A Linear-Time Algorithm to Find a Separator in a Graph Excluding a Minor

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Abstract. Let G be an n-vertex m-edge graph with weighted vertices. A pair of vertex sets $A, B \subseteq V(G)$ is a $\frac{2}{3}$-*separation* of *order* $|A \cap B|$ if $A \cup B = V(G)$, there is no edge between $A - B$ and $B - A$, and both $A - B$ and $B - A$ have weight at most $\frac{2}{3}$ the total weight of G. Let $\ell \in \mathbb{Z}^+$ be fixed. Alon et al. [1990] presented an algorithm that in $\mathcal{O}(n^{1/2}m)$ time, outputs either a K_ℓ-minor of G, or a separation of G of order $\mathcal{O}(n^{1/2})$. Whether there is a $\mathcal{O}(n + m)$-time algorithm for this theorem was left as an open problem. In this article, we obtain a $\mathcal{O}(n + m)$-time algorithm at the expense of a $\mathcal{O}(n^{2/3})$ separator. Moreover, our algorithm exhibits a trade-off between time complexity and the order of the separator. In particular, for any given $\epsilon \in [0, \frac{1}{2}]$, our algorithm outputs either a K_ℓ-minor of G, or a separation of G with order $\mathcal{O}(n^{(2-\epsilon)/3})$ in $\mathcal{O}(n^{1+\epsilon} + m)$ time. As an application we give a fast approximation algorithm for finding an independent set in a graph with no K_ℓ-minor.

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1. *Introduction*

This article presents a linear-time algorithm for finding a separator in graphs excluding a fixed minor.

A *separation* of a graph[1] G is a pair $\{A, B\}$ of vertex sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, and there is no edge between $A - B$ and $B - A$, as illustrated in Figure 1. The *order* of $\{A, B\}$ is $|A \cap B|$. The set $A \cap B$ is called a *separator* of G. A *weighting* of G is a function $w : V(G) \to \mathbb{R}^+$. Let $w(S) := \sum_{v \in S} w(v)$ for all $S \subseteq V(G)$, and let $w(G) := w(V(G))$. We say (G, w) is a *weighted graph*. A separation $\{A, B\}$ of a weighted graph (G, w) is a β*-separation* if $w(A - B) \leq \beta \cdot w(G)$ and $w(B - A) \leq \beta \cdot w(G)$.

A "separator theorem" is of the format: For some $0 < \beta < 1$ and $0 < \epsilon \leq 1$, every graph G from a certain family has a β-separation of order $\mathcal{O}(|G|^{1-\epsilon})$. Applications of separator theorems are numerous, and include VLSI circuit layout [Leiserson 1980], approximation algorithms using the divide-and-conquer paradigm [Chiba et al. 1981; Lipton and Tarjan 1980], solving sparse systems of linear equations [Lipton et al. 1979], pebbling games [Lipton and Tarjan 1980], and graph drawing [Dujmović and Wood 2004]. See the monograph by Rosenberg and Heath [2001] for more details.

A seminal theorem due to Lipton and Tarjan [1979] states that every weighted planar graph G has a $\frac{2}{3}$-separation of order $\mathcal{O}(|G|^{1/2})$ that can be computed in $\mathcal{O}(|G| + \|G\|)$ time. The importance of this result cannot be overstated, as suggested by the amount of effort that has gone into improving the constant in the $\mathcal{O}(|G|^{1/2})$ bound [Chung 1991; Djidjev 1982; Alon et al. 1994; Venkatesan 1987; Djidjev 1987]. Many other aspects of separators in planar graphs have been studied. For example, Miller [1986] proved that every 2-connected planar graph has a cycle separator, and Djidjev and Venkatesan [1997] improved the constants. Aleksandrov et al. [2006] and Djidjev [2000] considered separators in planar graphs whose order is measured in terms of associated vertex costs.

Djidjev and Gilbert [1999] considered separators in graphs with negative and multiple weights. Separators in certain geometric graphs have been studied by Miller et al. [1997] and Smith and Wormald [1998]. Plaisted [1990] developed a heuristic for finding separators in arbitrary graphs. Edge separators have been studied by Sýkora and Vrťo [1993] and Diks et al. [1993]. Alber et al. [2003] studied separators from the perspective of the theory of fixed parameter tractability. Approximation algorithms for separators are also well studied [Garg et al. 1999; Feige and Mahdian 2006; Arora et al. 2004; Amir et al. 2003; Even et al. 2000; Even et al. 1999; Bodlaender et al. 1995].

[1] We consider graphs G that are simple, finite, and undirected. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of G. Let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. For each vertex $v \in V(G)$, let $N(v) := \{w \in V(G) : vw \in E(G)\}$ be the set of neighbors of v. For each subgraph X of G, let $N(X) := \bigcup\{N(v) - V(X) : v \in V(X)\}$. For $n \in \mathbb{Z}^+$, let $[n] := \{1, 2, \ldots, n\}$.

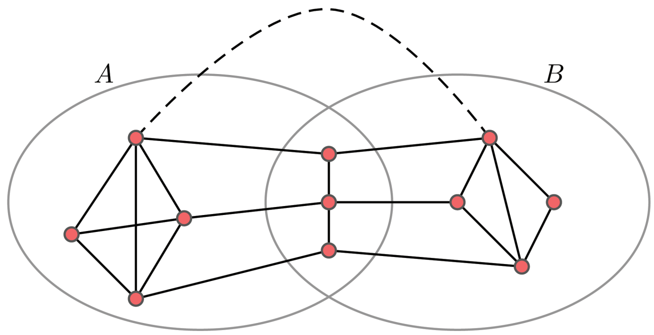

FIG. 1. A separation $\{A, B\}$.

The theorem of Lipton and Tarjan was generalized to graphs with genus γ by Gilbert et al. [1984] and Djidjev [1987, 1985b, 1981]. They proved that such graphs G have a separation of order $\mathcal{O}(\gamma^{1/2} \cdot |G|^{1/2})$, which can be computed in linear time [Djidjev 1985a; Aleksandrov and Djidjev 1996]. The special case of toroidal graphs was considered by Aleksandrov and Djidjev [1989].

Perhaps the most general setting for separator theorems is for graphs excluding a fixed minor, as studied by Alon et al. [1990b], Plotkin et al. [1994], Grohe [2003], and Demaine and Hajiaghayi [2008a, 2008b, 2005]. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges, in which case we say that G contains an *H-minor*. An *H-model* in G is a set of disjoint connected subgraphs $\{X_v : v \in V(H)\}$ indexed by the vertices of H, such that for every edge $vw \in E(H)$, there is an edge $xy \in E(G)$ with $x \in X_v$ and $y \in X_w$. Clearly G contains an H-minor if and only if G contains an H-model. For algorithmic purposes, we choose to work with H-models rather than H-minors. Graph classes defined by an exluded minor are often of interest. For example, the Kuratowski-Wagner theorem states that a graph is planar if and only if it contains no K_5-minor and no $K_{3,3}$-minor. Alon et al. [1990b] proved the following generalization of the Lipton-Tarjan separator theorem for graphs excluding an arbitrary minor.

THEOREM 1.1. [ALON ET AL. 1990B]. *There is an algorithm that, given $\ell \in \mathbb{Z}^+$ and a weighted graph (G, w), outputs either:*

(a) *a K_ℓ-model of G, or*
(b) *a $\frac{2}{3}$-separation of (G, w) of order at most $\ell^{3/2} \cdot |G|^{1/2}$*
in time $\mathcal{O}((\ell \cdot |G|)^{1/2} \cdot (|G| + \|G\|))$.

Suppose that ℓ is fixed. It follows from a result of Mader [1967] that Theorem 1.1 can be implemented in $\mathcal{O}(|G|^{3/2} + \|G\|)$ time; see Theorem 2.3. Alon et al. [1990b] left as an open problem whether linear $\mathcal{O}(|G| + \|G\|)$ time is possible. The main result of this article is the following partial answer to this question. We obtain linear time complexity at the expense of a slightly larger separator (and larger dependence on ℓ). Moreover, our algorithm exhibits a trade-off between time complexity (ranging from $\mathcal{O}(n)$ to $\mathcal{O}(n^{3/2})$) and the order of the separator (ranging from $\mathcal{O}(n^{2/3})$ to $\mathcal{O}(n^{1/2})$).

THEOREM 1.2. *There is an algorithm that, given $\epsilon \in [0, \frac{1}{2}]$, $\ell \in \mathbb{Z}^+$, and a weighted graph (G, w), outputs either:*

(a) *a K_ℓ-model of G, or*

(b) *a $\frac{2}{3}$-separation of (G, w) of order at most $\ell^{3/2} \cdot 2^{(\ell^2+4)/2} \cdot |G|^{(2-\epsilon)/3}$ in time $\mathcal{O}(\ell \cdot 2^{(3\ell^2+2\ell+6)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|)$.*

Note that for applications to divide-and-conquer algorithms a separation of order $\mathcal{O}(|G|^{1-\epsilon})$, for some constant $\epsilon > 0$, is all that is needed. For example, in Section 5 we apply Theorem 1.2 to obtain an approximation algorithm for the maximum independent set problem on graphs excluding a fixed minor that runs in near-linear time and has diminishing relative error. (A set of vertices I in a graph is *independent* if no two vertices in I are adjacent.) Theorem 1.2 has also recently been applied by Tazari and Müller-Hannemann [2009] and Yuster [2008] to obtain improved shortest-paths algorithms for graphs excluding a fixed minor, and by Yuster and Zwick [2007] to obtain the fastest known algorithm for finding a maximum matching in a graph excluding a fixed minor.

We now outline the idea behind the proof of Theorem 1.2 for fixed ℓ and with $\epsilon = 0$. Suppose that in $\mathcal{O}(|G|+\|G\|)$ time, we can find a partition $\{S_1, S_2, \ldots, S_{|G|^{2/3}}\}$ of $V(G)$, such that each S_i induces a connected subgraph of G with $\mathcal{O}(|G|^{1/3})$ vertices. Let H be the weighted graph obtained from G by contracting each subgraph $G[S_i]$ to a vertex v_i with weight $w(v_i) = w(S_i)$. Then apply Theorem 1.1 to H to obtain either a K_ℓ-model in H which defines a K_ℓ-model in G, or a $\frac{2}{3}$-separation $\{A, B\}$ of H with order $\mathcal{O}(|H|^{1/2}) = \mathcal{O}(|G|^{1/3})$, in which case $\{\bigcup\{S_i : v_i \in A\}, \bigcup\{S_i : v_i \in B\}\}$ is a $\frac{2}{3}$-separation of G with order $\mathcal{O}(|G|^{2/3})$. The time complexity is $\mathcal{O}(|H|^{3/2} + \|H\|) \subseteq \mathcal{O}(|G| + \|G\|)$.

The proof of Theorem 1.2 is actually a little different from this outline. In particular, the subgraphs $G[S_i]$ will not necessarily be connected. However, the partition of $V(G)$ will be "knitted" (see Section 4 for the definition), which will enable the output from Theorem 1.1 applied to H to be converted to the desired output for G. By relaxing the connectivity condition, we are able to prove that an appropriate partition exists.

In Section 2 we give an algorithmic version of a theorem of Mader [1967], which is used in Section 3 to prove an upper bound on the number of cliques in a graph excluding a minor. The main steps in the proof of Theorem 1.2 are presented in Section 4.

2. *Mader's Theorem*

Mader [1967] proved that every sufficiently dense graph contains a large complete graph as a minor. In this section we prove the following algorithmic version of this result. Note that Robertson and Seymour [1995, page 85] proved a similar result with quadratic time complexity.

THEOREM 2.1. *Given a graph G with $\|G\| \geq 2^{\ell-3} \cdot |G|$ for some $\ell \in \mathbb{Z}^+$, a K_ℓ-model in G can be computed in $\mathcal{O}(\ell(|G| + \|G\|))$ time.*

Note that if we ignore the time complexity, Theorem 2.1 is far from best possible. Kostochka [1982, 1984] and Thomason [1984] independently proved that if $\|G\| \in \Omega(\ell\sqrt{\log \ell} \cdot |G|)$ then G contains a K_ℓ-model. In particular, Thomason [2001] proved that if $\|G\| \geq (\delta + o(1))\ell\sqrt{\log \ell} \cdot |G|$, where $\delta = 0.319\ldots$ is a constant, then G contains a K_ℓ-model.

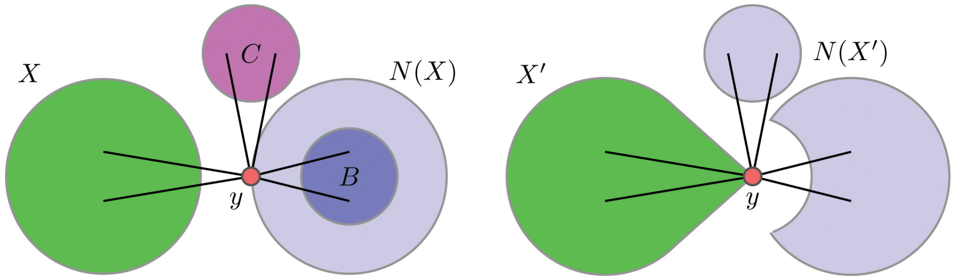

FIG. 2. Illustration of the proof of Lemma 2.2.

The proof of Theorem 2.1 is based on the following lemma.

LEMMA 2.2. *The following algorithm, given a graph G with $\|G\| \geq t \cdot |G|$ for some $t \in \mathbb{Z}^+$, outputs a connected nonempty induced subgraph X of G in time $\mathcal{O}(|G| + \|G\|)$, such that $G[N(X)]$ has minimum degree at least t.*

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let U be a component of G with ‖U‖ ≥ t · |U|
initialize X := G[{v}] for some vertex v ∈ V(U)
while some vertex y ∈ N(X) has degree at most t − 1 in G[N(X)] do
    X := G[V(X) ∪ {y}]
end while
output X
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PROOF. To prove the correctness of the algorithm it suffices to show that, upon termination, $X \neq U$ and $N(X) \neq \emptyset$, implying that $G[N(X)]$ has minimum degree at least t. We do so, by showing that the invariant

$$e(X) \leq t(|X| - 1) + |N(X)| \tag{1}$$

is maintained, where $e(X)$ is the number of edges of U with at least one endpoint in X. Certainly (1) holds when $X = \{v\}$, in which case $e(X) = |N(X)| = \deg(v)$. Now suppose that (1) holds for some subgraph X of U, and $y \in N(X)$ has degree at most $t - 1$ in $G[N(X)]$. Let $X' := G[V(X) \cup \{y\}]$. Partition $N(y) - V(X)$ into two sets, $B := N(y) \cap N(X)$ and $C := N(y) - (V(X) \cup N(X))$, as illustrated in Figure 2. Since $|B| \leq t - 1$ and $N(X') = (N(X) - \{y\}) \cup C$,

$$\begin{aligned} e(X') = e(X) + |B| + |C| &\leq t(|X| - 1) + |N(X)| + t - 1 + |C| \\ &= t \cdot |X| + |N(X')|. \end{aligned}$$

That is, (1) is satisfied for X'. Hence (1) is maintained throughout the algorithm. Now observe that $e(U) = \|U\| \geq t \cdot |U|$ and $N(U) = \emptyset$. Thus (1) is not satisfied for $X = U$. Hence, upon termination, $X \neq U$ and $N(X) \neq \emptyset$, and the algorithm computes X and $N(X)$ as claimed.

The algorithm can be implemented in $\mathcal{O}(|G| + \|G\|)$ time by maintaining the set $V(X)$, the set $N(X)$, the degree of each vertex in $G[N(X)]$, and a list L of the vertices in $N(X)$ with degree at most $t - 1$ in $G[N(X)]$. Whenever a vertex is moved from $N(X)$ into X or from $V(U) - (X \cup N(X))$ into $N(X)$, we traverse its list of neighbors, updating the degree within $N(X)$, and if necessary updating the list L. Thus, each list of neighbors is traversed $\mathcal{O}(1)$ times. Thus the algorithm can be implemented in $\mathcal{O}(|G| + \|G\|)$ time. We omit the routine description of the data structure manipulation necessary. $\square$

PROOF OF THEOREM 2.1. Theorem 2.1 is trivial for $\ell \leq 2$. Now assume that $\ell \geq 3$. Applying Lemma 2.2 with $t = 2^{\ell-3}$ (≥ 1), we obtain a nonempty connected subgraph X of G such that $G[N(X)]$ has minimum degree at least $2^{\ell-3}$. Thus $\|G[N(X)]\| \geq 2^{\ell-4}|N(X)|$. By induction, there is a $K_{\ell-1}$-model in $G[N(X)]$. Since every vertex in $N(X)$ is adjacent to some vertex in X, this $K_{\ell-1}$-model along with X forms a K_ℓ-model in G. There are ℓ applications of Lemma 2.2, each requiring $\mathcal{O}(|G| + \|G\|)$ time. □

Theorem 2.1 implies the following slightly faster version of Theorem 1.1 (for fixed ℓ).

THEOREM 2.3. *There is an algorithm that, given $\ell \in \mathbb{Z}^+$ and a weighted graph (G, w), outputs either:*

(a) *a K_ℓ-model of G, or*
(b) *a $\frac{2}{3}$-separation of (G, w) of order at most $\ell^{3/2} \cdot |G|^{1/2}$*
in time $\mathcal{O}(\ell \cdot 2^\ell \cdot |G|^{3/2} + \ell \cdot \|G\|)$.

PROOF. If $\|G\| \geq 2^{\ell-3}|G|$, then a K_ℓ-model in G can be found in $\mathcal{O}(\ell(|G| + \|G\|))$ time by Theorem 2.1. Otherwise $\|G\| < 2^{\ell-3}|G|$, and the result follows from Theorem 1.1. □

3. *Cliques in Graphs Excluding a Minor*

A critical aspect of the proof of our main result (Theorem 1.2) is an upper bound on the number of cliques in a graph excluding a given minor. We prove this bound in this section.

Let G be a graph. A *k-clique* of G is a (not necessarily maximal) set of k pairwise adjacent vertices of G. If every subgraph of G has a vertex of degree at most d, then G is *d-degenerate*. For example, Theorem 2.1 implies that a graph with no K_ℓ-minor is $2^{\ell-2}$-degenerate.

We have the following crude bound on the number of cliques in a degenerate graph; see Wood [2007] and Norine et al. [2006] for similar results.

LEMMA 3.1. *A d-degenerate graph G with no k-clique has fewer than $d^{k-1} \cdot |G|$ cliques.*

PROOF. Since G is d-degenerate, we can order the vertices so that each vertex v has at most d neighbors to the left of v. Thus for all $i \in [k-1]$, every vertex is the rightmost vertex of at most $\binom{d}{i-1} \leq d^{i-1}$ cliques on i vertices. Thus every vertex is the rightmost vertex of at most $\sum_{i=1}^{k-1} d^{i-1} < d^{k-1}$ cliques. The result follows. □

For example, a graph G with no K_ℓ-minor has fewer than $2^{(\ell-2)(\ell-1)} \cdot |G|$ cliques.

LEMMA 3.2. *Given a graph G with no k-clique and at least $2^{(\ell-2)(k-1)} \cdot |G|$ cliques for some $\ell, k \in \mathbb{Z}^+$, a K_ℓ-minor of G can be computed in $\mathcal{O}(\ell(|G| + \|G\|))$ time.*

PROOF. By Lemma 3.1 with $d = 2^{\ell-2}$, G is not $2^{\ell-2}$-degenerate. By Lemma A.1 in Appendix A, a subgraph H of G with minimum degree greater than $2^{\ell-2}$ can be computed in $\mathcal{O}(|G| + \|G\|)$ time. Now $\|H\| > 2^{\ell-3} \cdot |H|$.

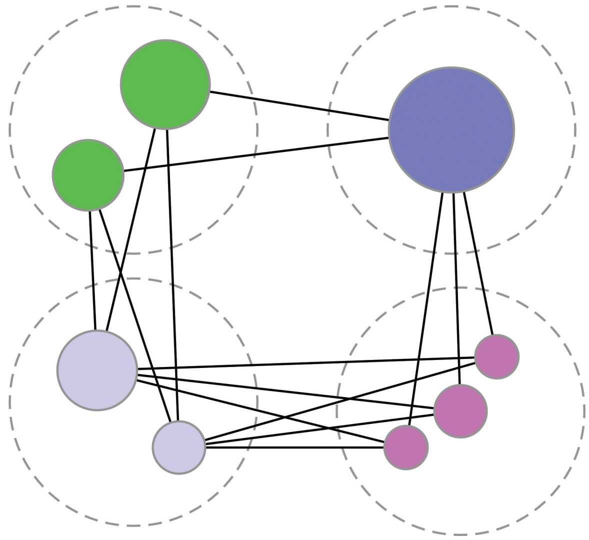

FIG. 3. A knitted C_4-partition; each disc represents a connected component of a part of the partition.

Thus, by Theorem 2.1, a K_ℓ-model in H, and hence in G, can be computed in $\mathcal{O}(\ell(|H| + \|H\|))$ time. $\square$

4. *Proof of Theorem 1.2*

Let G and H be graphs. An H-*partition* of G is a proper partition $\{S_v \subseteq V(G) : v \in V(H)\}$ of $V(G)$ indexed by the vertices of H, such that for all distinct $v, w \in V(H)$, we have $vw \in E(H)$ if and only if there is an edge of G between S_v and S_w. Let G_v denote the induced subgraph $G[S_v]$ for each $v \in V(H)$. An H-partition of G is *knitted* if for all distinct $v, w \in V(H)$, we have $vw \in E(H)$ if and only if there is an edge of G between each component of G_v and each component of G_w, as illustrated in Figure 3.

The following lemma, proved shortly, is the heart of the proof of our main result (Theorem 1.2).

LEMMA 4.1. *There is an algorithm that, given $\ell, k \in \mathbb{Z}^+$ and a graph G, outputs a knitted H-partition of G in time $\mathcal{O}(2^{2\ell} \cdot |G| + \|G\|)$, such that either:*

(a) *H contains a K_ℓ-model* (*which is also output*)*, or*

(b) *$|H| \leq 2^{\ell^2+2} \cdot |G| \cdot k^{-1}$, and $|G_x| < 2k$ for all $x \in V(H)$.*

Recall the main result of the article.

THEOREM 1.2. *There is an algorithm that, given $\epsilon \in [0, \frac{1}{2}]$, $\ell \in \mathbb{Z}^+$, and a weighted graph (G, w), outputs either:*

(a) *a K_ℓ-model of G, or*

(b) *a $\frac{2}{3}$-separation of (G, w) of order at most $\ell^{3/2} \cdot 2^{(\ell^2+4)/2} \cdot |G|^{(2-\epsilon)/3}$ in time $\mathcal{O}(\ell \cdot 2^{(3\ell^2+2\ell+6)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|)$.*

PROOF OF THEOREM 1.2 ASSUMING LEMMA 4.1. Apply Lemma 4.1 with $k = \lfloor |G|^{(1-2\epsilon)/3} \rfloor$. We obtain a knitted H-partition of G.

First suppose that case (a) in Lemma 4.1 holds. Thus H contains a K_ℓ-model $\{S_1, S_2, \ldots, S_\ell\}$, where each S_i is a connected subgraph of H. Choose a connected component Z_v of G_v for each $v \in V(H)$. For $i \in [\ell]$, let T_i be the induced subgraph $G[\bigcup\{V(Z_v) : v \in V(S_i)\}]$. Since the S_i subgraphs are pairwise disjoint, the T_i subgraphs are pairwise disjoint. Since each S_i is connected in H and each Z_v is connected in G, each T_i subgraph is connected in G. Since the S_i subgraphs are pairwise adjacent, $\{T_1, T_2, \ldots, T_\ell\}$ is a K_ℓ-model of G, and case (a) in Theorem 1.2 is satisfied.

Now assume that case (b) in Lemma 4.1 holds. Then

$$|H| \leq 2^{\ell^2+2} \cdot |G| \cdot k^{-1} \leq 2^{\ell^2+2} \cdot |G|^{2(1+\epsilon)/3},$$

and for all $x \in V(H)$,

$$|G_x| < 2k \leq 2|G|^{(1-2\epsilon)/3}.$$

Let $w(v) := w(G_v)$ for all $v \in V(H)$. Apply Theorem 2.3 to (H, w). The time complexity is

$$\begin{aligned}\mathcal{O}(\ell \cdot 2^\ell \cdot |H|^{3/2} + \ell \cdot \|H\|) &\subseteq \mathcal{O}(\ell \cdot 2^\ell \cdot (2^{\ell^2+2} \cdot |G|^{2(1+\epsilon)/3})^{3/2} + \ell \cdot \|G\|) \\ &\subseteq \mathcal{O}(\ell \cdot 2^{(3\ell^2+2\ell+6)/2} \cdot |G|^{1+\epsilon} + \ell \cdot \|G\|).\end{aligned}$$

We obtain either a K_ℓ-model of H, or a $\frac{2}{3}$-separation of H with order at most $\ell^{3/2} \cdot |H|^{1/2}$. In the first case, G contains a K_ℓ-model as proved before, and we are done.

Now assume that Theorem 2.3 gives a $\frac{2}{3}$-separation $\{A, B\}$ of (H, w) with order

$$\begin{aligned}|A \cap B| \leq \ell^{3/2} \cdot |H|^{1/2} &\leq \ell^{3/2} \cdot (2^{\ell^2+2} \cdot |G|^{2(1+\epsilon)/3})^{1/2} \\ &\leq \ell^{3/2} \cdot 2^{(\ell^2+2)/2} \cdot |G|^{(1+\epsilon)/3}.\end{aligned}$$

Let $X := \bigcup\{V(G_v) : v \in A\}$ and $Y := \bigcup\{V(G_v) : v \in B\}$. Then $\{X, Y\}$ is a separation of G. Since $|G_v| < 2|G|^{(1-2\epsilon)/3}$ the order of this separation is

$$\begin{aligned}|X \cap Y| = \sum_{v \in A \cap B} |G_v| &\leq \ell^{3/2} \cdot 2^{(\ell^2+2)/2} \cdot |G|^{(1+\epsilon)/3} \cdot 2|G|^{(1-2\epsilon)/3} \\ &\leq \ell^{3/2} \cdot 2^{(\ell^2+4)/2} \cdot |G|^{(2-\epsilon)/3}.\end{aligned}$$

We have $w(X - Y) = w(A - B) \leq \frac{2}{3}w(H) = \frac{2}{3}w(G)$. Similarly $w(B - A) \leq \frac{2}{3}w(G)$. Therefore $\{X, Y\}$ is a $\frac{2}{3}$-separation of G. □

It remains to prove Lemma 4.1.

PROOF OF LEMMA 4.1. *Step* 1. *Initial Partition*: Using a linear-time breadth-first search algorithm, compute a maximal set $\mathcal{A}$ of pairwise disjoint subsets of $V(G)$, such that $G[S]$ is connected and $|S| = k$ for each $S \in \mathcal{A}$. Let $\mathcal{B}$ be the set of vertex sets of the connected components of $G - \bigcup\{S : S \in \mathcal{A}\}$. Then $\mathcal{A} \cup \mathcal{B}$ is a partition of $V(G)$.

Step 2. *Constuction of* H: Let H be the graph such that $\mathcal{A} \cup \mathcal{B}$ is an H-partition of G. Since G_v is connected for each $v \in V(H)$, this H-partition is knitted. Let $A := \{v \in V(H) : V(G_v) \in \mathcal{A}\}$ and $B := \{v \in V(H) : V(G_v) \in \mathcal{B}\}$. A vertex v of

H is *big* if $|G_v| \geq k$. A vertex v of H is *small* if $|G_v| < k$. By construction, every vertex in A is big, B is an independent set of H, and every vertex in B is small.

Step 3. *Partition of* B: Partition $B = C \cup D \cup E$ as follows.

$$C := \{v \in B : \deg_H(v) \geq 2^{\ell-2}\}$$
$$D := \{v \in B : \ell - 1 \leq \deg_H(v) < 2^{\ell-2}\}$$
$$E := \{v \in B : \deg_H(v) \leq \ell - 2\}$$

Suppose that $|C| \geq |A|$. Then $H[C \cup A]$ has at least $2^{\ell-2} \cdot |C|$ edges and at most $2|C|$ vertices. By Theorem 2.1, a K_ℓ-model of $H[C \cup A]$ can be computed in $\mathcal{O}(\ell \cdot |G|)$ time, and we are done. Now assume that $|C| < |A|$.

Step 4. *Assignment*: "Assign" vertices in $D \cup E$ to pairs of distinct vertices in A as follows. Let $\binom{A}{2} := \{\{x, y\} : x, y \in A \text{ and } x \neq y\}$ be the set of pairs of distinct vertices in A. Let Q be the bipartite graph with vertex set $V(Q) := \binom{A}{2} \cup (D \cup E)$, where $\{x, y\} \in \binom{A}{2}$ is adjacent to $v \in D \cup E$ in Q if and only if $x, y \in N_H(v)$. Since each vertex in $D \cup E$ has degree at most $2^{\ell-2}$ in H, each vertex in $D \cup E$ has degree at most $2^{2\ell-4}$ in Q, and Q can be constructed in $\mathcal{O}(2^{2\ell} \cdot |G|)$ time.

Now apply the following greedy algorithm to construct a maximal matching M in Q. (M need not be maximum.) Formally, M is a partial function from $V(Q)$ to $E(Q)$, with M initially undefined everywhere. For each vertex $v \in D \cup E$ in arbitrary order, if v is incident to an edge $\{\{x, y\}, v\} \in E(Q)$, such that no edge in M is incident to $\{x, y\}$, then add (one such edge) $\{\{x, y\}, v\}$ to M. Formally, if $M(\{x, y\})$ is undefined for some edge $e = \{\{x, y\}, v\} \in E(Q)$, then set $M(\{x, y\}) := M(v) := e$. We say that v is *assigned* to the pair $\{x, y\}$. Since each vertex in $D \cup E$ has degree at most $2^{2\ell-4}$ in Q, this step can be implemented in $\mathcal{O}(2^{2\ell} \cdot |G|)$ time.

Suppose that there is a vertex $v \in D$ that is not assigned; that is, $M(v)$ is undefined. Let $\{x_1, x_2, \ldots, x_d\}$ be the neighborhood of v. Then $d \geq \ell - 1$. Thus for all distinct $i, j \in [d]$, there is a distinct vertex $v_{i,j} \in D \cup E$ that is assigned to the pair $\{x_i, x_j\}$, and $v_{i,j}$ is adjacent to both x_i and x_j. In the graph obtained from H by contracting each edge $x_i v_{i,j}$, the subgraph $\{x_1, x_2, \ldots, x_d, v\}$ is a clique on $d+1 \geq \ell$ vertices. Thus H contains a K_ℓ-model, and we are done. This K_ℓ-model can be computed in $\mathcal{O}(2^{2\ell})$ time (since $d < 2^\ell$, and the vertex assigned to a given pair $\{x_i, x_j\}$ can be determined from M in $\mathcal{O}(1)$ time). Hence this step has time complexity $\mathcal{O}(|G| + 2^{2\ell})$. Now assume that every vertex in D is assigned.

Let E^* be the set of assigned vertices in E. Consider the graph obtained from $H[A \cup D \cup E^*]$ by contracting the edge vx for each $v \in D \cup E^*$ assigned to the pair $\{x, y\}$. This graph has $|A|$ vertices and at least $|D| + |E^*|$ edges. Thus if $|D| + |E^*| \geq 2^{\ell-3} \cdot |A|$, then by Theorem 2.1, H contains a K_ℓ-model that can be computed in $\mathcal{O}(\ell \cdot |G|)$ time, and we are done. Now assume that $|D| + |E^*| < 2^{\ell-3} \cdot |A|$.

In total, Step 4 has $\mathcal{O}(2^{2\ell} \cdot |G|)$ time complexity.

Step 5. *Handling Unassigned Vertices in* E: Partition

$$E - E^* = \bigcup\{P_1, P_2, \ldots, P_s\}$$

such that for all $u, v \in E - E^*$, we have $N(u) = N(v)$ if and only if both $u, v \in P_i$ for some $i \in [s]$. By Lemma A.2 in Appendix A, since every vertex in $E - E^*$ has degree at most $\ell-2$ in H, this partition can be computed in $\mathcal{O}(\ell \cdot |H|)$ time. For all $i \in [s]$, partition $P_i = \bigcup\{P_{i,1}, P_{i,2}, \ldots, P_{i,t_i}\}$ such that

$$k \le \left|\bigcup\{G_v : v \in P_{i,j}\}\right| < 2k \quad \text{for all } j \in [t_i - 1], \text{and}$$
$$\left|\bigcup\{G_v : v \in P_{i,t_i}\}\right| < k.$$

This is possible since $|G_v| < k$ for all $v \in P_i$, and can trivially be implemented in $\mathcal{O}(|H|)$ time.

We now determine a new partition of G indexed by a graph H' constructed from H. Collapse each set $P_{i,j}$ of vertices in H into a single vertex $p_{i,j}$ in H', whose associated subgraph in G is $G_{p_{i,j}} := \bigcup\{G_v : v \in P_{i,j}\}$. The parts A, C, D, and E^* remain unchanged in H'. Since the vertices in $P_{i,j}$ have the same neighborhood, $\{G_v : v \in V(H')\}$ is a knitted partition of G. Let $E_{\text{big}} = \{p_{i,j} : i \in [s],\ j \in [t_i - 1]\}$ and $E_{\text{small}} = \{p_{i,t_i} : i \in [s]\}$. Then every vertex in E_{big} is big and every vertex in E_{small} is small.

Suppose that $|E_{\text{small}}| \ge 2^{\ell^2} \cdot |A|$. Let X be the graph with vertex set A obtained by adding a clique with vertex set $N_{H'}(v)$ for each vertex $v \in E_{\text{small}}$. Since each such vertex v has degree at most ℓ, the graph X can be constructed in $\mathcal{O}(\ell^2|H'|)$ time.

We now use this auxillary graph X to show that, in this case, H' contains a K_ℓ-minor. By construction, X has $|A|$ vertices and at most $\ell^2 \cdot |H|$ edges, and since distinct vertices in E_{small} have distinct neighborhoods, X has at least $|E_{\text{small}}| \ge 2^{\ell^2} \cdot |A|$ cliques. Thus by Lemma 3.2, a K_ℓ-model of X can be computed in time $\mathcal{O}(\ell \cdot (|X| + \|X\|))$, which is $\mathcal{O}(\ell^3 \cdot |H|)$.

For every edge $x_i x_j$ in this K_ℓ-model in X, we have $x_i, x_j \in N(v)$ for some $v \in E_{\text{small}}$. Since v is not assigned, there is a vertex $u \in D \cup E^*$ assigned to $\{x_i, x_j\}$, and u is adjacent to both x_i and x_j. In particular, $M(\{x_i, x_j\}) = \{\{x_i, x_j\}, u\}$ and u can be computed in $\mathcal{O}(1)$ time. Since u is not in the K_ℓ-model, we can include u in the connected subgraph of the K_ℓ-model that contains x_i or x_j, to obtain a K_ℓ-model in $H'[A \cup D \cup E^*]$ (without the edge $x_i x_j$), and we are done. Now assume that $|E_{\text{small}}| < 2^{\ell^2} \cdot |A|$.

In total, Step 5 has time complexity $\mathcal{O}(\ell^2 \cdot |H| + \ell \cdot (|X| + \|X\|)) \le \mathcal{O}(\ell^3 \cdot |G|)$,

Step 6. *Wrapping Up*: As illustrated in Figure 4, we have now partitioned $V(H')$ into sets $A \cup E_{\text{big}}$ of big vertices, and sets $C \cup D \cup E^* \cup E_{\text{small}}$ of small vertices. We have proved that $|C| < |A|$, $|D| + |E^*| < 2^{\ell-3} \cdot |A|$, and $|E_{\text{small}}| < 2^{\ell^2} \cdot |A|$. Thus the number of small vertices is less than $(1 + 2^{\ell-3} + 2^{\ell^2}) \cdot |A|$. By definition, the number of big vertices in H' is at most $|G| \cdot k^{-1}$. In particular, $|A| \le |G| \cdot k^{-1}$. Thus

$$|H'| \le (1+2^{\ell-3}+2^{\ell^2}) \cdot |A| + |G| \cdot k^{-1} \le (2+2^{\ell-3}+2^{\ell^2}) \cdot |G| \cdot k^{-1} \le 2^{\ell^2+2} \cdot |G| \cdot k^{-1}.$$

Moreover, $|H'_v| < 2k$ for every vertex $v \in V(H')$.

The time complexity is $\mathcal{O}(\ell \cdot |G| + \|G\|)$ for Steps 1–3, plus $\mathcal{O}(2^{2\ell} \cdot |G|)$ for Step 4, plus $\mathcal{O}(\ell^3 \cdot |G|)$ for Step 5. Thus the total time complexity is $\mathcal{O}(2^{2\ell} \cdot |G| + \|G\|)$. □

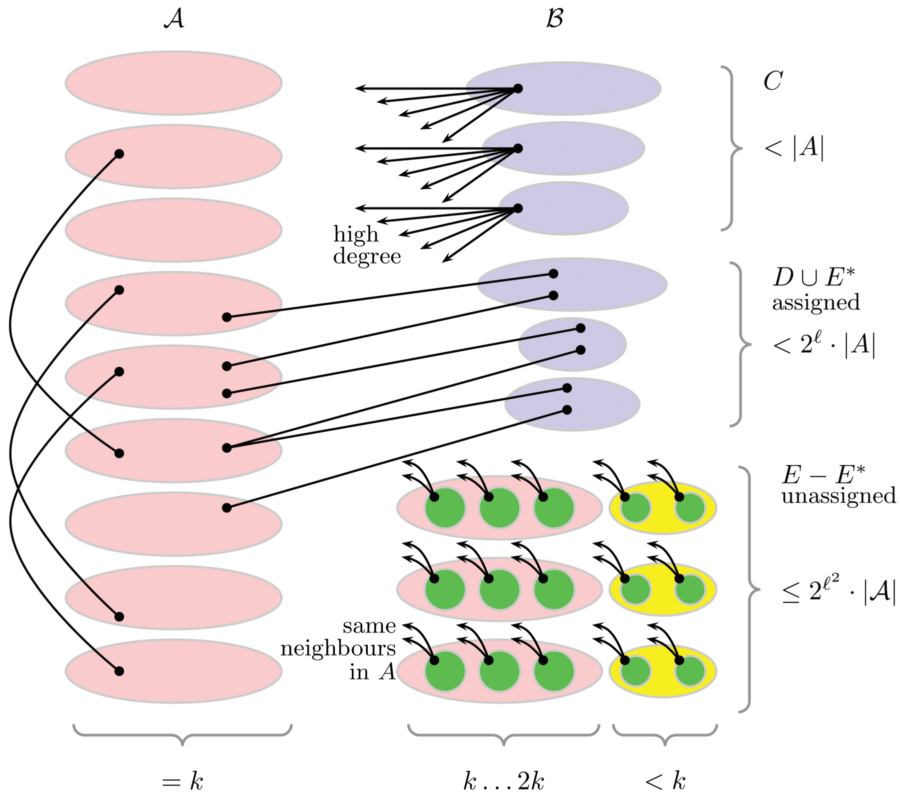

FIG. 4. The partition of $V(G)$ in the proof of Lemma 4.1.

5. *Application: Independent Sets*

The cardinality of a maximum independent set in a graph G is denoted by $\alpha(G)$. Determining whether $\alpha(G) \geq k$ is a classical $\mathcal{NP}$-complete problem, and is even hard to approximate in general [Engebretsen and Holmerin 2000; Håstad 1999]. On the other hand, Lipton and Tarjan [1980] showed that separators can be used as the basis for an approximation algorithm for finding independent sets in planar graphs. Using similar ideas, Alon et al. [1990a] outlined an $\mathcal{O}(|G|^{1/2} \cdot \|G\|)$-time approximation algorithm for finding an independent set in a graph excluding a fixed minor. We improve the time complexity of their algorithm to nearly linear as follows.

THEOREM 5.1. *For fixed ℓ, there is an algorithm that, given a graph G with no K_ℓ-minor, computes an approximation to the maximum independent set of G with relative error $\mathcal{O}((\log\log|G|)^{-1/3})$ in time $\mathcal{O}(|G|\log|G| + \|G\|)$.*

The proof of Theorem 5.1 depends on the following lemma.

LEMMA 5.2. *For fixed ℓ, the following algorithm, given $\epsilon \in [0, 1]$ and a weighted graph (G, w) with no K_ℓ-minor and total weight $w(G) \leq 1$, outputs a set S of $\mathcal{O}(|G|^{2/3}\epsilon^{-1/3})$ vertices of G in time $\mathcal{O}(|G|\log|G| + \|G\|)$, such that every connected component of $G - S$ has weight at most ϵ.*

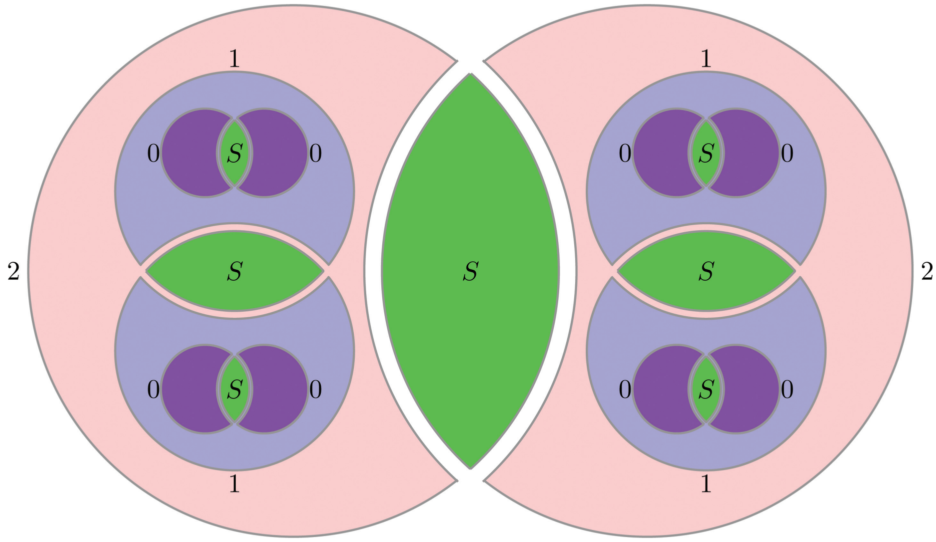

FIG. 5. Illustration of the computation of S in Lemma 5.2.

if $\epsilon \leq |G|^{-1}$ **then**
  $S := V(G)$
else
  $S := \emptyset$
  while there is a component P of $G - S$ with weight exceeding ϵ **do**
    let $\{A, B\}$ be a separation of P determined by Theorem 1.2 (with $\epsilon = 0$)
    $S := S \cup (A \cap B)$
  end while
end if
output S

PROOF. If $\epsilon \leq |G|^{-1}$ then $S := V(G)$ satisfies the requirements. Now assume that $\epsilon > |G|^{-1}$. Consider a component P of $G - S$ at some stage of the algorithm. If P is a component of $G - S$ at the termination of the algorithm, then we say P has *level* 0. Otherwise Theorem 1.2 was applied to P at same stage, to obtain a separation $\{A, B\}$ of P. Thus $w(A - B) \leq \frac{2}{3}w(P)$ and $w(B - A) \leq \frac{2}{3}w(P)$. Each component of $P - (A \cap B)$ is also a component of $G - S$ at some stage of the algorithm. Define the *level* of P to be 1 plus the maximum level of a component of $P - (A \cap B)$. Observe that two components with the same level are disjoint.

Each level 1 component has weight greater than ϵ, and in general, each level-i component has weight at least $(\frac{3}{2})^{i-1}\epsilon$. Since the total weight of G is at most 1, there are at most $(\frac{2}{3})^{i-1}\epsilon^{-1}$ level-i components. Let k be the maximum level. Then $1 \leq (\frac{2}{3})^{k-1}\epsilon^{-1} \leq (\frac{2}{3})^{k-1}|G|$, which implies that $k \leq 1 + \log_{3/2}|G|$. Since the time complexity of Theorem 1.2 is linear for fixed ℓ, and since two components at the same level are disjoint, the total time complexity is $\mathcal{O}(|G| \log |G| + \|G\|)$.

It remains to prove the upper bound on $|S|$. Let $P_1, P_2, \ldots, P_t$ be the components at level i. By Theorem 1.2, the number of vertices added to S by splitting $P_1, P_2, \ldots, P_t$ is at most $\mathcal{O}(\sum_{j=1}^{t} |P_j|^{2/3})$. We have $t \leq (\frac{2}{3})^{i-1}\epsilon^{-1}$ and $\sum_{j=1}^{t} |P_j| \leq |G|$. For fixed t, the sum $\sum_{j=1}^{t} |P_j|^{2/3}$, subject to $\sum_{j=1}^{t} |P_j| \leq |G|$, is maximized when $|P_j| = |G| \cdot t^{-1}$ for all j. Thus

$$\sum_{j=1}^{t} |P_j|^{2/3} \leq \sum_{j=1}^{t} (|G| \cdot t^{-1})^{2/3} = t^{1/3} \cdot |G|^{2/3} \leq \left(\left(\tfrac{2}{3}\right)^{i-1} \epsilon^{-1} \right)^{1/3} \cdot |G|^{2/3}.$$

Hence

$$|S| \in \mathcal{O}\left(\sum_{i=1}^{k} \left(\tfrac{2}{3}\right)^{(i-1)/3} \cdot \epsilon^{-1/3} \cdot |G|^{2/3} \right) \subseteq \mathcal{O}(|G|^{2/3} \epsilon^{-1/3}). \qquad \square$$

PROOF OF THEOREM 5.1. Apply Lemma 5.2 with $\epsilon := (\log_2 \log_2 |G|) \cdot |G|^{-1}$, and with each vertex having weight $|G|^{-1}$. We obtain a set S of $\mathcal{O}(|G| \cdot (\log \log |G|)^{-1/3})$ vertices of G such that every component of $G - S$ has weight at most ϵ; that is, every component of $G - S$ has at most $\log_2 \log_2 |G|$ vertices. In each component of $G - S$, find a maximum independent set by checking every subset of the vertices. Let I be the union of the independent sets obtained. Then I is an independent set of G.

The restriction of a maximum independent set of G to a component of $G - S$ is at most as large as the restriction of I to the same component. Thus

$$\alpha(G) - |I| \leq |S| \in \mathcal{O}(|G| \cdot (\log \log |G|)^{-1/3}).$$

Duchet and Meyniel [1982] proved that $\alpha(G) \geq |G|/2\ell$. Thus the relative error $(\alpha(G) - |I|)/\alpha(G) \in \mathcal{O}((\log \log |G|)^{-1/3})$.

The computation of S takes $\mathcal{O}(|G| \log |G| + \|G\|)$ time by Lemma 5.2.

For each component P of $G - S$ the second step of the algorithm takes $\mathcal{O}(|P| \cdot 2^{|P|})$ time. Thus in total, the second step takes $\mathcal{O}(\sum_P |P| \cdot 2^{|P|})$ time, which is maximized when all components P have the same maximal number of vertices; that is, when $|P| = \log_2 \log_2 |G|$. Hence the second step takes $\mathcal{O}(|G| \cdot 2^{|P|}) = \mathcal{O}(|G| \log |G|)$ time. $\square$

Appendix

A. *More Algorithmic Details*

This apendix provides details for some elementary algorithms used in the article.

LEMMA A.1. *The following algorithm, given a graph G that is not d-degenerate (for some $d \in \mathbb{R}^+$), outputs a subgraph H of G in time $\mathcal{O}(|G| + \|G\|)$, such that H has minimum degree greater than d.*

```
while there is a vertex v of degree at most d in G do
    delete v from G
end while
output G
```

PROOF. The assumption that G is not d-degenerate means that some subgraph of G has minimum degree greater than d. The algorithm finds such a subgraph since a vertex of degree at most d is in no subgraph of G with minimum degree greater than d. Thus upon termination of the algorithm, the remaining subgraph has minimum degree greater than d.

The algorithm can be implemented in $\mathcal{O}(|G| + \|G\|)$ time by maintaining the degree of each vertex in the current graph, and by maintaining a set L of vertices with degree at most d (represented as a boolean function that indicates whether a given vertex is in L in $\mathcal{O}(1)$ time). Clearly L can be initialized in $\mathcal{O}(|G| + \|G\|)$ time. When deleting a vertex v from G, only a neighbor of v needs its degree to be updated, and only a neighbor of v might need to be added to L. Thus when deleting v, these data structures can be maintained in $\mathcal{O}(\deg(v))$ time. Thus the total time complexity is $\mathcal{O}(|G| + \|G\|)$. □

LEMMA A.2. *There is an algorithm that takes as input a graph G and a set $X \subseteq V(G)$ with $\deg(v) \leq k$ for every vertex $v \in X$, and outputs a partition $S_1, \dots, S_k$ of X such that $v, w \in S_i$ if and only if $N(v) = N(w)$ for all $i \in [k]$. The time complexity is $\mathcal{O}(k \cdot |X|)$.*

PROOF. The following algorithm determines a partial function $f : 2^{V(G)} \to 2^X$, such that $f(S)$ is defined if and only there is a vertex $v \in X$ with $N_G(v) = S$, and in this case, $f(S) = \{v \in X : N_G(v) = S\}$. The set T is the set of all sets $S \subset V(G)$ for which $f(S)$ is defined.

```
T := ∅
for each vertev v ∈ X do
    S := N_G(v)
    if f(S) is defined then
        f(S) := f(S) ∪ {v}
    else
        T := T ∪ {S}
        f(S) := {v}
    end if
end for
for S ∈ T do
    output f(S)
end for
```

Since $\deg(v) \leq k$ for every vertex $v \in X$, we have $|S| \leq k$, and thus it takes $\mathcal{O}(k)$ time to execute each command inside the loops. The inner steps of each loop are executed $\mathcal{O}(|X|)$ times. Thus the total time complexity is $\mathcal{O}(k \cdot |X|)$. □

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