ROOTED K_4-MINORS

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ABSTRACT. Let a, b, c, d be four vertices in a graph G. A *K_4-minor rooted* at a, b, c, d consists of four pairwise-disjoint pairwise-adjacent connected subgraphs of G, respectively containing a, b, c, d. We characterise precisely when G contains a K_4-minor rooted at a, b, c, d by describing six classes of obstructions, which are the edge-maximal graphs containing no K_4-minor rooted at a, b, c, d. The following two special cases illustrate the full characterisation: (1) A 4-connected non-planar graph contains a K_4-minor rooted at a, b, c, d for every choice of a, b, c, d. (2) A 3-connected planar graph contains a K_4-minor rooted at a, b, c, d if and only if a, b, c, d are not on a single face.

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1. INTRODUCTION

Let G and H be graphs[1]. An *H-minor*[2] in G is a set $\{G_x : x \in V(H)\}$ of pairwise disjoint connected subgraphs of G indexed by the vertices of H, such that if $xy \in E(H)$ then some vertex in G_x is adjacent to some vertex in G_y. Each subgraph G_x is called a *branch set* of the minor. A complete graph K_t-minor in G is *rooted* at distinct vertices $v_1, \dots, v_t \in V(G)$ if $v_1, \dots, v_t$ are in distinct branch sets. For brevity, we say that a K_t-minor rooted at $\{v_1, \dots, v_t\}$ is a *$\{v_1, \dots, v_t\}$-minor*. Rooted minors are a significant tool in Robertson and Seymour's graph minor theory [12], and a number of recent papers have studied rooted minors in their own right [4, 7, 21, 22]. Rooted minors are analogous to H-linked graphs for subdivisions; see [2, 8, 9]. This paper considers the question:

When does a given graph contain a K_4-minor rooted at four nominated vertices?

Theorem 15 answers this question by describing six classes of obstructions, which are the edge-maximal graphs containing no K_4-minor rooted at four nominated vertices. The flavour of this result is best introduced by first considering the 3- and 4-connected cases, which are addressed in Sections 3 and 4. First, we survey some definitions and results from the literature that will be employed later in the paper.

2. BACKGROUND

The question of when does a graph contain a K_3-minor rooted at three nominated vertices was answered by Wood and Linusson [22].

Lemma 1 ([22])**.** *For distinct vertices a, b, c in a graph G, either:*

- *G contains an $\{a, b, c\}$-minor, or*
- *for some vertex $v \in V(G)$ at most one of a, b, c are in each component of $G - v$.*

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[1]We consider finite, simple, undirected graphs.

[2]This definition of minor is a more concrete version of the standard definition: H is a *minor* of G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges.

Note that in this lemma it is possible that $v \in \{a, b, c\}$.

For distinct vertices s_1, t_1, s_2, t_2 in a graph G, an (s_1t_1, s_2t_2)-*linkage* consists of an s_1t_1-path and an s_2t_2-path that are disjoint. Seymour [14] and Thomassen [17] independently proved that there is essentially one obstruction for the existence of a linkage, as we now describe; see [3, 5, 6, 10, 15, 16, 18, 20] for related results.

For a graph H, let H^+ denote a graph obtained from H as follows: for each triangle T of H, add a possibly empty clique X_T disjoint from H and adjacent to each vertex in T. We consider H^+ to be implicitly defined by the graph H and the cliques X_T. An (a, b, c, d)-*web* is a graph H^+, where H is an embedded planar graph with outerface (a, b, c, d), such that each internal face of H is a triangle, and each triangle of H is a face. An $\{a, b, c, d\}$-*web* is an (a, b, c, d)-web for some linear ordering (a, b, c, d). That is, in an $\{a, b, c, d\}$-web the vertex ordering around the outerface is not specified.

Lemma 2 ([14, 17]). *For distinct vertices s_1, t_1, s_2, t_2 in a graph G, either:*

- *G contains an (s_1t_1, s_2t_2)-linkage, or*
- *G is a spanning subgraph of an (s_1, s_2, t_1, t_2)-web.*

Lemma 2 implies the following result, first proved by Jung [5].

Lemma 3 ([5]). *For distinct vertices s_1, s_2, t_1, t_2 in a 4-connected graph G, either:*

- *G contains an (s_1t_1, s_2t_2)-linkage, or*
- *G is planar and s_1, s_2, t_1, t_2 are on some face in this order.*

Lemma 3 makes sense since every 3-connected planar graph has a unique planar embedding up to the choice of outerface [19]. We implicitly use this fact throughout the paper.

We now describe our first obstruction for a graph to contain a rooted K_4-minor.

Lemma 4. *Every (a, b, c, d)-web G contains no $\{a, b, c, d\}$-minor.*

First proof. Since G is an (a, b, c, d)-web, G contains no (ac, bd)-linkage [14, 17]. But if G contains a K_4-minor A, B, C, D respectively rooted at a, b, c, d, then some ac-path (contained in $A \cup C$) is disjoint from some bd-path (contained in $B \cup D$). Thus G contains no $\{a, b, c, d\}$-minor. □

Second proof. Suppose G contains an $\{a, b, c, d\}$-minor. Since G is connected, we may assume that every vertex is in some branch set. Contracting each edge with both endpoints in the same branch set produces an outerplanar K_4, which is a contradiction. □

We will need the following result by Dirac [1].

Lemma 5 ([1]). *For every set S of k vertices in a k-connected graph G, there is a cycle in G containing S.*

3. The 4-Connected Case

The following result characterises when a 4-connected graph contains a rooted K_4-minor. It is analogous to Lemma 3.

Theorem 6. *For distinct vertices a, b, c, d in a 4-connected graph G, either:*

- *G contains an $\{a, b, c, d\}$-minor, or*
- *G is planar and a, b, c, d are on a common face.*

Proof. Lemma 4 implies that if G contains an $\{a,b,c,d\}$-minor, then the second outcome does not occur. To prove the converse, assume that G is non-planar, or if G is planar then a,b,c,d are not on a common face. Since G is 4-connected, by Lemma 5, G contains a cycle C through a,b,c,d. Without loss of generality, a,b,c,d appear in this order in C. By Lemma 3, G contains an (ac,bd)-linkage. The result follows from Lemma 7 below. □

Lemma 7. *Let C be a cycle in a graph G containing vertices a,b,c,d in this order. If G contains an (ac,bd)-linkage then G contains an $\{a,b,c,d\}$-minor.*

Proof. Let G be a counterexample firstly with $|V(G)|$ minimum and then with $|E(G)|$ minimum. If $V(G)=\{a,b,c,d\}$ then $G\cong K_4$. Now assume that $|V(G)|\geq 5$, and the result holds for graphs with less than $|V(G)|$ vertices, or with $|V(G)|$ vertices and less than $|E(G)|$ edges.

Let P be an ac-path disjoint from some bd-path Q. Let R_{ab} be the ab-path contained in C avoiding c and d. Similarly define R_{bc}, R_{cd} and R_{da}. If some vertex or edge x is not in $P\cup Q\cup C$, then $G-x$ is not a counterexample, and thus contains an $\{a,b,c,d\}$-minor. Now assume that $G=P\cup Q\cup C$. We show that contracting some edge gives a graph that satisfies the hypothesis.

Suppose that some vertex v has degree 2. For at least one edge e incident to v, the endpoints of e are not both in $\{a,b,c,d\}$. Thus the contraction G/e satisfies the hypothesis, and G/e and hence G contains an $\{a,b,c,d\}$-minor. Now assume that every vertex has degree at least 3. Thus $V(G)=V(C)=V(P\cup Q)$.

Colour P red, and colour Q blue. Suppose that consecutive vertices u and v in C receive the same colour. Then G/uv satisfies the hypothesis, as illustrated in Figure 1 in the case that u and v are red. By the choice of G, G/uv and thus G contains an $\{a,b,c,d\}$-minor. Now assume that the colours alternate around C. In particular, $|V(P)|=|V(Q)|$. If $P=ac$ then $Q=bd$ and and we are done. Now assume that P contains some internal vertex.

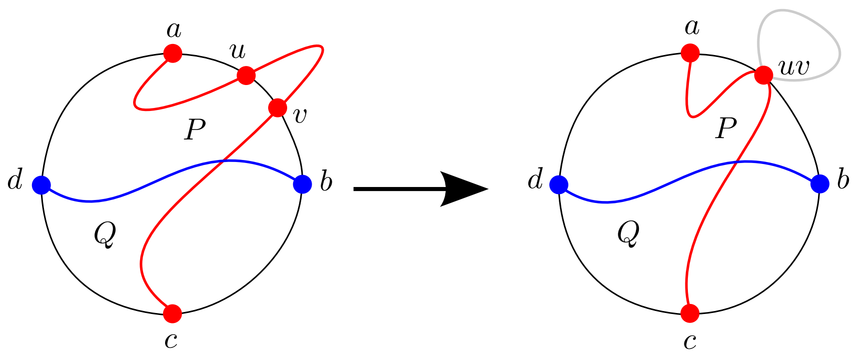

FIGURE 1. If consecutive vertices u and v in C receive the same colour then contract uv.

Let v be the neighbour of a in P, and let w be the neighbour of c in P. If v is in $R_{da}\cup R_{ab}$, then G/av satisfies the hypothesis, as illustrated in Figure 2. By the choice of G, G/av and thus G contains an $\{a,b,c,d\}$-minor. Now assume that $v\in R_{bc}\cup R_{cd}$. Similarly, $w\in R_{da}\cup R_{ab}$. Since P and Q are disjoint, $v\in R_{bc}\cup R_{cd}\setminus\{b,d\}$ and $w\in R_{da}\cup R_{ab}\setminus\{b,d\}$. Thus $v\neq w$. That is, P (and Q also) contains at least two internal vertices. Label v and a by "a". Label every other vertex in P by "c".

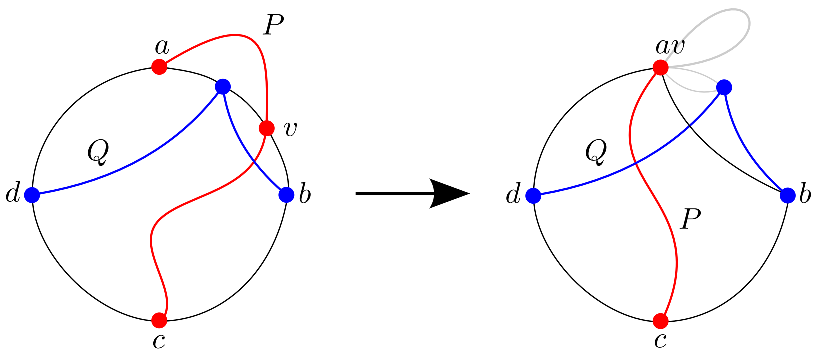

FIGURE 2. If v is in $R_{da} \cup R_{ab}$ then contract av.

Let x be the neighbour of v between v and c in $R_{bc} \cup R_{cd}$. Let y be the neighbour of a between w and a in $R_{da} \cup R_{ab}$. Since the colours around C alternate, x and y are in Q. Without loss of generality, b, x, y, d appear in this order in Q. Label the yd-subpath of Q by "d", and label the remaining vertices in Q (including x) by "b". Thus x, which is labelled "b", is adjacent to some vertex in Q labelled "d". The neighbours of x in C are labelled "a" and "c", and the neighbours of y in C are labelled "a" and "c". The sets of vertices labelled "a","b","c","d" form pairwise disjoint subpaths of P or Q respectively containing a, b, c, d. Thus contracting the vertices with the same label into a single vertex gives an $\{a, b, c, d\}$-minor in G, as illustrated in Figure 3. □

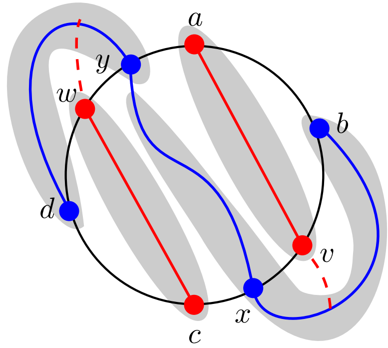

FIGURE 3. Construction of a rooted K_4-minor in Lemma 7.

4. The 3-Connected Case

We have the following characterisation for 3-connected graphs.

Theorem 8. *The following are equivalent for distinct vertices a, b, c, d in a 3-connected graph G:*

(1) *G contains an $\{a, b, c, d\}$-minor,*
(2) *G is not a spanning subgraph of an $\{a, b, c, d\}$-web,*
(3) *G contains an (ab, cd)-linkage, an (ac, bd)-linkage, and an (ad, bc)-linkage.*

Proof. Lemma 4 implies (1) $\Longrightarrow$ (2). Lemma 2 implies (2) $\Longrightarrow$ (3). It remains to prove (3) $\Longrightarrow$ (1). First suppose that some cycle C contains a, b, c, d. Without loss of generality assume that the order of the vertices in C is (a, b, c, d). Since G contains an (ac, bd)-linkage, by Lemma 7, G contains an $\{a, b, c, d\}$-minor. Now assume that no cycle contains a, b, c, d. By Lemma 5, since G is 3-connected, G contains a cycle C through a, b, c. Colour red the vertices in the ab-path in C that avoids c. Likewise colour blue the vertices in the bc-path in C that avoids a. And colour green the vertices in the ca-path in C that avoids b. Note that a, b and c each receive two colours. By Menger's Theorem there exists three paths from d to C, such that each path intersects C in one vertex, and any two of the paths only intersect at d. Colour each path with the colour of its vertex in C. If two paths receive the same colour, then we obtain a cycle through a, b, c, d, as illustrated in Figure 4(a). Now assume that no two paths receive the same colour. In this case we obtain an $\{a, b, c, d\}$-minor, as illustrated in Figure 4(b). □

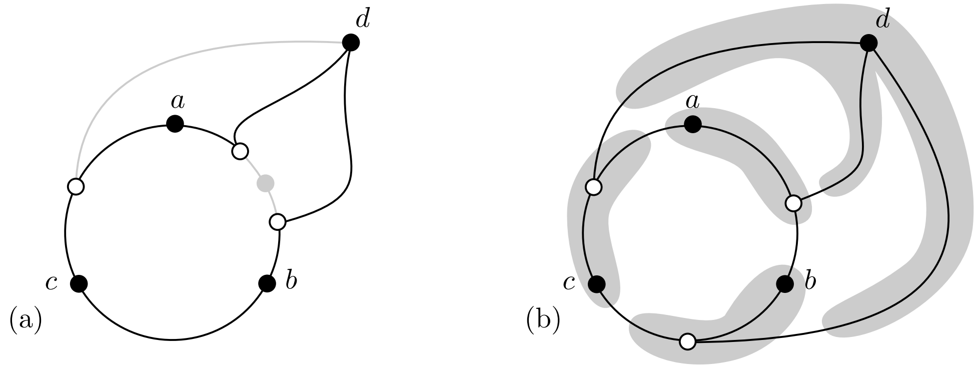

FIGURE 4. Finding a rooted K_4-minor in a 3-connected graph.

Note that Theorem 8 does not hold for 2-connected graphs. For example, $K_{2,3}$ with colour classes $\{a, b, c\}$ and $\{d, v\}$ contains an (ab, cd)-linkage, an (ac, bd)-linkage, and an (ad, bc)-linkage, but contains no $\{a, b, c, d\}$-minor.

Theorem 8 can be strengthened for 3-connected planar graphs.

Theorem 9. *For distinct vertices a, b, c, d in a 3-connected planar graph G, either:*

- *G contains an $\{a, b, c, d\}$-minor, or*
- *a, b, c, d are on a common face.*

Proof. If a, b, c, d are on a common face, then G is a spanning subgraph of an $\{a, b, c, d\}$-web; thus G contains no $\{a, b, c, d\}$-minor by Lemma 4. For the converse, assume that G contains no $\{a, b, c, d\}$-minor. By Theorem 8, G is a spanning subgraph of H^+ for some planar graph H with outerface $\{a, b, c, d\}$, such that every internal face of H is a triangle. Suppose that for some triangular face $T = (u, v, w)$ of H, at least two vertices $x, y \in X_T$ are adjacent in G to each of u, v, w. Let z be a vertex of H outside of T. There is such a vertex since the outerface has four vertices. Since G is 3-connected, there are three internally disjoint xz-paths, respectively passing through u, v, w. Thus G contains a subdivision of $K_{3,3}$ with colours classes $\{u, v, w\}$ and $\{x, y, z\}$. This contradiction proves that for each triangular face $T = (u, v, w)$ of H, at most one vertex in X_T is adjacent to each of u, v, w in G. If there is such a vertex $x \in X_T$ then move x into H. Observe that H remains planar: the face uvw is replaced by the faces $T_w = (u, v, x)$, $T_v = (u, w, x)$ and $T_u = (v, w, x)$. Each remaining vertex in X_T is now adjacent to at most two of u, v, w (and possibly x). Assign such a vertex to one of $X_{T_u}, X_{T_v}, X_{T_w}$ according to its neighbours in T. Repeat this step until $X_T = \emptyset$ for each triangle T of H. In this case, G is a spanning subgraph of H (not H^+), and a, b, c, d are on a common face of G. □

Corollary 10. *A planar triangulation contains an $\{a, b, c, d\}$-minor for all distinct vertices a, b, c, d.*

5. Reductions

This section describes a number of operations that simplify the search for rooted K_4-minors. The first motivates the definition of H^+.

Lemma 11. *Let a, b, c, d be distinct vertices in a graph H. For each graph H^+, we have H^+ contains an $\{a, b, c, d\}$-minor if and only if H contains an $\{a, b, c, d\}$-minor.*

Proof. Since H is a subgraph of H^+, if H contains an $\{a, b, c, d\}$-minor then so does H^+. For the converse, say A, B, C, D is a K_4-minor in H^+ rooted at a, b, c, d. Let $A' := A \cap H$. Define B', C', D' similarly. Suppose that A' intersects the clique X_T associated with some triangle T of H. Since T separates a and X_T, A' intersects T. Since the vertices in $A \cap T$ are pairwise adjacent, $A \cap H$ is a connected subgraph of H. If two branch sets, say A and B, are adjacent in X_T, then they both contain a vertex in T, and A' and B' are adjacent in H. Thus A', B', C', D' is a K_4-minor in H rooted at a, b, c, d. $\square$

A *separation* of a graph G is an ordered pair (G_1, G_2) of subgraphs of G such that $G = G_1 \bigcup G_2$, and $G_1 \not\subseteq G_2$ and $G_2 \not\subseteq G_1$. So there is no edge between $G_1 - G_2$ and $G_2 - G_1$. The *order* of (G_1, G_2) is $|V(G_1 \cap G_2)|$. If certain vertices in G are nominated, and there are s nominated vertices in G_1 and t nominated vertices in G_2, then (G_1, G_2) is an *(s, t)-separation.*

Lemma 12. *Let a, b, c, d be four nominated vertices in a 2-connected graph G. Let (G_1, G_2) be a $(2, 2)$-separation of G of order 2, such that $a, b \in V(G_1)$ and $c, d \in V(G_2)$. Let $\{u, v\} := V(G_1) \cap V(G_2)$. Let G'_i be the graph obtained from G_i by adding the edge uv. Then G contains an $\{a, b, c, d\}$-minor if and only if G'_1 contains an $\{a, b, u, v\}$-minor or G'_2 contains a $\{u, v, c, d\}$-minor.*

Proof. Since G is 2-connected, G'_2 can obtained from G by contracting G_1 onto the edge uv, and G'_1 can obtained from G by contracting G_2 onto uv. Thus, if G'_1 contains an $\{a, b, u, v\}$-minor or G'_2 contains a $\{u, v, c, d\}$-minor, then G contains an $\{a, b, c, d\}$-minor. For the converse, assume that G contains a K_4-minor A, B, C, D containing a, b, c, d respectively. Grow the branch sets until u and v are in $A \cup B \cup C \cup D$. Without loss of generality, u is in A. Thus v separates b from $\{c, d\}$ in $G - A$. Hence v is in B. Therefore $A \cap G_2, B \cap G_2, C, D$ is a $\{u, v, c, d\}$-minor of G_2. $\square$

Lemma 13. *Let G be a graph with four nominated vertices a, b, c, d, such that $N_G(a) = N_G(b) = \{u, v\}$ for some vertices $u, v \in V(G) \setminus \{a, b, c, d\}$. Let G' be the graph obtained from G by deleting a and b, and adding the edge uv. Then G contains an $\{a, b, c, d\}$-minor if and only if G' contains a $\{u, v, c, d\}$-minor.*

Proof. If G' contains a $\{u, v, c, d\}$-minor, then contracting the edges au and bv gives an $\{a, b, c, d\}$-minor in G. For the converse, say A, B, C, D is a K_4-minor in G respectively rooted at a, b, c, d. Grow the branch sets until u and v are in $A \cup B \cup C \cup D$. If u is in C then v separates $\{a, b\}$ and D, implying v is in D, in which case $A = \{a\}$ and $B = \{b\}$, and A and B are not adjacent. By symmetry, $\{u, v\} \cap (C \cup D) = \emptyset$. Thus $u, v \in A \cup B$. If $u, v \in A$ then A separates b and $C \cup D$. Thus $u \in A$ and $v \in B$, without loss of generality. Hence $A - a, B - b, C, D$ is a $\{u, v, c, d\}$-minor in G'. $\square$

6. Obstructions

Consider the following classes of graphs, each of which contains no K_4-minor rooted at the four nominated vertices. Each graph in each class is called an *obstruction*; see Figure 5.

Class $\mathcal{A}$: Let H be the graph consisting of an edge pq with p nominated, and three nominated vertices adjacent to both p and q. Let $\mathcal{A}$ be the class of all graphs H^+.

Class $\mathcal{B}$: Let H be the graph consisting of an edge pq, and four nominated vertices adjacent to both p and q. Let $\mathcal{B}$ be the class of all graphs H^+.

Class $\mathcal{C}$: Let H be the graph consisting of a triangle uvw, plus two nominated vertices adjacent to u and v, and two nominated vertices adjacent to v and w. Let $\mathcal{C}$ be the class of all graphs H^+.

Class $\mathcal{D}$: Let H be a planar graph with an outerface of four nominated vertices, such that every internal face is a triangle, and every triangle is a face. Let $\mathcal{D}$ be the class of all graphs H^+. (These are the webs.)

Class $\mathcal{E}$: Let H be a planar graph with outerface (p, q, r, s) where p and q are nominated, every internal face is a triangle, and every triangle is a face. Add to H two nominated vertices v and w adjacent to r and s. Let $\mathcal{E}$ be the class of all graphs H^+.

Class $\mathcal{F}$: Let H be a planar graph with outerface (p, q, r, s) where every other face is a triangle and every triangle is a face. Add to H two nominated vertices adjacent to p and q, and two nominated vertices adjacent to r and s. Let $\mathcal{F}$ be the class of all graphs H^+.

The *type* of a nominated vertex x in one of the above obstructions H^+ is defined as follows:

Type-1: $H^+ \in \mathcal{D} \cup \mathcal{E}$, and x is adjacent to some other nominated vertex in H.

Type-2: $H^+ \in \mathcal{A}$, and x has degree 4 in H.

Type-3: $H^+ \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{E} \cup \mathcal{F}$, and x is neither type-1 nor type-2; such a vertex x has degree 2 in H,

Lemma 14. *Every graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$ contains no K_4-minor rooted at the four nominated vertices.*

Proof. Lemma 4 implies the result for a class $\mathcal{D}$ obstruction. Let H^+ be an obstruction in some other class. By Lemma 11, it suffices to prove that H contains no $\{a, b, c, d\}$-minor, where a, b, c, d are the four nominated vertices.

If $H^+ \in \mathcal{A}$ then $H \cong K_{1,1,3}$, in which case contracting an edge incident to the one non-nominated vertex produces $K_4 - e$ or $K_{1,3}$, neither of which are K_4.

For $H^+ \in \mathcal{B} \cup \mathcal{C} \cup \mathcal{E} \cup \mathcal{F}$, Lemma 13 is applicable. In particular, $N_H(a) = N_H(b) = \{u, v\}$ for some vertices $u, v \in V(H) \setminus \{a, b, c, d\}$. Thus if H' is the graph obtained from H by deleting a and b, and adding the edge uv, then H^+ contains an $\{a, b, c, d\}$-minor if and only if H contains an $\{a, b, c, d\}$-minor if and only if H' contains a $\{u, v, c, d\}$-minor.

If $H^+ \in \mathcal{B}$ then $H' \cong K_4 - e$. Thus in each case, H' contains no $\{u, v, c, d\}$-minor, implying that H contains no $\{a, b, c, d\}$-minor. If $H^+ \in \mathcal{C}$ then $H' \in \mathcal{A}$, which has no $\{u, v, c, d\}$-minor as proved above. If $H^+ \in \mathcal{E}$ then $H' \in \mathcal{D}$, which has no $\{u, v, c, d\}$-minor by Lemma 4. If $H^+ \in \mathcal{F}$ then $H' \in \mathcal{E}$, which has no $\{u, v, c, d\}$-minor as proved above. □

7. Main Theorem

We now state and prove the main result of the paper. It characterises when a given graph contains a K_4-minor rooted at four nominated vertices.

Theorem 15. *For every graph G with four nominated vertices, either:*

- *G contains a K_4-minor rooted at the nominated vertices, or*
- *G is a spanning subgraph of a graph in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$*

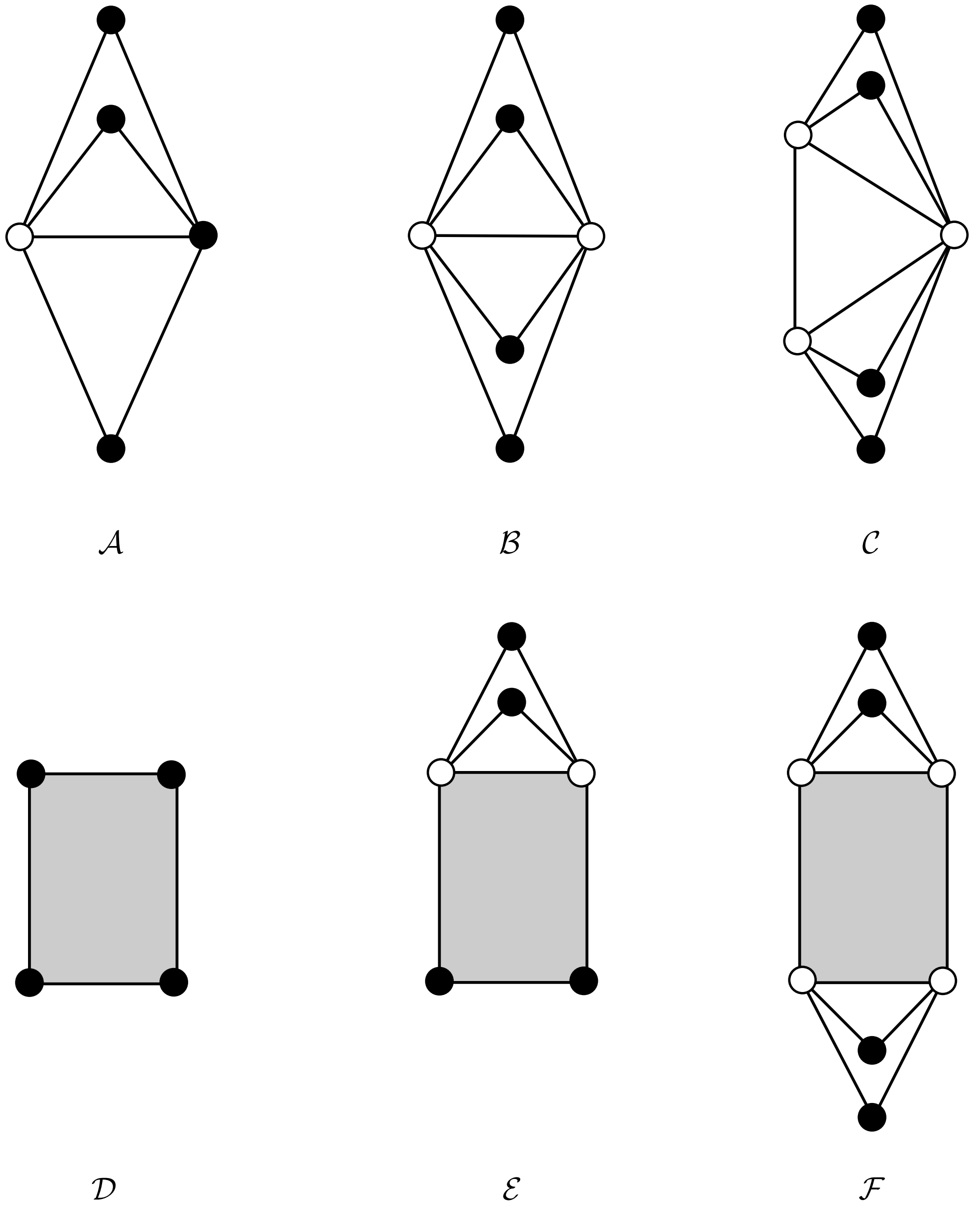

FIGURE 5. The obstructions. Nominated vertices are dark. Non-nominated vertices are white. Shaded regions represent a web. Adjacent to each triangle is an undrawn clique.

Proof. Lemma 14 proves that both outcomes are not simultaneously possible. Suppose on the contrary that for some graph G neither outcome occurs. That is, G contains no K_4-minor rooted at the nominated vertices, and G is not a spanning subgraph of a graph in $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}\cup\mathcal{E}\cup\mathcal{F}$. Choose G firstly with $|V(G)|$ minimum, and then with $|E(G)|$ maximum. Let a, b, c, d be the nominated vertices in G. If $|V(G)| = 4$ then G contains an $\{a, b, c, d\}$-minor if and only if $G \cong K_4$. Otherwise, G is a subgraph of K_4 minus an edge, which is in class $\mathcal{D}$. Now assume that $|V(G)| \geq 5$ and the result holds for every graph G' with $|V(G')| < |V(G)|$, or $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$. We proceed by considering the possible separations in G.

- Suppose there is a $(0, 4)$-separation (G_1, G_2) of order 0: If G_2 contains a K_4-minor rooted at the nominated vertices, then so does G. Otherwise, by the choice of G, G_2 is a spanning subgraph of an obstruction H^+. Adding $V(G_1)$ to X_T for some triangle T of H, we obtain an obstruction containing G as a spanning subgraph, as desired.

- Suppose there is a $(1,3)$-separation (G_1, G_2) of order 0: Let a be the nominated vertex in G_1. Let b, c, d be the nominated vertices in G_2. Thus G contains no ab-path. Hence G contains no $\{a, b, c, d\}$-minor. Let $H := K_4 - ad$ with $V(H) := \{a, b, c, d\}$. Let $X_{abc} := V(G_1) \setminus \{a\}$ and $X_{bcd} := V(G_2) \setminus \{b, c, d\}$. Hence G is a spanning subgraph of H^+, a class $\mathcal{D}$ obstruction.
- Suppose there is a $(2,2)$-separation (G_1, G_2) of order 0: Then as in the proof of the previous case, G contains no $\{a, b, c, d\}$-minor and G is a spanning subgraph of a class $\mathcal{D}$ obstruction.

Now assume that G is connected.

- Suppose that (G_1, G_2) is a $(0,4)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. If G_2 contains an $\{a, b, c, d\}$-minor then so does G, and we are done. Otherwise, by the choice of G, G_2 is a spanning subgraph of an obstruction H^+. Now, u is in $T \cup X_T$ for some triangle T of H. Add $V(G_1) \setminus \{u\}$ to X_T. The resulting graph H^+ is in the same class as the original H^+ and contains G as a spanning subgraph.
- Suppose that (G_1, G_2) is a $(1,3)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. Let a be the nominated vertex in $G_1 - G_2$. If G_2 contains an $\{u, b, c, d\}$-minor, then adding G_1 to the branch set that contains u gives an $\{a, b, c, d\}$-minor in G, and we are done. Otherwise, by the choice of G, G_2 is a spanning subgraph of an obstruction H^+, where u, b, c, d are nominated in G_2.

  If u is type-1, then u is in the outerface of H (as embedded in Figure 5). Let x and y be the two neighbours of u in this outerface. Add a into the outerface of H, adjacent to x, u and y. Thus axu and auy become internal faces of H. Let $X_{axu} := V(G_1) \setminus \{a, u\}$. The resulting graph H^+ contains G as a spanning subgraph, and is in the same class as the original H^+.

  If u is type-2, then H^+ is in class $\mathcal{A}$. Let x be the degree-4 neighbour of u in H. Add a to H adjacent to u and x, thus creating the triangle axu. Let $X_{axu} := V(G_1) \setminus \{a, u\}$. The resulting graph H^+ (with a nominated) is in class $\mathcal{B}$, and contains G as a spanning subgraph.

  If u is type-3, then u is in a unique triangle uxy in H. In H, delete u, add a adjacent to x and y, thus creating the triangle axy. Let $X_{axy} := V(X_{uxy}) \cup V(G_1) \setminus \{a\}$. The resulting graph H^+ (with a nominated) is in the same class as the original H^+, and contains G as a spanning subgraph.
- Suppose that (G_1, G_2) is a $(2,2)$-separation of order 1: Let $\{u\} := V(G_1 \cap G_2)$. Without loss of generality, $a, b \in V(G_1)$ and $c, d \in V(G_2)$. Let H be the planar graph with outerface (a, b, c, d), and one internal vertex u adjacent to a, b, c, d. Let $X_{abu} := V(G_1) \setminus \{a, b, u\}$ and $X_{cdu} := V(G_2) \setminus \{c, d, u\}$. The resulting graph H^+ is in class $\mathcal{D}$, and contains G as a spanning subgraph.
- Suppose that (G_1, G_2) is a $(1,4)$-separation of order 1: Without loss of generality, $a \in V(G_1)$ and $a, b, c, d \in V(G_2)$. If G_2 contains an $\{a, b, c, d\}$-minor then so does G. Otherwise, by the choice of G, G_2 is a spanning subgraph of an obstruction H^+. Now, a is in some triangle T of H. Add $V(G_1) \setminus \{a\}$ to X_T. The resulting graph H^+ is in the same class as the original H^+, and contains G as a spanning subgraph.
- Suppose that (G_1, G_2) is a $(2,3)$-separation of order 1: Without loss of generality, $a, b \in V(G_1)$ and $b, c, d \in V(G_2)$. Let $H := K_4 - ad$ where $V(H) := \{a, b, c, d\}$. Let $X_{abc} :=$

$V(G_1) \setminus \{a,b\}$ and $X_{bcd} := V(G_2) \setminus \{b,c,d\}$. The resulting graph H^+ is in class $\mathcal{D}$, and contains G as a spanning subgraph.

Now assume that G is 2-connected.

- Suppose there is a $(0,4)$-separation (G_1, G_2) of order 2, or a $(1,4)$-separation (G_1, G_2) of order 2, or a $(2,4)$-separation (G_1, G_2) of order 2: Let $\{u,v\} := V(G_1 \cap G_2)$. Let G' be the graph obtained by contracting G_1 onto the edge uv. (This is possible since G is 2-connected.) If G' contains an $\{a,b,c,d\}$-minor then so does G, and we are done. Otherwise, by the choice of G, G' is a spanning subgraph of an obstruction H^+. Since uv is an edge of G', we have $u, v \in T \cup X_T$ for some triangle T of H. Add $V(G_1) \setminus \{u,v\}$ to X_T. The resulting graph H^+ contains G as a spanning subgraph, and is in the same class as the original H^+.
- Suppose there is a $(2,3)$-separation (G_1, G_2) of order 2: Without loss of generality, a is the nominated vertex in $G_1 - G_2$, $\{u,b\} = V(G_1 \cap G_2)$, and c and d are the nominated vertices in $G_2 - G_1$. Let G' be the graph obtained by contracting G_1 onto the edge ub, and nominating u, b, c, d. (This is possible since G is 2-connected.)

  If G' contains a $\{u,b,c,d\}$-minor, then adding $G_1 - b$ to the branch set containing u gives an $\{a,b,c,d\}$-minor in G, and we are done. Otherwise, by the choice of G, G' is a spanning subgraph of some obstruction H^+. Since ub is an edge of G' and both u and b are nominated in G', H^+ is in class $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{E}$.

  If u is type-1, then ub is in the outerface of H (as embedded in Figure 5). Let x be the neighbour of u distinct from b in this outerface. Add a into the outerface of H adjacent to u, b, x, and let $X_{a,u,b} := V(G_1) \setminus \{a,b,u\}$. The resulting graph H^+ is in the same class as the original H^+, and contains G as a spanning subgraph.

  If u is type-2, then $H^+ \in \mathcal{A}$. Add a to H adjacent to u and b, thus creating the triangle aub. Let $X_{aub} := V(G_1) \setminus \{a,u,b\}$. The resulting graph H^+ is in class $\mathcal{E}$, and contains G as a spanning subgraph.

  Now assume that u is type-3. Thus ub is in one triangle ubx in H (since both u and b are nominated in G'). In H, delete u, add a adjacent to x and b creating the triangle axb, and let $X_{axb} := V(X_{ubx}) \cup V(G_1) \setminus \{a,b\}$. The resulting graph H^+ contains G as a spanning subgraph and is in the same class as the original H^+.
- Suppose there is a $(3,3)$-separation (G_1, G_2) of order 2: Without loss of generality, $a \in V(G_1 - G_2)$, $\{b,c\} = V(G_1 \cap G_2)$, and $d \in V(G_2 - G_1)$. Let $H := K_4 - ad$ where $V(H) := \{a,b,c,d\}$. Let $X_{abc} := V(G_1) \setminus \{a,b,c\}$ and $X_{bcd} := V(G_2) \setminus \{b,c,d\}$. The resulting graph H^+ is in class $\mathcal{D}$, and contains G as a spanning subgraph.
- Suppose there is a $(2,2)$-separation (G_1, G_2) of order 2: Let $\{u,v\} := V(G_1 \cap G_2)$. Let G'_i be the graph obtained from G_i by adding the edge uv. Since G is 2-connected, by Lemma 12, if G'_1 contains an $\{a,b,u,v\}$-minor or G'_2 contains a $\{u,v,c,d\}$-minor, then G contains an $\{a,b,c,d\}$-minor, and we are done. Otherwise, by the choice of G, each G'_i is a spanning subgraph of an obstruction H_i^+. Since the nominated vertices u and v are adjacent in G'_1 and G'_2, H_1^+ and H_2^+ are class $\mathcal{A}$, $\mathcal{D}$ or $\mathcal{E}$.

  Consider the case in which $H_1^+ \in \mathcal{D}$. Then the edge uv is either on the outerface of H_1 or is a diagonal of H_1. If uv is a diagonal of H_1 then $H_1 \cong K_4 - ab$ since every triangle of H_1 is a face of H_1. Similarly, if $H_2^+ \in \mathcal{D}$ and uv is a diagonal of H_2, then $H_2 \cong K_4 - cd$.

  Let H^+ be the graph obtained by identifying u, v in H_1^+ with u, v in H_2^+. Thus H^+ contains G as a spanning subgraph. By adding gray edges to H^+ as illustrated in Figure 6, we now show that H^+ is an obstruction. Consider the following cases:

- If $H_1^+ \in \mathcal{A}$ and $H_2^+ \in \mathcal{A}$ then $H^+ \in \mathcal{C}$.
- Say $H_1^+ \in \mathcal{A}$ and $H_2^+ \in \mathcal{D}$. If uv is on the outerface of H_2 then $H^+ \in \mathcal{E}$. Otherwise, uv is a diagonal of H_2, and $H^+ \in \mathcal{C}$.
- If $H_1^+ \in \mathcal{A}$ and $H_2^+ \in \mathcal{E}$ then $H^+ \in \mathcal{F}$.
- Say $H_1^+ \in \mathcal{D}$ and $H_2^+ \in \mathcal{D}$. If uv is on the outerface of H_1 and uv is on the outerface of H_2 then $H^+ \in \mathcal{D}$. If uv is a diagonal of H_1 and uv is on the outerface of H_2 then $H^+ \in \mathcal{E}$. Otherwise, uv is a diagonal of H_1 and uv is a diagonal of H_2, and $H^+ \in \mathcal{B}$.
- Say $H_1^+ \in \mathcal{E}$ and $H_2^+ \in \mathcal{D}$. If uv is on the outerface of H_2 then $H^+ \in \mathcal{E}$. Otherwise, uv is a diagonal of H_2, and $H^+ \in \mathcal{F}$.
- If $H_1^+ \in \mathcal{E}$ and $H_2^+ \in \mathcal{E}$ then $H^+ \in \mathcal{F}$.

Now assume that G is 2-connected and every separation of order 2 is a $(1,3)$-separation. Before addressing this case it will be convenient to first eliminate a particular separation of order 3.

- Suppose there is a separation (G_1, G_2) of order 3 with no nominated vertices in $G_2 - G_1$, such that $|V(G_2)| \geq 5$:

  Let $\{u, v, w\} := V(G_1 \cap G_2)$. We claim that G_2 contains a $\{u, v, w\}$-minor. If not, then by Lemma 1, there is a vertex x such that at most one of u, v, w is in each component of $G_2 - x$. Since $|V(G_2)| \geq 5$ there is a vertex $y \in V(G_2) \setminus \{u, v, w, x\}$. If y is in the same component of $G_2 - x$ as u, then $\{u, x\}$ is a cut-pair that forms a $(0,4)$-separation of order 2 in G. Thus y is not in the same component of $G_2 - x$ as u. Similarly, y is not in the same component of $G_2 - x$ as v or w. Thus x is a cut-vertex, which is a contradiction. Hence G_2 contains a $\{u, v, w\}$-minor. Let G' be the graph obtained from G_1 by adding the triangle uvw. Thus G' is a minor of G, and $|V(G')| < |V(G)|$. If G' contains an $\{a, b, c, d\}$-minor then so does G and we are done. Otherwise, by the choice of G, G' is a spanning subgraph of an obstruction H^+. The triangle uvw is contained in $T \cup X_T$ for some triangle T of H. Add $V(G_2) \setminus \{u, v, w\}$ to X_T. The resulting graph H^+ contains G as a spanning subgraph (since the neighbours of each vertex in $G_2 \setminus \{u, v, w\}$ are in G_2) and is of the same class as the original H^+.

Now assume that if (G_1, G_2) is a separation of order 3 with no nominated vertices in $G_2 - G_1$, then $|V(G_2)| = 4$. We consider the following two types of $(1,3)$-separations.

- Suppose there is a $(1,3)$-separation (G_1, G_2) of order 2, such that $|V(G_1)| \geq 4$, or $|V(G_1)| = 3$ and $G_1 \not\cong K_3$:

  Let a be the nominated vertex in $G_1 - G_2$. Let $\{u, v\} := V(G_1 \cap G_2)$. Let G' be the graph obtained from G_2 by adding the edge uv if it does not already exist, and by adding a new vertex a' adjacent to u and v, where a', b, c, d are nominated in G'. Observe that $|V(G')| < |V(G)|$ or if $|V(G')| = |V(G)|$ then $|E(G')| > |E(G)|$. Thus by the choice of G, G' contains an $\{a', b, c, d\}$-minor, or G' is a spanning subgraph of an obstruction H^+.

  First suppose that G' contains a K_4-minor A', B, C, D respectively rooted at a', b, c, d. Since a' has degree 2 in G', without loss of generality, u is in A'. Now $G_1 - v$ is connected, as otherwise v is a cut-vertex in G. Thus $A := (G_1 - v) \cup A'$ is connected and is disjoint from $B \cup C \cup D$. We claim that A, B, C, D is an $\{a, b, c, d\}$-minor in G. Clearly A, B, C, D respectively contain a, b, c, d. Since the edge uv was added to G', it may be that G' is not a minor of G. So this claim is not immediate. However, if uv is in G then G' is a minor of G, and A, B, C, D is a K_4-minor in G, and we are done. It remains to show

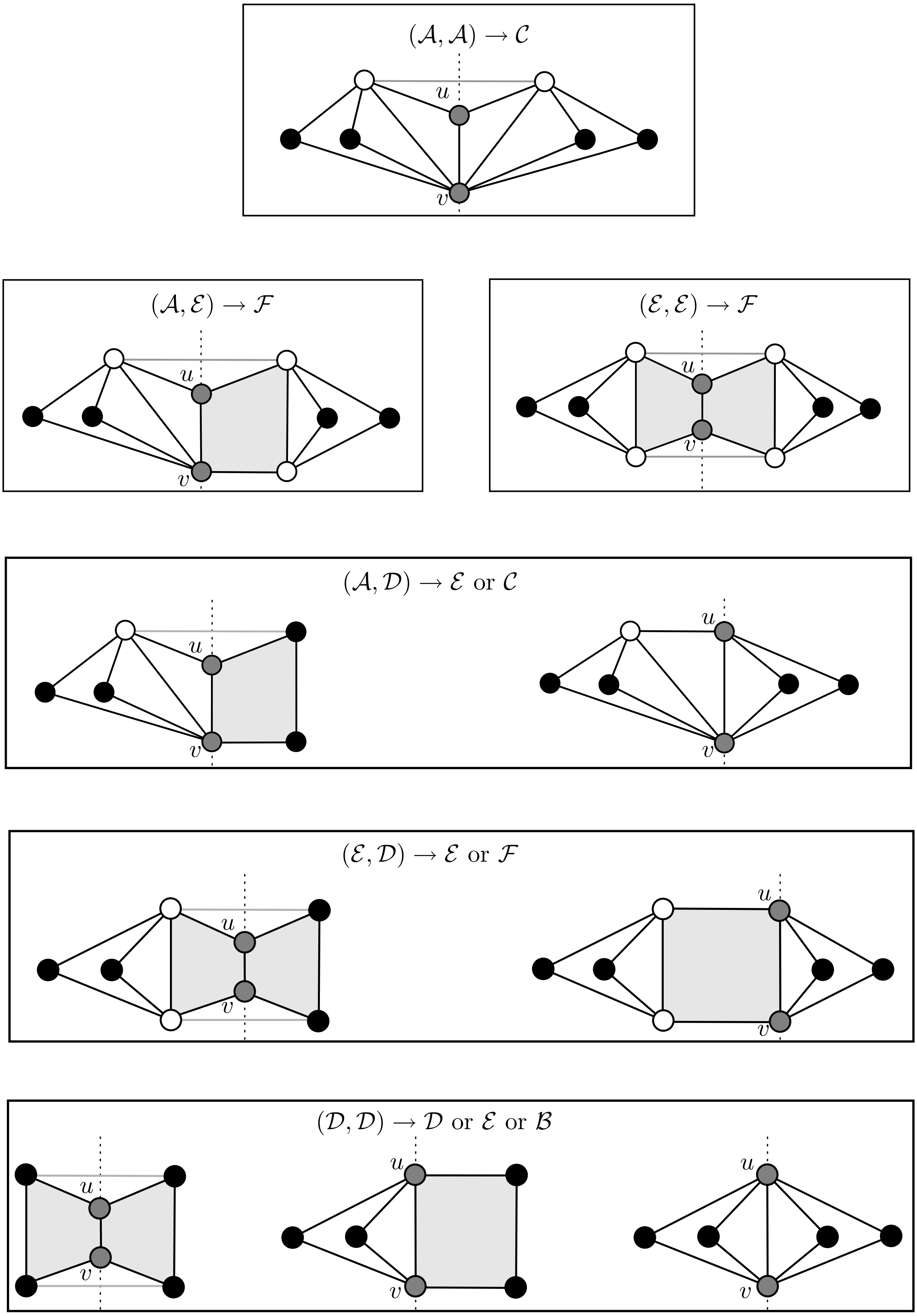

FIGURE 6. Constructions of new obstructions in the case of a $(2, 2)$-separation. Black vertices are nominated. Gray vertices are the cut-pair. White vertices are not nominated. Gray edges are inserted. Gray regions are webs.

that the edge uv is not needed for A, B, C, D to be a K_4-minor. Since u is in A, and A is connected, the only problem is if uv is the only edge between A and some other branch set, say B. But, since G is 2-connected, v has a neighbour in $G_1 - u - v$, which is a subgraph of A. This proves that A, B, C, D is an $\{a, b, c, d\}$-minor in G.

Now assume that G' is a spanning subgraph of some obstruction H^+. Thus $a', u, v \in T \cup X_T$ for some triangle T of H, and $a' \in T$. Rename a' as a in H, and add $V(G_1) \setminus \{a, u, v\}$ to X_T. The resulting graph H^+ is in the same class as the original H^+ and contains G as a spanning subgraph.

Now assume that if (G_1, G_2) is a separation of order 2, then $|V(G_1)| = 3$, the vertex in $G_1 - G_2$ is nominated, and $G_1 \cong K_3$ (since G is 2-connected).

- Suppose there is a $(1, 3)$-separation (G_1, G_2) of order 2: Let a be the nominated vertex in $G_1 - G_2$. Let $\{u, v\} := V(G_1 \cap G_2)$. Thus $G_1 \cong K_3$ with vertex set $\{a, u, v\}$.

  Let G_u be the graph obtained from G by contracting the edge au into u, and nominating u. Let G_v be the graph obtained from G by contracting the edge av into v, and nominating v. Each of G_u and G_v have four nominated vertices. Since a has degree 2 in G, G contains an $\{a, b, c, d\}$-minor if and only if G_u contains a $\{u, b, c, d\}$-minor or G_v contains a $\{v, b, c, d\}$-minor. Also observe that $G_u \cong G_v$; they only differ in one nominated vertex. For the time being, concentrate on G_u; we will return to G_v later.

  If G_u contains a $\{u, b, c, d\}$-minor, then G contains an $\{a, b, c, d\}$-minor, and we are done. Otherwise, by the choice of G, G_u is a spanning subgraph of an obstruction H^+. Since a class $\mathcal{A}$ obstruction has a $(2, 3)$-separation, and a class $\mathcal{B}, \mathcal{C}, \mathcal{E}$ or $\mathcal{F}$ obstruction has a $(2, 2)$-separation, H^+ is in class $\mathcal{D}$.

  If $|X_T| \geq 2$ for some triangle T of H then $(G - X_T, T \cup X_T)$ is a separation of order 3 with no nominated vertices in X_T, such that $|V(T \cup X_T)| \geq 5$, which is a contradiction. Thus $|X_T| \leq 1$. If $X_T = \{w\}$ then move w out of X_T into H; the resulting graph H^+ is in $\mathcal{D}$ and contains G_u as a spanning subgraph. Repeat this step until $X_T = \emptyset$ for each triangle T of H. Thus G_u is a spanning subgraph of H (not H^+), and G_u is planar. Since G_u was obtained from G by deleting a degree-2 vertex whose neighbours are adjacent, G is also planar.

  Since $H \in \mathcal{D}$, u is type-1. Let S be the set of degree-2 nominated vertices in G. Thus $a \in S \subseteq \{a, b, c, d\}$. Observe that G is almost 3-connected in the sense that the only cut-pairs are the neighbours of vertices in S, and in this case the cut-pair are adjacent. As illustrated in Figure 7, let $G^* := G - S$. A separation in G^* is a separation in G. Thus G^* is 3-connected and planar. Hence G^* has a unique planar embedding. Moreover, every planar embedding of G is obtained from the unique planar embedding of G^* by drawing each vertex $x \in S$ in one of the two faces that contain the edge between the two neighbours of x. In the planar embedding of G_u induced by the planar embedding of H, the nominated vertices u, b, c, d are on the outerface. Moreover, the unique planar embedding of G^* is obtained from this embedding of G_u by deleting $S \setminus \{a\}$.

  If the edge uv is on the outerface of G_u (as in Figure 7(a)), then draw a in the outerface of G_u adjacent to u and v, and possibly add edges between a and other nominated vertices to obtain an obstruction (in the same class as H) that contains G as a spanning subgraph.

  Now assume that uv is not on the outerface of G_u (as in Figure 7(b)). Recall that $G_u \cong G_v$, and v, b, c, d are nominated in G_v. Consider this embedding of G_u to be an embedding of G_v. The outerface of G_v contains b, c, d but not v.

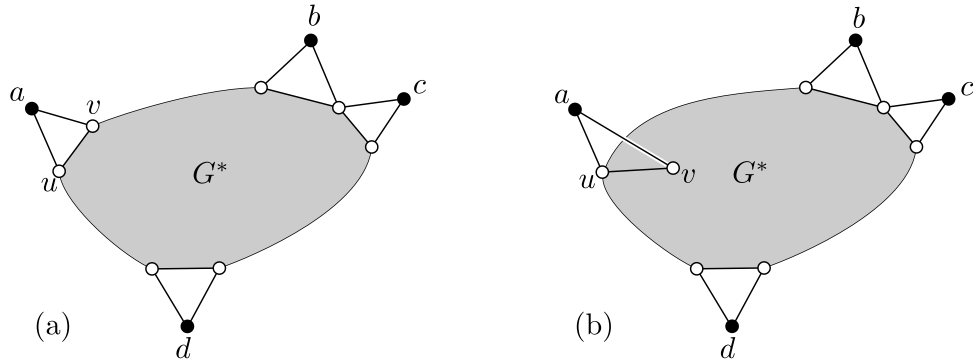

FIGURE 7. Illustration of G with a $(1,3)$-separation of order 2. Vertex a has degree 2, and b, c, d might have degree 2.

For $x \in \{b, c, d\}$, if $x \in S$ then choose a neighbour x' of x, otherwise let $x' := x$. If x and y are distinct vertices in S, then $N_G(x) \neq N_G(y)$, as otherwise G would contains a $(2,2)$-separation of order 2. Thus we may choose b', c', d' so that they are distinct. Each of b', c', d' are on the outerface of G_v. So v, b', c', d' are all distinct.

Consider v, b', c', d' to be nominated vertices in G^*. Consider the embedding of G^* formed from H. Then b', c', d' are on the outerface of G^*, but v is not. In a 3-connected planar graph, three vertices all appear on at most one face. Thus, no face of G^* contains all of v, b', c', d'. Thus by Theorem 9, G^* contains a $\{v, b'c', d'\}$-minor. Given that G^* can be obtained from G by contracting av, bb', cc' and dd', G contains an $\{a, b, c, d\}$-minor. (Here, if $b = b'$ then contracting bb' does nothing.)

Now assume that G is 3-connected. The result follows from Theorem 8, since a web is in class $\mathcal{D}$. □

8. Algorithmics

Robertson and Seymour [13] presented a $O(n^3)$ time algorithm that (for fixed t) tests whether a given n-vertex graph contains a K_t-minor rooted at t nominated vertices. We conjecture that for $t = 4$ there is a $O(n)$ time algorithm for this problem; see [3, 6, 11, 20] for related linear time algorithms.

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