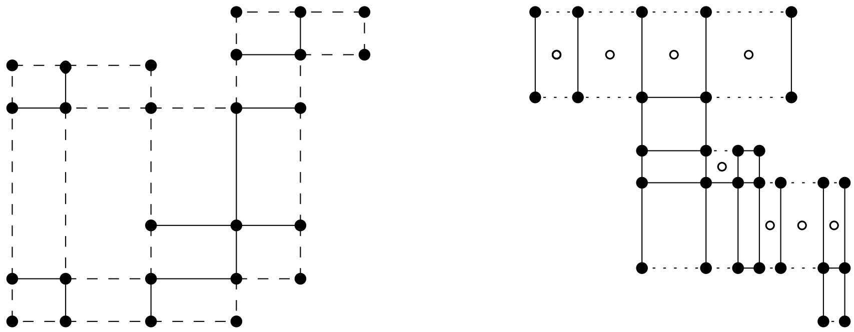

Figure 5: Left: a group of 4-cycles; in dashed lines, its red edges. Right: in dotted lines, the horizontal red edges associated to each value $x' \in x_m$ (the white points correspond to the centers of rectangles R_i such that $x' = x(m_i)$).

of F_4^1 have the same orientation. Without loss of generality, we assume that they are axis-aligned. For each 4-cycle $c_i \in F_4^1$, let R_i be the rectangle associated to c_i, and m_i be the center of R_i. We denote by $x(m_i)$ and $y(m_i)$ respectively the x and y coordinates of m_i. We define $x_m = \{x(m_i)\}_{c_i \in F_4^1}$ and $y_m = \{y(m_i)\}_{c_i \in F_4^1}$.

Let $x' \in x_m$. Then $x' = x(m_i)$, for some m_i center of a rectangle R_i associated to a cycle $c_i \in F_4^1$. If the top horizontal edge of this rectangle is red, we associate it to x'. Otherwise, there exists a rectangle R_j associated to a 4-cycle $c_j \in F_4^1$ whose bottom horizontal edge coincides with the top horizontal edge of R_i, and thus $x(m_j) = x(m_i)$. If the top horizontal edge of R_j is red, we associate it to x'. Otherwise we continue moving up and visiting rectangles such that the x coordinate of their center equals x' until we find an horizontal red edge, which we associate to x'. We repeat the same search starting from the bottom horizontal edge of R_i, and visiting rectangles downwards. We find another horizontal red edge that is associated to x'. By repeating this strategy for all values $x' \in x_m$, we conclude that the number of horizontal red edges of F_4^1 is greater than or equal to $2|x_m|$. (See Figure 5, right for an example.)

Now let $y' = y(m_k)$, for some m_k center of a rectangle R_k. If the vertical edges of R_k are red, we associate them to y'. Otherwise, we visit rectangles sharing vertical edges with R_k until we find two vertical red edges. This shows that the number of vertical red edges of F_4^1 is greater than or equal to $2|y_m|$. In total, the number of red edges of F_4^1 is at least $2(|x_m| + |y_m|)$.

Let us next bound $|F_4^1|$. There are $|x_m|$ possible values for the x coordinate of the center of some fixed R_i (associated to $c_i \in F_4^1$), and $|y_m|$ possible values for its y coordinate. Two distinct such rectangles have different centers, because they are empty by definition. Consequently, $|F_4^1| \leq |x_m||y_m|$. Since the arithmetic mean of two positive values is greater than or equal to their geometric mean, we have

$$\text{number of red edges of } F_4^1 \geq 2(|x_m| + |y_m|) \geq 4\sqrt{|x_m||y_m|} \geq 4\sqrt{|F_4^1|}.$$

⊠

Theorem 2.6. *Every modified Gabriel graph on n vertices has at most $4n - \frac{2\sqrt{n}}{3}$ edges.*

Proof. Let $G = \mathsf{MGG}(S)$. Suppose that p_ip_j is an edge of G that crosses some other edge p_lp_m. Since $\mathsf{O\text{-}DISC}(p_i, p_j) \cap S = \emptyset$, we have that $\widehat{p_ip_lp_j} \leq \pi/2$ and $\widehat{p_ip_mp_j} \leq \pi/2$. Analogously, $\widehat{p_lp_ip_m} \leq \pi/2$ and $\widehat{p_lp_jp_m} \leq \pi/2$. Since the sum of these four angles is π (p_i, p_l, p_j, and p_m form a quadrilateral), all these inequalities are actually equalities, and $\mathsf{O\text{-}DISC}(p_i, p_j) = \mathsf{O\text{-}DISC}(p_l, p_m)$. In particular, if $e_1, e_2, e_3 \in E(\mathsf{MGG}(S))$ and e_1 crosses e_2 and e_3, then e_2 and e_3 also cross. If $e_1, e_2, \ldots, e_k$ are pairwise crossing edges then there exists a circle c such that the endpoints of each e_i are at antipodal points on c, and each pair of edges e_i and e_j intersect at the center of c. Furthermore, there are no points of S in the interior of c. Add edges between points from the set of endpoints of $e_1, e_2, \ldots, e_k$ that are consecutive in c. Notice that these edges do not create crossings with the edges in G. We refer to the interior of the resulting $2k$-gon as a *crossing region.* If a crossing region is bounded by a cycle of 6 or more vertices, remove all the crossing edges and add chords of the cycle so that the region inside the cycle is triangulated (after this modification, the region is no longer called a *crossing region*). Finally, add edges to the region inside $CH(S)$ disjoint from the crossing regions so that each interior face is a triangle. Let G' be the resulting graph. By construction,

$$|E(G')| \geq |E(G)|.$$

The graph G' subdivides the interior of $CH(S)$ into crossing regions and regions bounded by triangles, and the boundaries of the crossing regions are given by rectangles. We denote by $|F_4|$ the number of such rectangles, and by $|F_3|$ the number of triangles. Let G'_{pl} be a graph obtained from G' by deleting one edge in each rectangle. Clearly, G'_{pl} is a triangulation. Therefore,

$$|E(G'_{pl})| = 3n - h - 3,$$
$$|E(G')| = 3n - h - 3 + |F_4|,$$

where h is the size of the convex hull of S. By Euler's formula,

$$|F_3| + 2|F_4| + n = |E(G'_{pl})| + 1.$$

Combining these equations, we obtain that

$$|F_3| = 2n - 2|F_4| - h - 2.$$

Now suppose, for the sake of contradiction, that $|E(G)| \geq 4n - \frac{2\sqrt{n}}{3}$. Then $|F_4| = |E(G')| - 3n + h + 3 \geq |E(G)| - 3n + h + 3 \geq n - \frac{2\sqrt{n}}{3} + h + 3 > n - \frac{2\sqrt{n}}{3}$. Consider the subgraph G'' of G' containing only the edges of the rectangles bounding the crossing regions. This graph satisfies the hypothesis of Lemma 2.5. Consequently, the number of edges of G'' that are edges of exactly one rectangle of this graph (which are called red edges) is at least $4\sqrt{|F_4|}$. Every red edge belongs either to the boundary of a triangle of G' or to the boundary of the convex hull of S. Thus the number of red edges is at most $3|F_3| + h$. Taking $4\sqrt{|F_4|} \leq 3|F_3| + h$, and substituting $|F_3| = 2n - 2|F_4| - h - 2$, we obtain $4\sqrt{|F_4|} \leq 6n - 6|F_4| - 2h - 6$. Since $|F_4| > n - \frac{2\sqrt{n}}{3}$, we get

$$0 < 4\sqrt{n} - 4\sqrt{n - \frac{2\sqrt{n}}{3}} - 2h - 6.$$

For all $n \geq 1$, we have that $4\sqrt{n} - 4\sqrt{n - \frac{2\sqrt{n}}{3}} \leq 2$. Then

$$0 < 4\sqrt{n} - 4\sqrt{n - \frac{2\sqrt{n}}{3}} - 2h - 6 \leq 2 - 2h - 6 < 0,$$

which is a contradiction. ⊠

We believe that this upper bound is not tight, and that it can be improved to $|E(\mathsf{MGG}(S))| \leq 4n - 6\sqrt{n} + 2$. This value is attained when S is a square grid of size $\sqrt{n} \times \sqrt{n}$, and n is a square number.

2.2 Minimum and maximum degree

It is obvious that every spanning tree T satisfies $\delta(T) = 1$. In the union of the minimum spanning trees the situation might be significantly different. Since every planar graph has minimum degree at most five, $\delta(\mathsf{U\text{-}MST}(S)) \leq 5$. We now prove that this result is best possible.

Proposition 2.7. *There exist arbitrarily large point sets $\mathcal{S}$ such that $\delta(\mathsf{U\text{-}MST}(\mathcal{S})) = 5$.*

Proof. We first produce a point set $\mathcal{S}'$ for which $\mathsf{U\text{-}MST}(\mathcal{S}')$ has three vertices of degree 3 and nine vertices of degree 5. The construction is illustrated in Figure 6.

We start with a circle C_3 around the origin, and six half-lines $l_1, l_2, \ldots, l_6$ with initial point at the origin, such that the angle between l_i and l_{i+1} is $\pi/3$. We place three points q_4^2, q_4^4, q_4^6 on the intersections of C_3 with l_2, l_4, l_6, respectively. We draw another circle C_2 around the origin. The circle should fit into the triangle $\triangle q_4^2 q_4^4 q_4^6$ and be at distance ε from it, where ε is a sufficiently small number. We place three points q_3^1, q_3^3, q_3^5 on the intersections of C_2 with l_1, l_3, l_5, respectively. Then we place three points q_2^2, q_2^4, q_2^6 on l_2, l_4, l_6, respectively, such that $|q_4^i q_2^i| = |q_4^i q_3^{i-1}|$, for $i = 2, 4, 6$. Finally, we place three points q_1^1, q_1^3, q_1^5 on l_1, l_3, l_5, respectively, such that $|q_3^i q_1^i| = |q_3^i q_2^{i+1}|$, for $i = 1, 3, 5$.

The union of the minimum spanning trees of $\mathcal{S}'$ is shown in Figure 6. We call $q_3^1, q_4^2, q_3^3, q_4^4$ q_3^5, q_4^6 the *outer* points of $\mathcal{S}'$; q_4^2, q_4^4, q_4^6 are also called *extremal* points.

If ε is zero, q_4^2, q_3^1, q_4^6 are collinear. By making ε sufficiently small, the radius r of the circle through the three outer points q_4^2, q_3^1, q_4^6 can be made arbitrarily large. We place four copies of $\mathcal{S}'$ on the corners of a large square as follows. We place one extremal point of each $\mathcal{S}'$ on a corner of the square, and the remaining points of each $\mathcal{S}'$ inside the square. The half-line l_6 of each copy of $\mathcal{S}'$ passes through the center of the square as shown in Figure 7. Consider two copies of $\mathcal{S}'$, $\mathcal{S}'_1$ and $\mathcal{S}'_2$ say, along a side of the square. Take a set of three outer points from $\mathcal{S}'_1$ and three outer points from $\mathcal{S}'_2$ that lie more or less along the same side of the square. We modify the length of the sides of the square so that the two circles of radius r defined by each one of these two triples of outer points have the same center. After this movement, there are four such centers outside the square. We place points $q_5^1, q_5^2, q_5^3, q_5^4$ on these centers. By making r large enough, if ρ is the cone with apex at q_5^1 and minimum angle that contains the edges between q_5^1 and its six closest outer points, then the amplitude of ρ can come arbitrarily close to $\pi/6$. This implies that the distance between q_5^1 and its 6 closest outer points is larger than the length

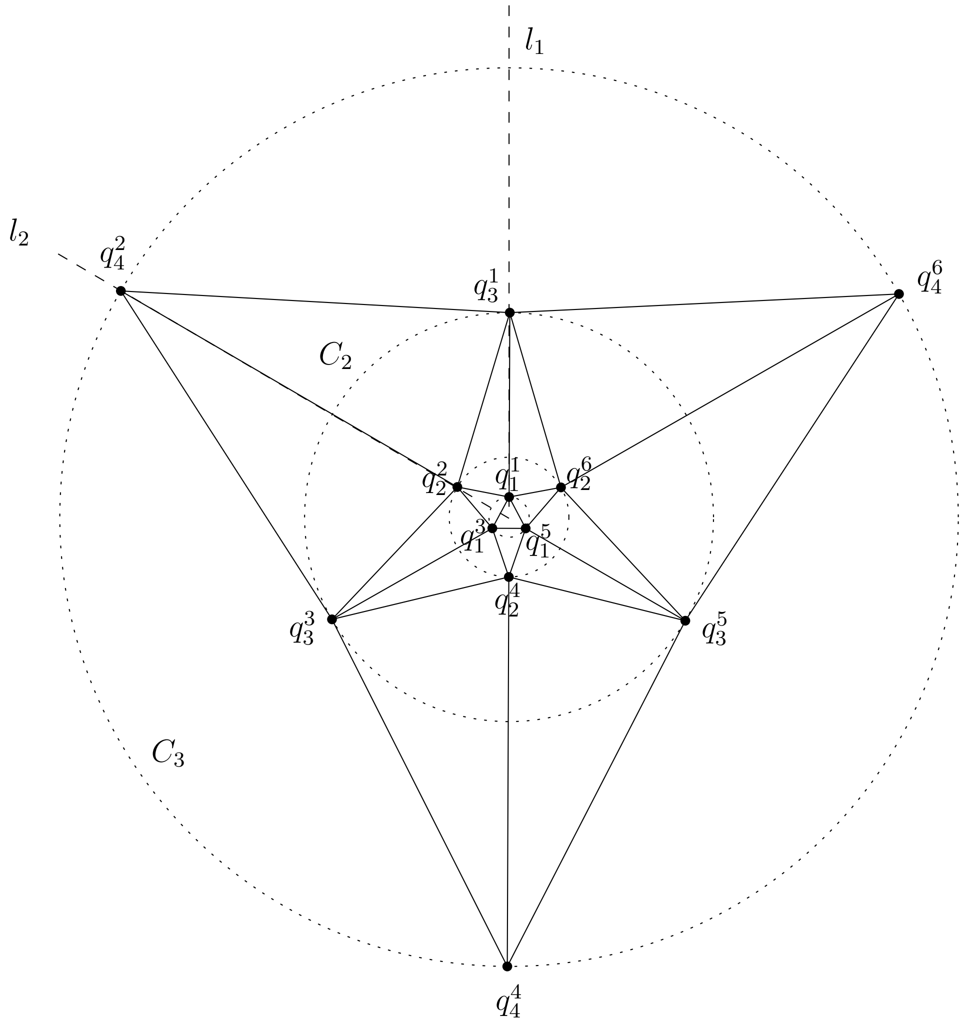

Figure 6: A point set $\mathcal{S}'$ for which $\mathsf{U\text{-}MST}(\mathcal{S}')$ has three vertices of degree 3 and nine vertices of degree 5.

of a side of the square. Thus when constructing minimum spanning trees, we first add an edge inside the square between two extremal points of different copies of $\mathcal{S}'$. Since there are four such edges, all of the same length, all four are in the union of the minimum spanning trees. After that we add edges connecting q_5^i to outer points of copies of $\mathcal{S}'$, for $i = 1, 2, 3, 4$. The result is a point set $\mathcal{S}$ consisting of 52 points such that $\mathsf{U\text{-}MST}(\mathcal{S})$ has minimum degree 5.

We point out that we can obtain arbitrarily large point sets such that the minimum degree of the union of the minimum spanning trees is 5 by placing along a line copies of the previous point set and making the distance between two consecutive copies large enough. ⊠

Notice that the previous example also shows that the bounds $\delta(\mathsf{RNG}(S)) \leq 5$, $\delta(\mathsf{GG}(S)) \leq 5$, and $\delta(\mathsf{DG}(S)) \leq 5$ are tight. This was already known for the case of the Gabriel and Delaunay graphs (see [27]). As for the relative neighborhood graph, the fact that there exist examples where all vertices have degree five or greater disproves a conjecture by Cimikowski (see [13]) and settles one of the open problems in [13].

In the same paper it is shown that the minimum degree of any relatively closest graph is

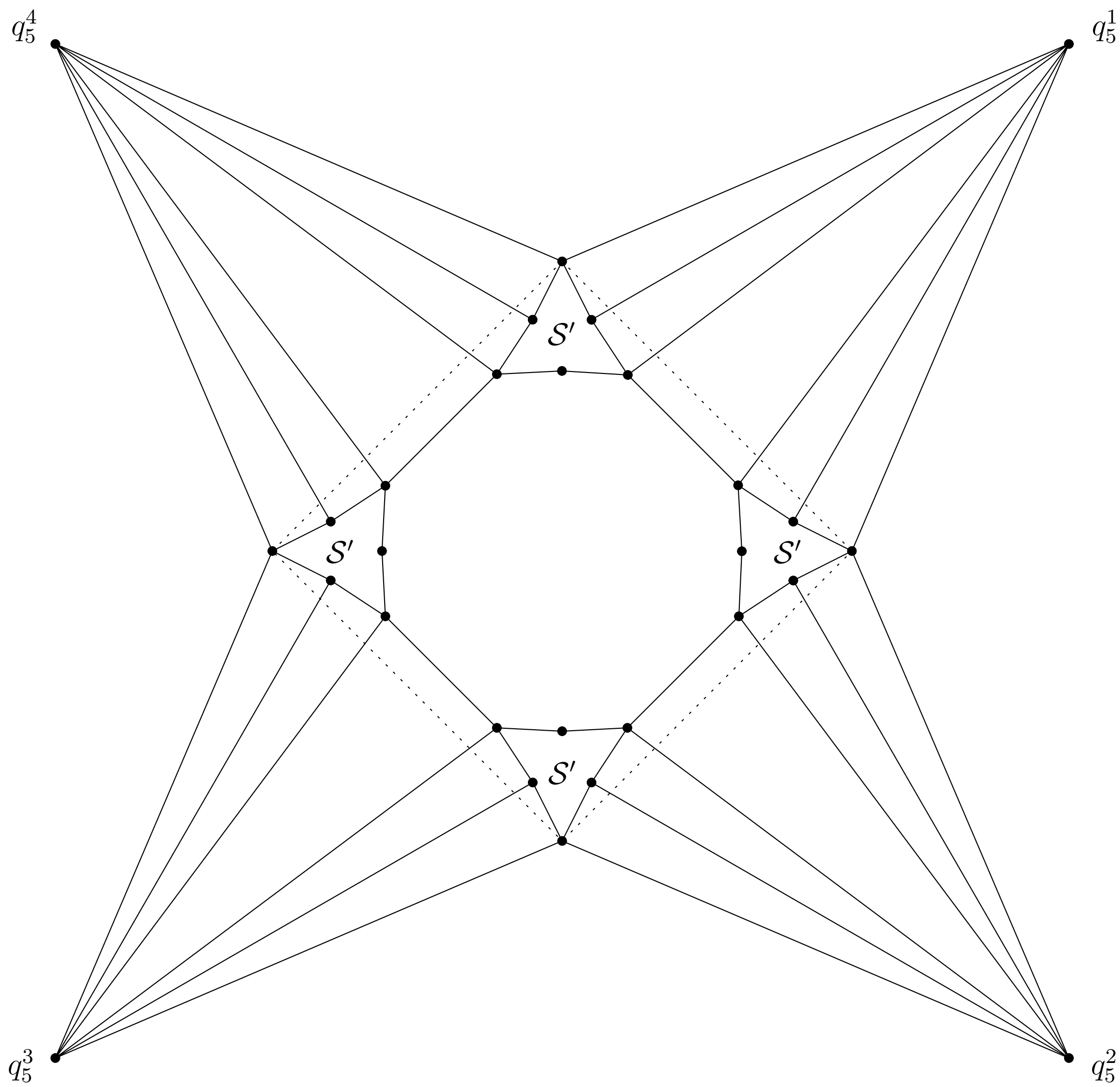

Figure 7: A point set $\mathcal{S}$ for which all vertices of $\mathsf{U\text{-}MST}(\mathcal{S})$ have degree at least 5. For the sake of clarity, points in the copies of $\mathcal{S}'$ that are not extremal or outer have been suppressed from the figure.

not greater than 3. The example in Figure 4 illustrates that this result can not be strengthened because all vertices have degree 3 or 4.

We finally consider the minimum degree of the modified Gabriel graph. Since the number of edges of this graph is strictly smaller than $4n$, by the handshaking lemma, $\delta(\mathsf{MGG}(S)) \leq 7$. Our best example is the following:

Proposition 2.8. *There exist arbitrarily large point sets $\mathcal{S}$ such that $\delta(\mathsf{MGG}(\mathcal{S})) = 6$.*

Proof. The general structure of the point set $\mathcal{S}$ is shown in Figure 8. In the figure, all points have degree 6 or greater, except for the ones on the boundary of eight empty dodecagons that look almost like regular hexagons. In order to increase the degree of these vertices, we add 42 points in the interior of each dodecagon as in Figure 9. ⊠

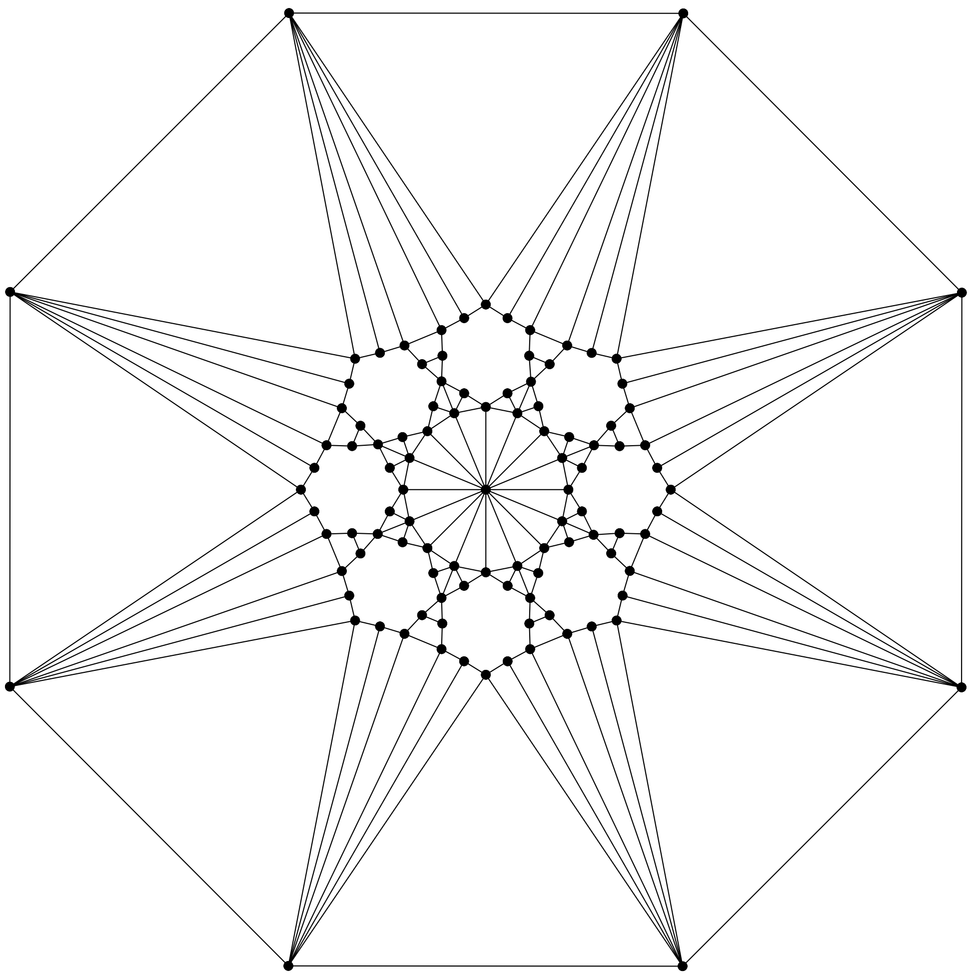

Figure 8: A point set $\mathcal{S}$ for which all vertices of $\mathsf{MGG}(\mathcal{S})$ have degree at least 6. Points in the interior of the empty dodecagons have been suppressed from the figure for the sake of clarity; they can be seen in Figure 9.

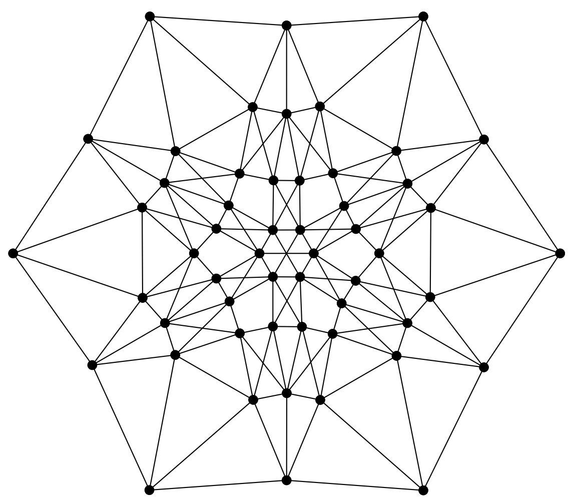

Figure 9: The interior of each empty dodecagon in Figure 8 is filled with extra points in this way.

Some of the constructions we have seen in this section are complicated because all vertices in the graphs needed to have some fixed degree. In contrast, when considering the maximum degree the situation is simpler because only one vertex needs to have high degree.

It is well known (see, for example, [29]) that every minimum spanning tree T has maximum degree $\Delta(T) \leq 6$ and this bound is tight.

We next show that $\mathsf{U\text{-}MST}$, RNG, GG, MGG, and DG might contain a vertex of maximum degree, i.e., a vertex of degree $n-1$. Indeed an easy example of this fact, already given in [27] for the case of GG, consists of placing $n-1$ points of $\mathcal{S}$ on a circle centered at $p \in \mathcal{S}$. In these graphs p is adjacent to all the other vertices of $\mathcal{S}$. Let us point out that, in the case of GG, MGG, and DG, we can slightly modify this configuration to obtain a vertex p of degree $n-1$ in a more general position of the points. More precisely, the $n-1$ points that are adjacent to p can be positioned at increasing distances from p as follows: after placing the first point q_1, we place a new point q_2 very close to q_1 so that $|pq_2| > |pq_1|$, q_2 it to the right of $\overrightarrow{pq_1}$, q_2 is outside the circle with diameter $\overline{pq_1}$, and q_2 is on the same side as p of the perpendicular line to $\overline{pq_1}$ passing through q_1. We repeat the same procedure to place $q_3, q_4 \ldots$

It only remains to consider the maximum degree of the relatively closest graph. In [13] it is proved that the only complete bipartite graphs that may be relatively closest graphs are $K_{2,2}$ and $K_{1,n}$ for $1 \leq n \leq 5$. Due to Lemma 2.4, a vertex of degree six in a relatively closest graph would give rise to a $K_{1,6}$. Thus $\Delta(\mathsf{RCG}(P)) \leq 5$. This bound is attained by the relatively closest graph of a regular pentagon and its center.

2.3 Chromatic and clique numbers

We start by giving bounds for the chromatic number. This is trivial for the minimum spanning tree, because every spanning tree T satisfies $\chi(T) = 2$.

As $\mathsf{U\text{-}MST}(S)$, $\mathsf{RNG}(S)$, $\mathsf{GG}(S)$, and $\mathsf{DG}(S)$ are plane graphs, by the 4-color theorem [4, 5], the chromatic number of these graphs is at most 4. Next we present an example of a 4-chromatic $\mathsf{U\text{-}MST}$. Let $q_1, q_2, \ldots, q_l$ be the vertices of a regular l-gon where $l > 9$ is odd. For all the even indices i, let q'_i be the point that is symmetric to q_i with respect to the segment $\overline{q_{i-1}q_{i+1}}$. Let $\mathcal{S}$ consist of all these points. The union of the minimum spanning trees of $\mathcal{S}$ is shown in Figure 10. Let us try to color $\mathsf{U\text{-}MST}(\mathcal{S})$ with three colors: if, for example, we assign color 1 to q_1, then q_2 is colored 2 and q'_2 is colored 3 or viceversa, q_3 is colored 1... and we continue coloring this way until we reach q_l. Since this vertex has three neighbors each of which with a different color, we need a fourth color to complete the coloring. Notice that this example also shows that there exist 4-chromatic relative neighborhood graphs, Gabriel graphs, and Delaunay graphs. Furthermore, it can be modified in order to avoid more than three concyclic points.

Let us next look at the relatively closest graph. Since every triangle-free planar graph is 3-colorable [21], Lemma 2.4 implies that every relatively closest graph is 3-colorable (as already observed in [13]). To obtain a 3-chromatic relatively closest graph, consider the set $\mathcal{S}$ of vertices of a regular n-gon, where n is an odd number greater than 3. In $\mathsf{RCG}(\mathcal{S})$ each pair of consecutive vertices are adjacent, so we need three colors to color this graph. As above, this configuration can be perturbed so that no four points are concyclic.

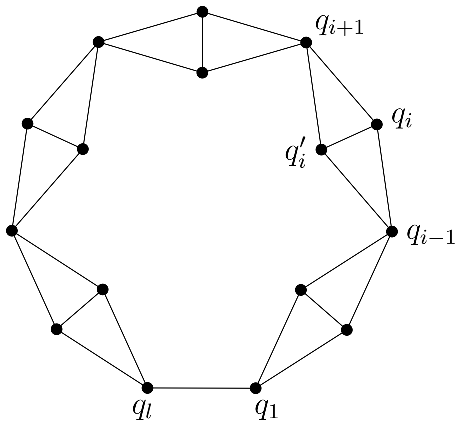

Figure 10: A 4-chromatic U-MST.

Finally, we consider the chromatic number of the modified Gabriel graph. We have seen that these graphs always contain a vertex of degree at most 7. Observe that, if p_ip_j is an edge of $\mathsf{MGG}(S)$, this edge is also present in $\mathsf{MGG}(S \setminus \{p_l\})$ for any $p_l \in S$ ($p_l \neq p_i, p_j$). Thus, if $\mathsf{MGG}(S) \setminus S'$ is an induced subgraph of $\mathsf{MGG}(S)$ on n' vertices, then it is a subgraph of $\mathsf{MGG}(S \setminus S')$ and it contains a vertex of degree 7 or less. Therefore we can color $\mathsf{MGG}(S)$ with 8 colors using the minimum degree greedy algorithm [16].

Unfortunately, in this case our lower and upper bounds leave a not insignificant gap, as we have not been able to find a modified Gabriel graph having chromatic number larger than the clique of size 4's (see below).

To end this section, let us study the cliques of maximum size in these graphs.

Clearly, $\omega(T) = 2$ for every spanning tree T. In [32] and [27] respectively it is proved that relative neighborhood graphs and Gabriel graphs have no 4-cliques. This also implies that $\omega(\mathsf{U\text{-}MST}(S)) \leq 3$. These bounds are tight (see examples above). As for the relatively closest graph, since it is triangle-free, the maximum number of vertices in a clique is 2, and this bound is best possible. The modified Gabriel graph and the Delaunay graph might contain cliques of larger size, as we will immediately see.

Proposition 2.9. *For every point set S, $\omega(\mathsf{MGG}(S)) \leq 4$. This bound is tight.*

Proof. Firstly we show that in any modified Gabriel graph the 4-cliques are not plane. Suppose, for the sake of contradiction, that the graph contains a 4-clique whose four vertices form a triangle $\triangle p_ip_jp_k$ with an interior point p_l. Then p_l sees one of the edges of the triangle, for example p_ip_j, with an angle greater than $\pi/2$. This contradicts the fact that p_ip_j is an edge of the graph. Consequently, the four vertices of the 4-clique are in convex position and form a crossing. As seen in the proof of Theorem 2.6, in this situation the four vertices are concyclic and form an empty rectangle.

Therefore, if a modified Gabriel graph had a clique of size greater than 4, all the vertices of the clique would be concyclic and each group of four would form an empty rectangle. But, given three vertices of a rectangle, the fourth vertex is forced. This contradiction yields the the first statement of the proposition.

To see that this bound is best possible, notice that the quadrangular grid has many 4-cliques. ☒

Finally, it is well known that Delaunay graphs do not contain cliques of size 5 because they are plane. On the other hand, the simplest example of a triangle with an interior point shows that they might contain cliques of size 4.

3 Order-k graphs

In the second part of this paper we consider higher order proximity graphs. These graphs have not received as much attention as their order-0 counterparts and, in particular, some graphs from the previous section have not even been generalized to an order-k version. Here we only study k-SNNG, k-NNG, k-RNG, k-GG, and k-DG, which are the higher order proximity graphs from our family that have been contemplated before in the literature. Moreover, they satisfy

$$(k+1)\text{-}\mathsf{SNNG}(S) \subseteq (k+1)\text{-}\mathsf{NNG}(S) \subseteq k\text{-}\mathsf{RNG}(S) \subseteq k\text{-}\mathsf{GG}(S) \subseteq k\text{-}\mathsf{DG}(S)^3,$$

which makes it easier to compare them.

In order to make the analysis simpler, and also because otherwise some graphs are not well-defined, throughout the section we make some non-degeneracy assumptions. We assume that point sets S are in *general position* in an extended sense: no three points are collinear, no four points are concyclic and, for each $p \in S$, the set of its k nearest points in S is well-defined, i.e., the kth nearest neighbor of p is unique, for any $k \geq 1$. We will denote by *k-distance* of p, $k - \text{dist}(p)$, the distance to the k-nearest neighbor of p.

3.1 Number of edges

Recall that k-NNG(S) is a directed graph. When counting its number of edges, though, bidirectional edges are counted once. The *in-degree* of a vertex p is the number of edges pointing to p, while its *out-degree* is the number of edges emanating from p. The *degree* of p in k-NNG(S) is defined as the sum of its in-degree and its out-degree, minus the number of bidirectional edges incident to p.

Proposition 3.1. *For every point set S,*

$$\frac{kn}{2} \leq |E(k\text{-}\mathsf{NNG}(S))| \leq kn - \binom{k+1}{2},$$
$$\binom{k+1}{2} \leq |E(k\text{-}\mathsf{SNNG}(S))| \leq \frac{kn}{2}.$$

These bounds are tight.

Proof. It is clear that $\frac{kn}{2} \leq |E(k\text{-}\mathsf{NNG}(S))| \leq kn$ and $0 \leq |E(k\text{-}\mathsf{SNNG}(S))| \leq \frac{kn}{2}$ because each vertex in k-NNG(S) has out-degree k. Two of these bounds can be improved to $|E(k\text{-}\mathsf{NNG}(S))| \leq$

[3] As indicated before, by $(k+1)$-SNNG$(S) \subseteq (k+1)$-NNG$(S) \subseteq k$-RNG(S) we mean that the undirected graph resulting from suppressing the directions of the edges of $(k+1)$-NNG(S) satisfies these relations.

$kn - \binom{k+1}{2}$ and $|E(k\text{-}\mathsf{SNNG}(S))| \geq \binom{k+1}{2}$ by showing that every k-nearest neighbor graph contains at least $\binom{k+1}{2}$ bidirectional edges.

Let $p_1, p_2, \ldots, p_n$ be the points in S sorted by increasing k-distance (choose any order if there are ties). Consider a vertex p_i, with $i \leq k$, and an edge of k-$\mathsf{NNG}(S)$ of the form $\overrightarrow{p_i p_j}$. We claim that, if $\overrightarrow{p_j p_i} \notin k$-$\mathsf{NNG}(S)$, then $j < i$. Indeed, if $\overrightarrow{p_j p_i} \notin k$-$\mathsf{NNG}(S)$, then $|p_j p_i| > k - \text{dist}(p_j)$. Since $|p_i p_j| \leq k - \text{dist}(p_i)$, we have that $k - \text{dist}(p_j) < k - \text{dist}(p_i)$ and $j < i$. Consequently, from the set of k edges of k-$\mathsf{NNG}(S)$ having p_i as the origin, at most $i-1$ are not bidirectional. This implies that every k-nearest neighbor graph has a set of $\sum_{i=1}^{k}(k-i+1)$ different edges that are bidirectional.

The bounds $|E(k\text{-}\mathsf{NNG}(S))| \geq \frac{kn}{2}$ and $|E(k\text{-}\mathsf{SNNG}(S))| \leq \frac{kn}{2}$ are attained when all edges in the k-nearest neighbor graph are bidirectional, as is the case in Example 3.2 below. The bounds $|E(k\text{-}\mathsf{NNG}(S))| \leq kn - \binom{k+1}{2}$ and $\leq |E(k\text{-}\mathsf{SNNG}(S))| \geq \binom{k+1}{2}$ are attained by the construction described in Example 3.3. ⊠

We next describe two examples that will be used throughout the paper.

Example 3.2. Let n be a multiple of $k+1$. Let $\mathcal{S}$ be a set of n points grouped into sets of size $k+1$, and such that these groups are at a sufficiently large distance from each other. In k-$\mathsf{NNG}(S)$ these groups form cliques of size $k+1$, and there are no edges between two different groups. Thus, in particular, all edges are bidirectional.

Example 3.3. Let $\mathcal{S}$ be a set $q_1, q_2, \ldots, q_n$ of almost collinear and exponentially spaced points, that is, where $|q_i q_{i+1}| = C|q_i q_{i-1}|$, for some fixed constant $C \geq 2$. When looking at k-$\mathsf{NNG}(\mathcal{S})$, there are two groups of vertices. If $i \leq k+1$, the k nearest neighbors of q_i are the points q_j, with $j \leq k+1$ and $j \neq i$. If $i \geq k+1$, the k nearest neighbors of q_i are the points q_j, with $j \in \{i-k, i-k+1, \ldots, i-1\}$. Hence k-$\mathsf{NNG}(\mathcal{S})$ contains exactly $\binom{k+1}{2}$ bidirectional edges, namely, those of the form $q_i q_j$, with $i, j \leq k+1$. On the other hand, $(k+1)$-$\mathsf{NNG}(\mathcal{S}) = k$-$\mathsf{RNG}(\mathcal{S}) = k$-$\mathsf{GG}(\mathcal{S})$, so in this case k-$\mathsf{RNG}(\mathcal{S})$ and k-$\mathsf{GG}(\mathcal{S})$ contain $(k+1)n - \binom{k+2}{2}$ edges.

Recall that $(k+1)$-$\mathsf{NNG}(\mathsf{S}) \subseteq k$-$\mathsf{RNG}(S) \subseteq k$-$\mathsf{GG}(S)$. Hence the lower bound on the number of edges of k-$\mathsf{NNG}(S)$ immediately yields that, for every point set S, $|E(k\text{-}\mathsf{RNG}(S))| \geq \frac{(k+1)n}{2}$ and $|E(k\text{-}\mathsf{GG}(S))| \geq \frac{(k+1)n}{2}$. An example of a k-RNG and a k-GG with a relatively small number of edges is given in Example 3.3.

The number of edges of k-$\mathsf{DG}(S)$ has been studied in [1]. In this paper they show that, if $k < \frac{n}{2} - 1$, then $|E(k\text{-}\mathsf{DG}(S))| \geq (k+1)n$. In the next proposition we give a point set whose k-DG has a small number of edges.

Proposition 3.4. *For any even values of k and $n \geq 4k+4$, there exists a point set $\mathcal{S}$ such that $|E(k\text{-}\mathsf{DG}(\mathcal{S}))| \leq \frac{3kn}{2} + 2k^2 + \frac{5n}{2} - 4k - 6$.*

Proof. We place $n/2$ points in an horizontal line l_p such that the distance between two consecutive points is always the same; from left to right, we denote them by $p_1, p_2, \ldots, p_{n/2}$. We place the remaining $n/2$ points in another horizontal line l_q below l_p such that each point in l_p has a counterpart in l_q with the same abscissa; from left to right, we denote the new points by $q_1, q_2, \cdots, q_{n/2}$.

Let i be such that $k+2 \leq i \leq n-k-1$ and let $G = k\text{-}\mathsf{DG}(\mathcal{S})$. We next show that $d_G(q_i) \leq 3k+5$; notice that, by symmetry, we will also have that $d_G(p_i) \leq 3k+5$. In k-$\mathsf{DG}(\mathcal{S})$ the point q_i is adjacent to $2k+2$ points in l_q, namely, $q_{i-k-1}, q_{i-k}, \ldots, q_{i-1}$ and $q_{i+1}, q_{i+2}, \ldots, q_{i+k+1}$. To see which points in l_p are adjacent to q_i, we start by showing that $q_i p_{i-\frac{k}{2}-2} \notin E(k\text{-}\mathsf{DG}(\mathcal{S}))$. Suppose, for the sake of contradiction, that there exists a circle C through q_i and $p_{i-\frac{k}{2}-2}$ that contains at most k points from $\mathcal{S}$ in its interior. Perturb C keeping it incident to q_i and $p_{i-\frac{k}{2}-2}$ until it goes through a third point of $\mathcal{S}$; let C' be the resulting circle. Notice that C' contains at most k points from $\mathcal{S}$ in its interior. Additionally, by the symmetry of $\mathcal{S}$, C' goes through four points of $\mathcal{S}$ (otherwise it would contain too many points of $\mathcal{S}$): $p_{i-\frac{k}{2}-2}$, another point in l_p which we denote by p_j, q_i, and another point in l_q which we denote by q_t. We have that $t = j - \frac{k}{2} - 2$, and thus C' contains $p_{i-\frac{k}{2}-1}, p_{i-\frac{k}{2}}, \ldots, p_{j-1}$ and $q_{j-\frac{k}{2}-1}, q_{j-\frac{k}{2}}, \ldots, q_{i-1}$, that is, $k+2$ points (see Figure 11, left). This yields a contradiction. By analogous arguments, q_i is not adjacent to p_r for $r < i - \frac{k}{2} - 2$, and also for $r \geq i + \frac{k}{2} + 2$. Therefore q_i can only be adjacent to $p_{i-\frac{k}{2}-1}, p_{i-\frac{k}{2}}, \ldots, p_{i+\frac{k}{2}+1}$ and we conclude that $d_G(q_i) \leq 3k+5$.

Next let i be such that $i < k+2$. In l_q, q_i is adjacent to $q_1, q_2, \ldots, q_{i-1}$ and $q_{i+1}, q_{i+2}, \ldots, q_{i+k+1}$. Regarding connections to points in l_p, using similar arguments to those in the preceding paragraph, we see that q_i is not adjacent to p_r for $r \geq i+k+1$. Consequently, $d_G(q_i) \leq 4k+2$. Analogously, $d_G(q_i) \leq 4k+2$ for $i > n-k-1$, and $d_G(p_i) \leq 4k+2$ for $i < k+2$ and $i > n-k-1$.

In summary, $\sum_{q \in \mathcal{S}} d_G(q) \leq (n-4k-4)(3k+5) + (4k+4)(4k+2) = 3kn + 4k^2 + 5n - 8k - 12$. Notice that $\mathcal{S}$ can be perturbed so that it becomes non-degenerate. ⊠

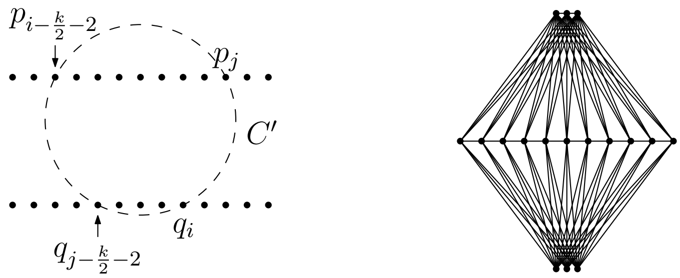

Figure 11: Sets of points whose k-DG have $\frac{3kn}{2} + o(kn)$ (left) and $3kn + o(kn)$ (right) edges.

Let us next look at the maximum number of edges of k-RNG, k-GG, and k-DG. In [1] it is proved that $|E(k\text{-}\mathsf{DG}(S))| \leq 3(k+1)n - 3(k+1)(k+2)$. Due to the fact that $k\text{-}\mathsf{GG}(S) \subseteq k\text{-}\mathsf{DG}(S)$, the graph k-$\mathsf{GG}(S)$ inherits the upper bound: $|E(k\text{-}\mathsf{GG}(S))| \leq 3(k+1)n - 3(k+1)(k+2)$. As for k-RNG, in Proposition 3.12 we will see that the vertices of k-RNG have degree at most $5(k+1)$. This yields that $|E(k\text{-}\mathsf{RNG}(S))| \leq \frac{5(k+1)n}{2}$. The number of edges of k-RNG was also considered in [9], but under no non-degeneracy assumption. As a consequence, their upper bound $|E(k\text{-}\mathsf{RNG}(S))| \leq 9(k+1)n$ is higher.

Examples of k-RNG and k-GG with a large number of edges are described in the next proposition.

Proposition 3.5. *If $\omega(1) \leq k \leq o(n)$, there exists a set $\mathcal{S}$ of n points such that $|E(k\text{-}\mathsf{RNG}(\mathcal{S}))| = \frac{3\pi}{4\pi - 3\sqrt{3}} nk + o(nk)$ and $|E(k\text{-}\mathsf{GG}(\mathcal{S}))| = 2nk + o(nk)$.*

Proof. Let $\omega(1) \leq d \leq o(\sqrt{n})$. Let $\mathcal{S}$ be a set of n points arranged in a slightly perturbed unit square grid of size $\sqrt{n} \times \sqrt{n}$, so that the points are in general position. First note that two points $q_i, q_j \in \mathcal{S}$ not close to the boundary are neighbors in k-$\mathsf{RNG}(\mathcal{S})$ if the area of $\mathsf{O\text{-}LENS}(q_i, q_j)$ is $k + \Theta(1)$, that is, if their distance is at most $\sqrt{k/(2\pi/3 - \sqrt{3}/2)} + \Theta(1)$. Let d be the value of this distance. For the points close to the boundary, that is, at distance at most d from it, their neighbors in k-$\mathsf{RNG}(\mathcal{S})$ consist of those points in $\mathcal{S}$ inside a circle of radius at most $\sqrt{2}d$. Thus all points in $\mathcal{S}$ except for those close to the boundary have degree $\pi k/(2\pi/3 - \sqrt{3}/2) + \Theta(1)$ in k-$\mathsf{RNG}(\mathcal{S})$. Since only a sublinear number of points in $\mathcal{S}$ are close to the boundary, we conclude that $|E(k\text{-}\mathsf{RNG}(\mathcal{S}))| = \frac{3\pi}{4\pi - 3\sqrt{3}} nk + o(nk) \simeq 1.28nk + o(nk)$.

Except for a similar analysis for the points close to the boundary, two points in $\mathcal{S}$ are neighbors in k-$\mathsf{GG}(\mathcal{S})$ if their distance is at most $2\sqrt{k/\pi} + \Theta(1)$. Hence all but a sublinear number of points in $\mathcal{S}$ have degree $4k$ in k-$\mathsf{GG}(\mathcal{S})$, and $|E(k\text{-}\mathsf{GG}(\mathcal{S}))| = 2nk + o(nk)$ ⊠

For k-DG we provide a different construction:

Proposition 3.6. *For any $k \geq 0$ and $n \geq 2k + 3$, there exists a point set $\mathcal{S}$ such that $|E(k\text{-}\mathsf{DG}(\mathcal{S}))| \geq 3nk - 9k^2 + 2n - 11k - 4$.*

Proof. Refer to Figure 11 (right). The number of points in the upper group is $k + 1$, and the lower group contains the same number of points. The remaining points are placed in the middle group.

In k-$\mathsf{DG}(\mathcal{S})$, the points in the upper group form a clique because they can be covered with a disk that does not contain any other point of $\mathcal{S}$. Analogously, the points in the lower group form a clique as well. Let $q_1, q_2, \ldots, q_{n-2k-2}$ be the points in the middle group sorted from left to right. Each point q_i is connected to all upper and lower points, since the circles through q_i with center at the vertical line through q_i do not contain any point q_j such that $j \neq i$. The point q_i is also connected to its k predecessors and k successors in the middle group if it has enough points to its left and right. Consequently, if $G = k$-$\mathsf{DG}(\mathcal{S})$, then $\sum_{q \in \mathcal{S}} d_G(q) \geq 2(k+1)(n-k-2) + (n-4k-2)(4k+2) = 6nk - 18k^2 + 4n - 22k - 8$.

This construction can be perturbed so that the non-degeneracy assumptions are satisfied. ⊠

3.2 Minimum and maximum degree

We start analyzing the minimum degree.

Proposition 3.7. *For every point set S, any vertex of k-$\mathsf{NNG}(S)$ has degree k or larger, and at least one vertex has degree exactly k.*

Proof. Since every vertex in k-$\mathsf{NNG}(S)$ has out-degree k, it has degree at least k. Let p be a vertex of k-$\mathsf{NNG}(S)$ with maximum k-distance. Then p has no incoming edges that are not bidirectional. Thus the degree of p in k-$\mathsf{NNG}(S)$ is k. ⊠

In the case of k-SNNG, all vertices have degree k or smaller. We can obtain a k-SNNG where all vertices have degree k by considering the set in Example 3.2.

For the minimum degree of k-RNG we use the following lemma:

Lemma 3.8. *In any angular sector with apex $p \in S$ and amplitude $\alpha \leq \pi/3$, the only points that can be connected to p in k-RNG(S) are the $k+1$ closest points to p that are contained in the sector.*

Proof. Let $p_1, p_2, \ldots$ be the points of S that are contained in the sector sorted by increasing distance to p. For each $i \geq 2$, the points $p_1, p_2, \ldots, p_{i-1}$ are contained in the intersection of the two disks centered at p and p_i with radius $|pp_i|$. (See Figure 12, left.) Consequently, p and p_i are not connected in k-RNG(S) for $i - 1 > k$. ⊠

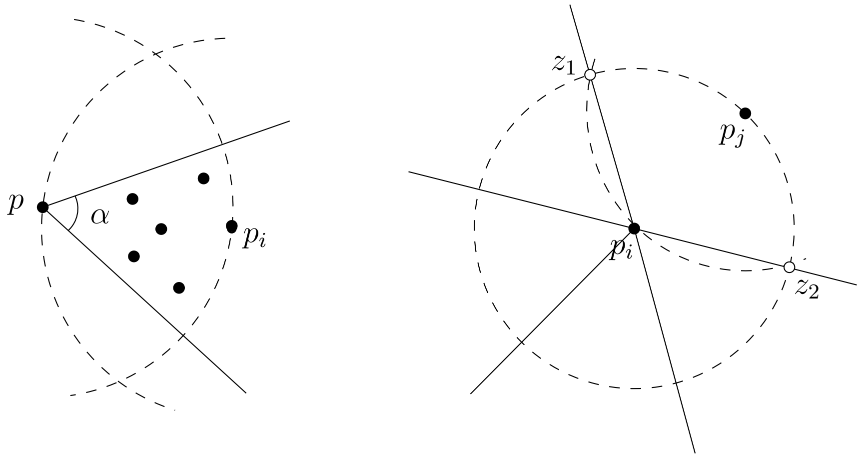

Figure 12: Left: an angular sector with apex p and amplitude $\alpha \leq \pi/3$. Right: each of the five angular sectors contains at most $k+1$ points that are connected to p_i in k-RNG(S).

As a corollary, we have that any left-most, right-most, upper-most, or lower-most point of S has degree not greater than $3k+3$ in k-RNG(S). Thus $\delta(k\text{-RNG}(S)) \leq 3k+3$. We now show a construction where all vertices have degree $2k+2$ or more:

Proposition 3.9. *For any $k \geq 0$ and $n \geq 3k+4$, there exists a set $\mathcal{S}$ of n points such that $\delta(k\text{-RNG}(\mathcal{S})) = 2k+2$.*

Proof. Let $\mathcal{S} = \{q_1, q_2, \ldots, q_n\}$ denote the set of vertices of a slightly perturbed regular n-gon (see Figure 13). In k-RNG$(\mathcal{S})$ each point q_i is adjacent to $\{q_{i-k-1}, q_{i-k}, \ldots, q_{i-1}, q_{i+1}, q_{i+2}, \ldots, q_{i+k+1}\}$ (arithmetic is taken modulo n). Hence all vertices have degree $2k+2$. ⊠

The upper bounds for the maximum number of edges of k-GG(S) and k-DG(S) and the handshaking lemma yield that $\delta(k\text{-GG}(S)) \leq 6k+5$ and $\delta(k\text{-DG}(S)) \leq 6k+5$. Point sets for which k-GG and k-DG have large minimum degree are described in Examples 3.10 and 3.11, respectively.

Example 3.10. Let n be a multiple of $3k+3$. We construct a set $\mathcal{S}$ of n points consisting of copies of the following set of $3k+3$ points originally described in [3]. Let P, Q, R be the vertices of an equilateral triangle. Let $\widehat{PQ}$ be the arc of the circle centered at R and having endpoints

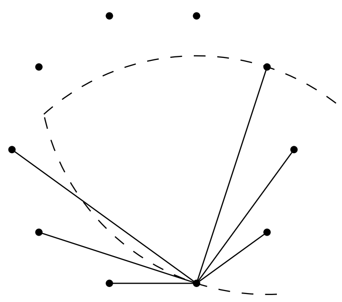

Figure 13: Point set of Proposition 3.9 for $k = 2$.

P, Q. Let $\widehat{QR}$ and $\widehat{RP}$ be defined analogously. We place $k+1$ points $p_1, p_2, \dots, p_{k+1}$ on $\widehat{PQ}$ and close to P, $k+1$ points $q_1, q_2, \dots, q_{k+1}$ on $\widehat{QR}$ and close to Q, and $k+1$ points $r_1, r_2, \dots, r_{k+1}$ on $\widehat{RP}$ and close to R (see Figure 14, left). If the positions of the points are carefully chosen, this group of $3k+3$ points forms a clique in k-$\mathsf{GG}(\mathcal{S})$ (see [3] for details). Thus, in particular, each vertex of k-$\mathsf{GG}(\mathcal{S})$ has degree at least $3k+2$ and the chromatic number of the graph is $3k+3$ or greater.

Example 3.11. We describe a set of n points $\mathcal{S}$, where n is multiple of $4k+4$, that is formed by copies of the following set of $4k+4$ points, described for the first time in [22]. The points P, Q, R are the vertices of an equilateral triangle, T is the midpoint of R and Q, and S lies on the vertical line through R and satisfies $|RS| = |RP|$ (see Figure 14, right). There are $k+1$ points $p_1, p_2, \dots, p_{k+1}$ on the segment $\overline{RP}$ and close to P, $k+1$ points $q_1, q_2, \dots, q_{k+1}$ on $\overline{PQ}$ and close to Q, $k+1$ points $r_1, r_2, \dots, r_{k+1}$ on $\overline{QR}$ and close to R, and $k+1$ points $s_1, s_2, \dots, s_{k+1}$ on $\overline{ST}$ and close to S. It can be easily shown that this set of $4k+4$ points forms a clique in k-$\mathsf{DG}(\mathcal{S})$. Therefore the minimum degree and the chromatic number of k-$\mathsf{DG}(\mathcal{S})$ are at least $4k+3$ and $4k+4$, respectively.

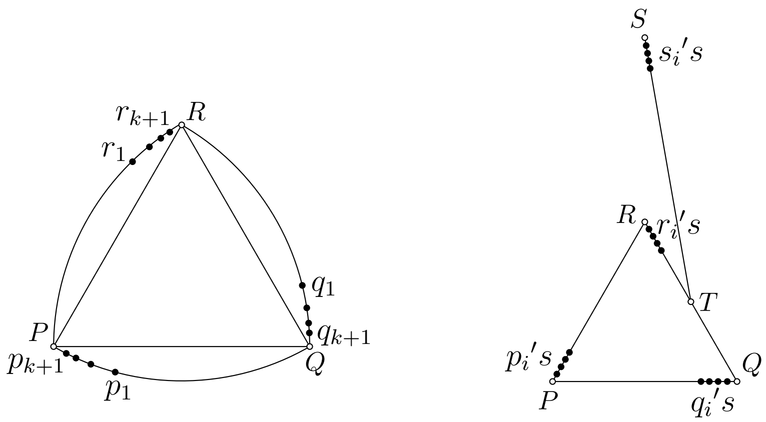

Figure 14: Left: a set of $3k+3$ points whose k-GG is the complete graph. Right: a set of $4k+4$ points whose k-DG is the complete graph.

We next study the maximum degree. As in the previous section, we give the exact value of the maximum degree of all graphs.

We start with the maximum degree of k-RNG. A straightforward application of Lemma 3.8 yields that the degree of any vertex in k-$\mathsf{RNG}(S)$ is at most $6(k+1)$. In the next proposition we improve this bound:

Proposition 3.12. *For every point set S, the vertices of k-$\mathsf{RNG}(S)$ have degree at most $5(k+1)$.*

Proof. Let p_i be a point in S, and let p_j be the neighbor of p_i in k-$\mathsf{RNG}(S)$ at largest distance from p_i. We define z_1 and z_2 as the intersection points between the circle centered at p_i with radius $|p_ip_j|$, and the circle centered at p_j with the same radius (see Figure 12, right). Since $p_ip_j \in k$-$\mathsf{RNG}(S)$, $\mathsf{O\text{-}LENS}(p_i, p_j)$ contains at most k points from S. Furthermore, as p_j is the furthest neighbor to p_i, indeed the angular sector with apex p_i and amplitude $\widehat{z_2p_iz_1}$ contains at most $k+1$ points that are adjacent to p_i. To complete the proof, we observe that the plane can be divided into the previous angular sector and an angular sector with apex p_i and amplitude $4\pi/3$, which in turn can be divided into four angular sectors of amplitude $\pi/3$. By Lemma 3.8, each one of these four sectors contains at most $k+1$ points that are connected to p_i in k-$\mathsf{RNG}(S)$. Thus we conclude that the degree of p_i is no greater than $5(k+1)$. ⊠

Since k-$\mathsf{NNG}(S) \subseteq (k-1)$-$\mathsf{RNG(S)}$, we also have that $\Delta(k\text{-}\mathsf{NNG}(S)) \leq 5k$. To see that $\Delta(k\text{-}\mathsf{NNG}(S)) \leq 5k$ and $\Delta(k\text{-}\mathsf{RNG}(S)) \leq 5(k+1)$ are tight, let $q_1, q_2, \ldots, q_5$ be the vertices of a regular pentagon and c be its center. Clearly, we can place a set of $k-1$ points around each vertex q_i in such a way that c is the kth nearest neighbor of all the points in the construction and, consequently, it has degree $5k$ in k-$\mathsf{NNG}(S)$ and $(k-1)$-$\mathsf{RNG(S)}$. The point set as described can be perturbed to attain general position.

Let us make a final remark on k-GG and k-DG. In the previous section we have described a point set in general position whose GG and DG contain a vertex of degree $n-1$. In k-GG and k-DG the same vertex has maximum degree as well.

3.3 Chromatic and clique numbers

We apply the minimum degree greedy algorithm (see, for example, [16]) to obtain upper bounds on the chromatic number of these graphs. In order to do so, we need information on the minimum degree of the induced subgraphs.

Remark 3.13. Let S be a set of points and $G \in \{k\text{-}\mathsf{NNG}, k\text{-}\mathsf{SNNG}, k\text{-}\mathsf{RNG}, k\text{-}\mathsf{GG}, k\text{-}\mathsf{DG}\}$. If p_ip_j is an edge of $G(S)$, this edge is also present in $G(S \setminus \{p_l\})$ for any $p_l \in S$ ($p_l \neq p_i, p_j$). Thus, if $G(S) \setminus S'$ is an induced subgraph of $G(S)$, then it is a subgraph of $G(S \setminus S')$ and $\delta(G(S) \setminus S') \leq \delta(G(S \setminus S'))$. Now $G(S \setminus S')$ is a proximity graph and $\delta(G(S \setminus S')) \leq f(k)$, where different functions $f(k)$ have been given in the previous subsection.

Applying the minimum degree greedy algorithm, hence, we obtain that $\chi(k\text{-}\mathsf{NNG}(S)) \leq k+1$, $\chi(k\text{-}\mathsf{SNNG}(S)) \leq k+1$, $\chi(k\text{-}\mathsf{RNG}(S)) \leq 3k+4$, $\chi(k\text{-}\mathsf{GG}(S)) \leq 6(k+1)$, and $\chi(k\text{-}\mathsf{DG}(S)) \leq 6(k+1)$. The last two bounds have also been given in [7].

We can construct a k-NNG and a k-SNNG with chromatic number $k+1$ by placing the points as in Example 3.2. On the other hand, we can easily obtain cliques of size $k+2$ in a k-RNG (and, consequently, a k-RNG with chromatic number $k+2$ or greater) by placing $k+2$ initial points anywhere, drawing all lenses defined by pairs of these points, and placing the remaining $n-(k+2)$ points outside the union of the lenses. Finally, the k-Gabriel graph of the set in

Example 3.10 has chromatic number greater than or equal to $3k + 3$, whereas the chromatic number of the k-Delaunay graph of the construction in Example 3.11 is at least $4k + 4$ (see also [7]).

Let us look at the clique number.

Proposition 3.14. *For every point set S, $\omega(k\text{-}\mathsf{NNG}(S)) \leq k + 1$. This bound is tight.*

Proof. Let $p_1, p_2, \ldots, p_l$ be the vertices in a clique of maximum size in k-$\mathsf{NNG}(S)$ sorted in increasing order of k-distance (choose any order if there are ties). If for some $h < l$ we have that $\overrightarrow{p_h p_l} \in k\text{-}\mathsf{NNG}(S)$, then $|p_h p_l| \leq k - \text{dist}\,(p_h) \leq k - \text{dist}\,(p_l)$ and $\overrightarrow{p_l p_h} \in k\text{-}\mathsf{NNG}(S)$. Thus in the clique p_l has no incoming edges that are not bidirectional. Consequently, $l \leq k + 1$.

A clique number of $k + 1$ is attained by the construction in Example 3.3. ⊠

As a corollary, we obtain that $\omega(k\text{-}\mathsf{SNNG}(S)) \leq k + 1$. We have seen in previous examples that this bound is tight.

Proposition 3.15. *For every point set S, $\omega(k\text{-}\mathsf{RNG}(S)) \leq k + 2$. There exist examples where this value is achieved.*

Proof. Let $S_\omega = \{p_1, p_2, \ldots, p_l\}$ be the vertices in a clique of maximum size in k-$\mathsf{RNG}(S)$. Let i, j be indices such that $\{p_i, p_j\}$ is a diametral pair of this set. Then all the remaining points in S_ω lie in $\mathsf{C\text{-}LENS}(p_i, p_j)$. In fact, since we assume that the set of k nearest points in S of p_i and p_j has cardinality k for any $k \geq 1$, we have that all points in $S_\omega \setminus \{p_i, p_j\}$ lie in $\mathsf{O\text{-}LENS}(p_i, p_j)$. But there are at most k such points, because $p_i p_j \in k\text{-}\mathsf{RNG}(S)$. Therefore $l \leq k + 2$.

As for the second part of the statement, when discussing the chromatic number of k-RNG we have described a simple way to obtain cliques of size $k + 2$ in this graph. ⊠

The problem of delimiting the size of the maximum clique that might be a subgraph of k-$\mathsf{DG}(S)$ is closely related to the following open problem (see [33]): what is the largest number $\Pi(n)$ such that for every set S of n points in the plane, there exist two points $p_i, p_j \in S$, where every circle (the interior and the boundary) containing p_i and p_j contains $\Pi(n)$ points of S? It is known that $\Pi(n) \geq \frac{n+1}{2} - \sqrt{\frac{(n-2)^2-1}{12}} \geq \frac{n}{4.74}$ [19]. On the other hand, $\Pi(n) \leq \lceil \frac{n}{4} \rceil + 1$ because there exists a set $\mathcal{S}$ of n points in the plane such that, for every pair $q_i, q_j \in \mathcal{S}$, there exists a circle containing them that contains at most $\lceil \frac{n}{4} \rceil - 1$ additional points of $\mathcal{S}$ [22] (in fact, this is Example 3.11). As a corollary, we obtain

Corollary 3.16. *For every point set S, we have $\omega(k\text{-}\mathsf{DG}(S)) \leq 4.74k + 14$. For any $n \geq 4$ and $k \leq \lceil \frac{n}{4} \rceil - 1$, there exist sets of n points in the plane whose k-Delaunay graph contains a clique of size $4k + 4$.*

Proof. Assume, for the sake of contradiction, that there exists a set S such that k-$\mathsf{DG}(S)$ contains a clique of size greater than or equal to $4.74k + 15$. Let S' be a subset of cardinality $n' = 4.74k + 15$ of the vertices of this clique. By Remark 3.13, k-$\mathsf{DG}(S')$ is the complete graph. Since $\Pi(n') \geq \frac{n'}{4.74}$, there exists $p_i, p_j \in S'$ such that every circle containing p_i and p_j contains at least $\frac{n'}{4.74} - 2$ additional points of S'. Consequently, for all values of k' such that $k' < \frac{n'}{4.74} - 2$ we have that $p_i p_j \notin k'$-$\mathsf{DG}(S')$. Now $k \approx \frac{n'}{4.74} - 3.16 < \frac{n'}{4.74} - 2$, which yields the contradiction.

The second part of the statement follows from the example proving $\Pi(n) \leq \lceil \frac{n}{4} \rceil + 1$, described in [22]. ⊠

Analogously, the question of determining the size of the maximum clique that might be a subgraph of k-$\mathsf{GG}(S)$ is related to a variant of the problem of delimiting $\Pi(n)$ that consists of restricting the set of circles though $p_i, p_j \in S$ to the diametral circle. This variant is solved. In particular, it is known [3] that for every set S of n points in the plane, there exist two points $p_i, p_j \in S$ such that $|\mathsf{C\text{-}DISC}(p_i, p_j) \cap (S \setminus \{p_i, p_j\})| \geq \lceil \frac{n}{3} \rceil - 1$. Additionally, there exists a set $\mathcal{S}$ of n points in the plane such that, for every pair $q_i, q_j \in \mathcal{S}$, $|\mathsf{C\text{-}DISC}(q_i, q_j) \cap (\mathcal{S} \setminus \{q_i, q_j\})| \leq \lceil \frac{n}{3} \rceil - 1$ [3] (this is Example 3.10). From this we can derive the exact size of a maximal clique in k-GG :

Corollary 3.17. *For every point set S, $\omega(k\text{-}\mathsf{GG}(S)) \leq 3k + 3$. For any $n \geq 3$ and $k \leq \lceil \frac{n}{3} \rceil - 1$, there exist sets of n points in the plane whose k-Gabriel graph contains a clique of size $3k + 3$.*

4 Concluding remark

We have reviewed graph-theoretic properties of some proximity graphs of the family of the Delaunay graph, and presented new bounds. The natural open problem is to close the gaps between the lower and upper bounds that do not match.

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