

BETTER BOUNDS FOR POSET DIMENSION AND BOXICITY

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ABSTRACT. The *dimension* of a poset P is the minimum number of total orders whose intersection is P . We prove that the dimension of every poset whose comparability graph has maximum degree Δ is at most $\Delta \log^{1+o(1)} \Delta$. This result improves on a 30-year old bound of Füredi and Kahn and is within a $\log^{o(1)} \Delta$ factor of optimal. We prove this result via the notion of boxicity. The *boxicity* of a graph G is the minimum integer d such that G is the intersection graph of d -dimensional axis-aligned boxes. We prove that every graph with maximum degree Δ has boxicity at most $\Delta \log^{1+o(1)} \Delta$, which is also within a $\log^{o(1)} \Delta$ factor of optimal. We also show that the maximum boxicity of graphs with Euler genus g is $\Theta(\sqrt{g \log g})$, which solves an open problem of Esperet and Joret and is tight up to a constant factor.

1. INTRODUCTION

1.1. Poset dimension and degree. The *dimension* of a poset P , denoted by $\dim(P)$, is the minimum number of total orders whose intersection is P . Let $\dim(\Delta)$ be the maximum dimension of a poset whose comparability graph has maximum degree at most Δ . Several bounds on $\dim(\Delta)$ have been proved in the literature. In unpublished work, Rödl and Trotter proved the first upper bound, $\dim(\Delta) \leq 2\Delta^2 + 2$; see [44, pp. 165–166] for the proof. Füredi and Kahn [25] improved this result to

$$(1) \quad \dim(\Delta) \leq O(\Delta \log^2 \Delta).$$

On the other hand, Erdős, Kierstead, and Trotter [17] proved the lower bound,

$$(2) \quad \dim(\Delta) \geq \Omega(\Delta \log \Delta).$$

Both these proofs use probabilistic methods. The problem of narrowing the gap between (1) and (2) was described as “an important topic for further research” by Erdős et al. [17]; Trotter [45] speculated that the lower bound could be improved and wrote “a really new idea will be necessary to improve the upper bound—if this is at all possible”; and Wang [48, p. 52] described the problem as “one of the most challenging (and probably quite difficult) problems in dimension theory.”

Our first contribution is the following result, which is the first improvement to the Füredi–Kahn upper bound in 30 years and shows that (2) is sharp to within a $(\log \Delta)^{o(1)}$ factor:

$$(3) \quad \dim(\Delta) = \Delta \log^{1+o(1)} \Delta.$$

A more precise result is given below (see Theorem 15).

Received by the editors June 17, 2018, and, in revised form, August 8, 2019.

2010 *Mathematics Subject Classification.* Primary 05C62, 06A07.

The first author was supported by a Leverhulme Trust Research Fellowship.

The second author was supported by the Australian Research Council.

1.2. Boxicity and degree. We prove (3) via the notion of boxicity. The *boxicity* of a (finite undirected) graph G , denoted by $\text{box}(G)$, is the minimum integer d , such that G is the intersection graph of boxes in \mathbb{R}^d . Here a *box* is a Cartesian product $I_1 \times I_2 \times \cdots \times I_d$, where $I_i \subseteq \mathbb{R}$ is an interval for each $i \in [d]$. So a graph G has boxicity at most d if and only if there is a set of d -dimensional boxes $\{B_v : v \in V(G)\}$ such that $B_v \cap B_w \neq \emptyset$ if and only if $vw \in E(G)$. Note that a graph has boxicity 1 if and only if it is an interval graph. It is easily seen that every graph has finite boxicity.

Let $\text{box}(\Delta)$ be the maximum boxicity of a graph with maximum degree Δ . It is easily seen that $\text{box}(2) = 2$, and Adiga and Chandran [2] proved that $\text{box}(3) = 3$. Chandran, Francis, and Sivadasan [9] proved the first general upper bound of $\text{box}(\Delta) \leq 2\Delta^2 + 2$, which was improved to $\Delta^2 + 2$ by Esperet [19]. A breakthrough was made by Adiga, Bhowmick, and Chandran [1], who noted the following connection to poset dimension:

$$(4) \quad \frac{1}{2} \text{box}(\Delta - 1) \leq \dim(\Delta) \leq 2 \text{box}(\Delta).$$

Thus $\dim(\Delta) = \Theta(\text{box}(\Delta))$. For the sake of completeness, we prove (4) in Propositions 8 and 9.

Adiga et al. [1] concluded from (1), (2), and (4) that

$$\Omega(\Delta \log \Delta) \leq \text{box}(\Delta) \leq O(\Delta \log^2 \Delta).$$

We improve the upper bound, giving the following result, which is equivalent to (3):

$$(5) \quad \text{box}(\Delta) = \Delta \log^{1+o(1)} \Delta.$$

Again, a more precise result is given below (see Theorem 14).

1.3. Boxicity and genus. Next consider the boxicity of graphs embeddable in a given surface. Schlegel [40] proved that every outerplanar graph has boxicity at most 2. Thomassen [43] proved that every planar graph has boxicity at most 3 (generalised to “cubicity” by Felsner and Francis [24]). Esperet and Joret [22] proved that every toroidal graph has boxicity at most 7, improved to 6 by Esperet [21]. The *Euler genus* of an orientable surface with h handles is $2h$. The *Euler genus* of a non-orientable surface with c cross-caps is c . The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds (with no crossings). Esperet and Joret [22] proved that every graph with Euler genus g has boxicity at most $5g + 3$. Esperet [20] improved this upper bound to $O(\sqrt{g} \log g)$ and also noted that there are graphs of Euler genus g with boxicity $\Omega(\sqrt{g \log g})$, which follows from the result of Erdős et al. [17] mentioned above. See [21] for more on the boxicity of graphs embedded in surfaces.

The second contribution of this paper is to improve the upper bound to match the lower bound up to a constant factor (see Theorem 16). We conclude that the maximum boxicity of a graph with Euler genus g is

$$(6) \quad \Theta(\sqrt{g \log g}).$$

Furthermore, the implicit constant in (6) is not large: the upper bound in Theorem 16 is $(12 + o(1))\sqrt{g \log g}$.

1.4. Boxicity and layered treewidth. The third contribution of the paper is to prove a new upper bound on boxicity in terms of layered treewidth, which is a graph parameter recently introduced by Dujmović, Morin, and Wood [15] (see Section 5). This generalises the known bound in terms of treewidth and leads to generalisations of known results for graphs embedded in surfaces where each edge is in a bounded number of crossings.

1.5. Related work. The present paper can be considered to be part of a body of research connecting poset dimension and graph structure theory. Several recent papers [28–32, 35, 42, 47] show that structural properties of the cover graph of a poset lead to bounds on its dimension. Finally, we mention the following relationships between boxicity and chromatic number. Graphs with boxicity 1 (interval graphs) are perfect. Asplund and Grünbaum [4] proved that graphs with boxicity 2 are χ -bounded. But Burling [7] constructed triangle-free graphs with boxicity 3 and unbounded chromatic number.

2. TOOLS

Roberts [38] introduced boxicity and proved the following two fundamental results.

Lemma 1 ([38]). *For all graphs G, G_1, \dots, G_r such that $G = G_1 \cap \dots \cap G_r$,*

$$\text{box}(G) \leq \sum_{i=1}^r \text{box}(G_i).$$

Note that Lemma 1 is proved trivially by taking Cartesian products.

Lemma 2 ([38]). *Every n -vertex graph has boxicity at most $\lfloor \frac{n}{2} \rfloor$.*

Note that Trotter [46] characterised those graphs for which equality holds in Lemma 2.

A graph is *k-degenerate* if every subgraph has a vertex of degree at most k . Note that 1-degenerate graphs (that is, forests) have boxicity at most 2, but 2-degenerate graphs have unbounded boxicity, since the 1-subdivision of K_n is 2-degenerate and has boxicity $\Theta(\log \log n)$ [22]. Adiga, Chandran, and Matthew [3] proved the following bound. Throughout this paper, all logarithms are natural unless otherwise indicated.

Lemma 3 ([3]). *Every k -degenerate graph on n vertices has boxicity at most $(k + 2)\lceil 2e \log n \rceil$.*

The following lemma, due to Espert [21], is the starting point for our work on embedded graphs.

Lemma 4 ([21]). *Every graph G with Euler genus g has a set X of at most $60g$ vertices such that $G - X$ has boxicity at most 5.*

Let $[n] := \{1, 2, \dots, n\}$. For our purposes, a *permutation* of a set X is a bijection from X to $[|X|]$. A set $\{\pi_1, \dots, \pi_p\}$ of permutations of a set X is *r-suitable* if for every r -subset S of X and for every element $x \in S$, there is permutation π_i such that $\pi_i(x) < \pi_i(y)$ for all $y \in S \setminus \{x\}$. This definition was introduced by Dushnik [16]; see [8, 13, 41] for further results on suitable sets. Spencer [41] attributes the following result to Hajnal. We include the proof for completeness and so that the dependence on k is absolutely clear (since Spencer assumed that k is fixed).

Lemma 5 ([41]). *For every $k \geq 2$ and $n \geq 10^4$ there is a k -suitable collection of permutations of size at most $k2^k \log \log n$.*

Proof. A sequence S_1, \dots, S_r of subsets of $[s]$ is t -scrambling if for every set $I \subseteq [r]$ with $|I| \leq t$ and every $A \subseteq I$, we have

$$(7) \quad \bigcap_{i \in A} S_i \cap \bigcap_{j \in I \setminus A} ([s] \setminus S_j) \neq \emptyset.$$

For $s \geq t \geq 1$, let $M(s, t)$ be the maximum cardinality of a t -scrambling family of subsets of $[s]$. Note that $M(s, t)$ is monotone increasing in s (and trivially $M(s, t) \leq 2^s$).

Claim. Let $s \geq t \geq 1$. If m is a positive integer such that

$$(8) \quad 2^t \binom{m}{t} (1 - 2^{-t})^s < 1,$$

then $M(s, t) \geq m$.

Proof of Claim. Choose subsets S_1, \dots, S_m of $[s]$ independently and uniformly at random. For any t -set $I \subseteq [m]$ and any $A \subseteq I$, the probability that (7) is not satisfied is $(1 - 2^{-t})^s$. There are $\binom{m}{t}$ choices for I and then 2^t choices for A , so taking a union bound, the probability that there is some pair (I, A) such that (7) is not satisfied is at most the left hand side of (8) (note that it is enough to consider sets I of size exactly t). Since this is smaller than 1, we are done. \square

Given n and k , choose s minimal so that $2^{M(s, k-1)} \geq n$. Let $M := M(s, k - 1)$, let S_1, \dots, S_M be a $(k - 1)$ -scrambling set of subsets of $[s]$, and let Q_1, \dots, Q_n be distinct subsets of $[M]$. We define orders $<_1, \dots, <_s$ on $[n]$ as follows: for $a, b \in [n]$, let $j(a, b) = \min(Q_a \Delta Q_b)$; then $a <_i b$ if either

- $i \in S_{j(a,b)}$ and $j(a, b) \in Q_a$, or
- $i \notin S_{j(a,b)}$ and $j(a, b) \in Q_b$.

(Note that if $S_i = [M]$, then this gives the lex order on the Q_s , and if $S_i = \emptyset$ it is reverse lex.)

Now $<_1, \dots, <_s$ is a k -suitable collection of orders on $[n]$. This is straightforward but a little tricky: given a set B of k elements of $[n]$ and $b \in B$, let $I := \{\min(Q_h \Delta Q_b) : h \in B \setminus \{b\}\}$, let $A = I \cap Q_b$, and choose an element i of the intersection on the left hand side of (7). Consider the order $<_i$. It is enough to show that $b <_i h$ for each $h \in B \setminus \{b\}$. Given $h \in B \setminus \{b\}$, let $q = j(b, h) = \min(Q_b \Delta Q_h)$. If $q \in Q_b$, then $q \in A$ and so $i \in S_q$, and therefore $b <_i h$. If $q \notin Q_b$, then $q \notin A$ and so $i \notin S_q$, and again $b <_i h$.

How big is s ? By the choice of s and monotonicity, $M(s - 1, k - 1) < \log_2 n$. The left hand side of (8) is less than

$$(9) \quad (2em/t)^t \exp(-s2^{-t}) = \left(\frac{2em \exp(-s/t2^t)}{t} \right)^t,$$

and so

$$M(s, t) \geq \frac{t}{2e} e^{s/t2^t},$$

as setting m equal to the right hand side of this expression leaves (9) less than 1. Thus, bounding $M(s - 1, k - 1)$, we have

$$\frac{k - 1}{2e} \exp\left(\frac{s - 1}{(k - 1)2^{k-1}}\right) - 1 \leq M(s - 1, k - 1) \leq \log_2 n,$$

and so

$$s \leq 1 + (k - 1)2^{k-1} \log\left(\frac{2e}{k - 1} \log_2(2n)\right),$$

which is at most $k2^k \log \log n$ for $k \geq 2$ and $n \geq 10^4$. □

We will also use the Lovász Local Lemma:

Lemma 6 ([18]). *Let E_1, \dots, E_n be events in a probability space, each with probability at most p and mutually independent of all but at most D other events. If $4pD \leq 1$, then with positive probability, none of E_1, \dots, E_n occur.*

For a graph G and set $X \subseteq V(G)$, the graph $G[X]$ with vertex set X and edge set $\{vw \in E(G) : v, w \in X\}$ is called the subgraph of G induced by X . Let $G\langle X \rangle$ be the graph obtained from G by adding an edge between every pair of non-adjacent vertices at least one of which is not in X .

Lemma 7. $\text{box}(G\langle X \rangle) = \text{box}(G[X])$.

Proof. Given a d -dimensional box-representation of $G\langle X \rangle$, delete the boxes representing the vertices in $V(G) \setminus X$ to obtain a d -dimensional box-representation of $G[X]$. Thus $\text{box}(G[X]) \leq \text{box}(G\langle X \rangle)$. Given a d -dimensional box-representation of $G[X]$, for every vertex x in $V(G) \setminus X$, add a box with interval \mathbb{R} in every dimension (so that it meets all other boxes). We obtain a d -dimensional box-representation of $G\langle X \rangle$. Thus $\text{box}(G\langle X \rangle) \leq \text{box}(G[X])$. □

For a graph G and disjoint sets $X, Y \subseteq V(G)$, the graph $G[X, Y]$ with vertex set $X \cup Y$ and edge set $\{vw \in E(G) : v \in X, w \in Y\}$ is called the bipartite subgraph of G induced by X, Y . For non-adjacent vertices $v \in X$ and $w \in Y$, we say vw is a *non-edge* of $G[X, Y]$. Let $G\langle X, Y \rangle$ be the graph obtained from G by adding an edge between distinct vertices v and w whenever $v, w \in V(G) \setminus X$ or $v, w \in V(G) \setminus Y$.

For the sake of completeness we now establish the relationship between $\text{box}(\Delta)$ and $\text{dim}(\Delta)$ from (4). In fact, Proposition 8 is a slight strengthening.

Proposition 8. $\text{box}(\Delta) \leq \text{dim}(\Delta + 1)$.

Proof. Let G be a graph with maximum degree Δ . Let P be the poset on $V(G) \times \{0, 1\}$ where $(u, i) \preceq_P (v, j)$ if and only if $i = 0$ and $j = 1$, and $u = v$ or $uv \in E(G)$. Let $d := \text{dim}(P)$. Let $\preceq_1, \dots, \preceq_d$ be total orders on $V(G) \times \{0, 1\}$ whose intersection is P . For each vertex v of G , let B_v be the box $[x_1(v, 0), x_1(v, 1)] \times \dots \times [x_d(v, 0), x_d(v, 1)]$, where $x_i(v, j)$ is the position of (v, j) in \preceq_i . Let v, w be distinct vertices in G . If $vw \in E(G)$, then $(v, 0), (w, 0) \prec_i (v, 1), (w, 1)$ for each $i \in [d]$, implying that $B_v \cap B_w \neq \emptyset$. If $vw \notin E(G)$, then $(v, 1) \prec_i (w, 0)$ for some $i \in [d]$, implying that $(v, 0) \prec_i (v, 1) \prec_i (w, 0) \prec_i (w, 1)$ and $B_v \cap B_w = \emptyset$. Thus $\{B_v : v \in V(G)\}$ is a box representation for G , and $\text{box}(G) \leq \text{dim}(P)$. Since the comparability graph of P has maximum degree $\Delta + 1$, we have $\text{box}(\Delta) \leq \text{dim}(\Delta + 1)$. □

The next lemma uses the following characterisation of poset dimension due to Füredi and Kahn [25]: Let P be a poset with ground set X . Then $\dim(P)$ equals the minimum integer k for which there are total orders $\preceq_1, \dots, \preceq_k$ of X , with the property that for all incomparable pairs (x, y) of P there is an order \preceq_i such that $x \prec_i z$ for all $z \in X$ with $y \preceq_P z$.

Proposition 9. $\dim(\Delta) \leq 2 \operatorname{box}(\Delta)$.

Proof. Let P be a poset such that comparability graph G of P has maximum degree Δ . Let $\{[\ell_1(x), r_1(x)] \times \dots \times [\ell_d(x), r_d(x)] : x \in V(G)\}$ be a d -dimensional box representation of G , where $d := \operatorname{box}(G) \leq \operatorname{box}(\Delta)$. For $i \in [d]$, let \preceq'_i be the total order on X determined by $(\ell_i(x) : x \in X)$ ordered right-to-left, and let \preceq''_i be the total order on X determined by $(r_i(x) : x \in X)$ ordered left-to-right. Let (x, y) be an incomparable pair of P . Thus $xy \notin E(G)$, and $r_i(x) < \ell_i(y)$ or $r_i(y) < \ell_i(x)$ for some $i \in [d]$. First suppose that $r_i(x) < \ell_i(y)$. Consider $z \in X$ such that $y \preceq_P z$. Thus $y = z$ or $yz \in E(G)$, implying that $r_i(z) \geq \ell_i(y)$. Hence $r_i(x) < \ell_i(y) \leq r_i(z)$. By construction, $x \prec''_i z$ as desired. Now assume that $r_i(y) < \ell_i(x)$. Consider $z \in X$ such that $y \preceq_P z$. Thus $y = z$ or $yz \in E(G)$, implying that $r_i(y) \geq \ell_i(z)$. Hence $\ell_i(z) \leq r_i(y) < \ell_i(x)$. By construction, $x \prec'_i z$, as desired. By the above characterisation, $\dim(P) \leq 2d = 2 \operatorname{box}(G)$, and $\dim(\Delta) \leq 2 \operatorname{box}(\Delta)$. \square

3. BOUNDED DEGREE

The first ingredient in our proof is the following colouring result that bounds the number of monochromatic neighbours of each vertex. A very similar result was proved by Hind, Molloy, and Reed [27]; they required the additional property that the colouring is proper but had $k = \max\{(d+1)\Delta, e^3 \Delta^{1+1/d}/d\}$, which is too much for our purposes.

Lemma 10. *For every graph G with maximum degree $\Delta > 0$ and for all integers $d \geq 1$ and $k \geq \frac{(4d+4)^{1/d} e}{d} \Delta^{1+1/d}$, there is a partition V_1, \dots, V_k of $V(G)$ such that $|N_G(v) \cap V_i| \leq d$ for each $v \in V(G)$ and $i \in [k]$.*

Proof. Colour each vertex of G independently and randomly with one of k colours. Let V_1, \dots, V_k be the corresponding colour classes. For each set S of exactly $d+1$ vertices in G , such that $S \subseteq N_G(v)$ for some vertex $v \in V(G)$, introduce an event which holds if and only if $S \subseteq V_i$ for some $i \in [k]$. Each such event has probability $p := k^{-d}$. The colour on one vertex affects at most $\Delta \binom{\Delta-1}{d}$ events. Thus each event is mutually independent of all but at most D other events, where

$$D := (d+1)\Delta \binom{\Delta-1}{d} \leq (d+1)\Delta \left(\frac{e\Delta}{d}\right)^d = (d+1) \left(\frac{e}{d}\right)^d \Delta^{d+1}.$$

It follows that $4pD \leq 1$. By Lemma 6, with positive probability, no event occurs. Thus the desired partition exists. \square

Note that an example in [27], due to Noga Alon, shows that the value of k in Lemma 10 is within a constant factor of optimal. Lemma 10 leads to our next lemma. A similar result was used by Füredi and Kahn [25] in their work on poset dimension.

Corollary 11. *For every graph G with maximum degree $\Delta \geq 2$ and for all integers $d \geq 100 \log \Delta$ and $k \geq \frac{3\Delta}{d}$, there is a partition V_1, \dots, V_k of $V(G)$ such that $|N_G(v) \cap V_i| \leq d$ for each $v \in V(G)$ and $i \in [k]$.*

Proof. Since $d \geq 100 \log \Delta \geq 69$, we have $(4d+4)^{1/d} e \leq 2.95$ and $d \geq \log^{-1}(\frac{3}{2.95}) \log \Delta$ and $\Delta^{1/d} \leq \frac{3}{2.95}$. Thus $\frac{(4d+4)^{1/d} e}{d} \Delta^{1+1/d} \leq \frac{2.95}{d} \Delta^{1+1/d} \leq \frac{3\Delta}{d} \leq k$. The result follows from Lemma 10. \square

The next lemma is a key new idea in our proof. Its proof is a straightforward application of the Lovász Local Lemma.

Lemma 12. *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most d and vertices in B have degree at most Δ . Let r, t, ℓ be positive integers such that*

$$\ell \geq e \left(\frac{ed}{r+1} \right)^{1+1/r} \quad \text{and} \quad t \geq \log(4d\Delta).$$

Then there exist t colourings c_1, \dots, c_t of B , each with ℓ colours, such that for each vertex $v \in A$, for some colouring c_i , each colour is assigned to at most r neighbours of v under c_i .

Proof. For $i \in [t]$ and for each vertex $w \in B$, let $c_i(w)$ be a random colour in $[\ell]$. Let X_v be the event that for each $i \in [t]$, some set of $r + 1$ neighbours of v is monochromatic under c_i . The probability that there is a monochromatic set of at least $r + 1$ neighbours of v under c_i is

$$\binom{\deg(v)}{r+1} \ell^{-r} \leq \binom{d}{r+1} \ell^{-r} \leq \left(\frac{ed}{r+1} \right)^{r+1} \ell^{-r} \leq e^{-1}.$$

Thus $\mathbb{P}(X_v) \leq e^{-t}$. Observe that X_v is mutually independent of all but at most $d\Delta$ other events. By assumption, $4e^{-t}d\Delta \leq 1$. By Lemma 6, with positive probability no event X_v occurs. Therefore, the desired colourings exist. \square

Lemma 13. *Let G be a bipartite graph with bipartition $\{A, B\}$, where vertices in A have degree at most d and vertices in B have degree at most Δ for some $\Delta \geq d \geq 2$. Let $G' = G\langle A, B \rangle$ be the graph obtained from G by adding a clique on A and a clique on B . Then, as $d \rightarrow \infty$,*

$$\text{box}(G') \leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}.$$

Proof. Let $r := \lceil \sqrt{\log d} \rceil$, let $\ell := \lceil e \left(\frac{ed}{r+1} \right)^{1+1/r} \rceil$, and let $t := \lceil \log(4d\Delta) \rceil$. As $d \rightarrow \infty$, we may assume that d is large.

By Lemma 12, there exist t colourings c_1, \dots, c_t of B , each with ℓ colours, such that for each vertex $v \in A$, for some colouring c_j , each colour is assigned to at most r neighbours of v . Let $\{A_j : j \in [t]\}$ be a partition of A such that for each $v \in A_j$, at most r neighbours of v are assigned the same colour under c_j . Assume that $[\ell]$ is the set of colours used by each c_j .

Our aim is to construct a box representation for each of the graphs $G\langle A_i, B \rangle$ and then take their intersection using Lemma 1. In light of Lemma 7, it is enough to concentrate on the subgraphs $G[A_i \cup B]$. To handle $G[A_i \cup B]$, we further decompose B according to c_1, \dots, c_t as follows. For each $j \in [t]$ and each colour $\alpha \in [\ell]$, let $B_{j,\alpha} := \{w \in B : c_j(w) = \alpha\}$. Let $G_{j,\alpha} := G\langle A_j, B_{j,\alpha} \rangle$. Note that $G' = \bigcap_{j,\alpha} G_{j,\alpha}$.

We now bound the boxicity of $G_{j,\alpha}$. Let H be the graph with vertex set $B_{j,\alpha}$, where distinct vertices $x, y \in B_{j,\alpha}$ are adjacent in H whenever x and y have a common neighbour in A_j . Since each vertex in A_j has at most r neighbours in $B_{j,\alpha}$, the graph H has maximum degree at most $r\Delta$. Thus $\chi(H) \leq h := r\Delta + 1$.

Let X_1, \dots, X_h be the colour classes in a proper colouring of H . For $q \in [h]$, let \overrightarrow{X}_q denote an arbitrary linear ordering of X_q . Let \overleftarrow{X}_q be the reverse of \overrightarrow{X}_q . Since we may assume that d is large, Lemma 5 shows that there exists a set of $(r+1)$ -suitable permutations π_1, \dots, π_p of $[h]$ for some $p \leq (r+1)2^{r+1} \log \log(h)$.

For each $a \in [p]$, we introduce two 2-dimensional representations of $G_{j,\alpha}$. Let σ_a be the ordering $\overrightarrow{X_{\pi_a(1)}, \overrightarrow{X_{\pi_a(2)}, \dots, \overrightarrow{X_{\pi_a(h)}}$ of $B_{j,\alpha}$. Similarly, let σ'_a be the ordering $\overleftarrow{X_{\pi_a(1)}, \overleftarrow{X_{\pi_a(2)}, \dots, \overleftarrow{X_{\pi_a(h)}}$ of $B_{j,\alpha}$. For each vertex x in B , say x is the b_x -th vertex in σ_a and x is the b'_x -th vertex in σ'_a . Then represent x by the box with corners $(-\infty, +\infty)$ and $(2b_x, 2b_x)$. For each vertex $v \in A_j$, if v has no neighbours in B , then represent v by the point $(2|B|, -2|B|)$. Otherwise, if x is the leftmost neighbour of v in σ_a and y is the rightmost neighbour of v in σ_a , then represent v by the box with corners $(\infty, -\infty)$ and $(2b_x - 1, 2b_y + 1)$, as illustrated in Figure 1. Now, in two new dimensions introduce the following representation. Represent each x in $B_{j,\alpha}$ by the box with corners $(-\infty, +\infty)$ and $(2b'_x, 2b'_x)$. For each vertex $v \in A_j$, if v has no neighbours in B , then represent v by the point $(2|B|, -2|B|)$; otherwise, if x is the leftmost neighbour of v in σ'_a and y is the rightmost neighbour of v in σ'_a , then represent v by the box with corners $(\infty, -\infty)$ and $(2b'_x - 1, 2b'_y + 1)$. In each of these four dimensions, add every vertex in $V(G) \setminus (B_{j,\alpha} \cup A_j)$ with interval \mathbb{R} . Observe that A and B are both cliques in this representation.

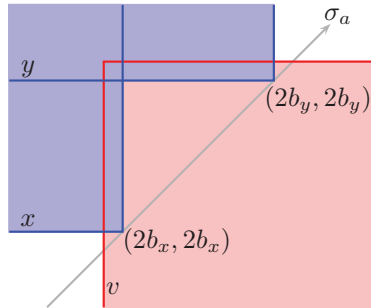


FIGURE 1. Representation of $G_{j,\alpha}$ with respect to σ_a .

By construction, for every edge vw of $G_{j,\alpha}$ the boxes of v and w intersect in every dimension. Now consider a non-edge zv of $G_{j,\alpha}$ with $z \in B_{j,\alpha}$ and $v \in A_j$. Let C be the set of integers $q \in [h]$ such that some neighbour of v is in X_q . Thus $|C| \leq r$. Say z is in $X_{q'}$. First suppose that $q' \notin C$. Since $|C \cup \{q'\}| \leq r+1$, for some permutation π_a , we have $\pi_a(q') < \pi_a(q)$ for each $q \in C$. Let x be the leftmost neighbour of v in σ_a . Thus $b_z < b_x$, and in the first 2-dimensional representation corresponding to π_a , the right hand side of the box representing z is to the left of the left hand side of the box representing v , as illustrated in Figure 2(a). Thus the boxes representing v and z do not intersect. Now assume that $q' \in C$. By construction, there is exactly one neighbour x of v in $X_{q'}$. Since $|C| \leq r$, for some permutation π_a , we have $\pi_a(q') \leq \pi_a(q)$ for each $q \in C$. If $z < x$ in $\overrightarrow{X}_{q'}$, then $b_z < b_x$, and as argued above and illustrated in Figure 2(b), the boxes representing v and z do not intersect. Otherwise, $z < x$ in $\overleftarrow{X}_{q'}$. Then $b'_z < b'_x$, and in the second 2-dimensional representation corresponding to π_a , the right hand side of the box

representing z is to the left of the left hand side of the box representing v . Hence the boxes representing v and z do not intersect. Therefore $\text{box}(G_{j,\alpha}) \leq 4p$.

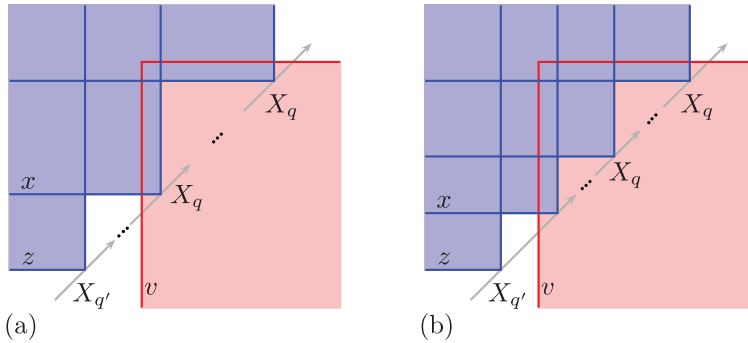


FIGURE 2. Proof for the representation of $G_{j,\alpha}$.

By Lemma 1,

$$\begin{aligned} \text{box}(G') &\leq 4t\ell p \\ &\leq 4\lceil \log(4d\Delta) \rceil \left[e \left(\frac{ed}{r+1} \right)^{1+1/r} \right] (r+1)2^{r+1} \log \log(r\Delta + 1). \end{aligned}$$

Since $\log(4d\Delta) \leq (1 + o(1)) \log(d\Delta)$ and $e^{1+1/r} \leq (1 + o(1))e$ and $\log(r\Delta + 1) \leq (1 + o(1)) \log(\Delta)$,

$$\begin{aligned} \text{box}(G') &\leq (8e^2 + o(1)) \log(d\Delta) \left(\frac{d}{r+1} \right)^{1+1/r} (r+1)2^r \log \log(\Delta) \\ &\leq (60 + o(1)) \log(d\Delta) \log \log(\Delta) d^{1+1/r} \frac{2^r}{(r+1)^{1/r}} \\ &\leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) \left(d^{1/r} 2^r \right) \\ &\leq (60 + o(1)) d \log(d\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}. \end{aligned}$$

□

We now prove our first main result.

Theorem 14. For every graph G with maximum degree Δ , as $\Delta \rightarrow \infty$,

$$\text{box}(G) \leq (180 + o(1)) \Delta \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \log \log \Delta.$$

Proof. Let $d := \lceil 100 \log \Delta \rceil$ and let $k := \lceil \frac{3\Delta}{d} \rceil$. By Corollary 11, there is a partition V_1, \dots, V_k of $V(G)$ such that $|N_G(v) \cap V_i| \leq d$ for each $v \in V(G)$ and $i \in [k]$. Note that

$$\text{box}(G) = \bigcap_i G\langle V_i \rangle \cap G\langle V_i, V(G) \setminus V_i \rangle.$$

Since $G[V_i]$ has maximum degree at most d , by the result of [19], the graph $G[V_i]$ has boxicity at most $d^2 + 2$. By Lemma 7,

$$\text{box}(G\langle V_i \rangle) \leq d^2 + 2.$$

Let $G_i := G[V_i, V(G) \setminus V_i]$. Every vertex in $V(G) \setminus V_i$ has degree at most d in G_i . Let G'_i be obtained from G_i by adding a clique on V_i and a clique on $V(G) \setminus V_i$. By Lemmas 7 and 13 and since $\log(d\Delta) \leq (1 + o(1)) \log \Delta$,

$$\text{box}(G[V_i, V(G) \setminus V_i]) \leq \text{box}(G'_i) \leq (60 + o(1)) d \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}}.$$

Applying Lemma 1 again,

$$\begin{aligned} \text{box}(G) &\leq k(d^2 + 2) + (60 + o(1)) k d \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log d}} \\ &\leq (9 + o(1))(\Delta \log \Delta) + (180 + o(1)) \Delta \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \\ &\leq (180 + o(1)) \Delta \log(\Delta) \log \log(\Delta) (2e)^{\sqrt{\log \log \Delta}}. \end{aligned}$$

□

Since $(2e)^{\sqrt{\log \log \Delta}} \log \log \Delta \leq \log^{o(1)} \Delta$, Theorem 14 implies (5). More precisely,

$$\text{box}(\Delta) = \Delta(\log \Delta)e^{O(\sqrt{\log \log \Delta})}.$$

Theorem 14 and the result of [1] mentioned in Section 1 imply the following quantitative version of (3).

Theorem 15. *For every poset P whose comparability graph has maximum degree Δ , as $\Delta \rightarrow \infty$,*

$$\dim(P) \leq (360 + o(1)) \Delta \log(\Delta) (2e)^{\sqrt{\log \log \Delta}} \log \log \Delta.$$

Again, with (2), this gives

$$\dim(\Delta) = \Delta(\log \Delta)e^{O(\sqrt{\log \log \Delta})}.$$

4. EULER GENUS

We now prove our second main result.

Theorem 16. *For every graph G with Euler genus g , as $g \rightarrow \infty$,*

$$\text{box}(G) \leq (12 + o(1))\sqrt{g \log g}.$$

Proof. By Lemma 4, G contains a set X of at most $60g$ vertices such that $\text{box}(G - X) \leq 5$. First suppose that $|X| < 10^4$. Deleting one vertex reduces boxicity by at most 1. Thus $\text{box}(G) \leq \text{box}(G - X) + |X| \leq 10^4 + 5$, and we are done since $g \rightarrow \infty$. Now assume that $|X| \geq 10^4$.

Let $G_1 := G(V(G) \setminus X)$. Let Y be the set of vertices in $G - X$ with exactly one or exactly two neighbours in X . Let $G_2 := \langle X, Y \rangle$. By Lemma 5, there is a 3-suitable set of permutations π_1, \dots, π_p of X for some $p \leq 24 \log \log |X|$. For each π_i we introduce two dimensions, as illustrated in Figure 3. Represent each vertex $w \in X$ by the box with corners $(-\infty, +\infty)$ and $(2\pi_i(w), 2\pi_i(w))$. For each vertex $v \in Y$, if x and y are respectively the leftmost and rightmost neighbours of v in π_i , then represent v by the box with corners $(2\pi_i(x) - 1, 2\pi_i(y) + 1)$ and $(+\infty, -\infty)$. Observe that X and Y are both cliques in this representation. If $vw \in E(G)$ and $v \in Y$ and $w \in X$, then the box representing v intersects the box representing w . Consider a non-edge vz with $v \in Y$ and $z \in X$. Since π_1, \dots, π_p is 3-suitable and $\deg_X(v) \leq 2$, for some i , we have $\pi_i(z) < \pi_i(x)$ for each $x \in N_G(v)$. Thus, for the 2-dimensional representation defined with respect to π_i , the boxes representing v and z do not intersect.

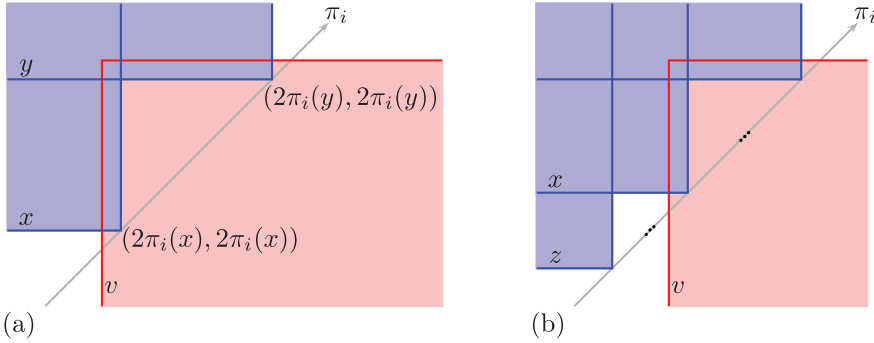


FIGURE 3. Representation of G_2 with respect to π_i .

Add each vertex in $G - (X \cup Y)$ to every dimension with interval \mathbb{R} . We obtain a box representation of G_2 . Thus $\text{box}(G_2) \leq 48 \log \log |X| \leq 48 \log \log(1000g)$.

Let Z be the set of vertices in $G - X$ with at least three neighbours in X . Let $G_3 := G \setminus (X \cup Z)$. Observe that $G = G_1 \cap G_2 \cap G_3$.

To bound $\text{box}(G_3)$, we first bound $\text{box}(H)$, where $H := G[X \cup Z]$. The number of edges in $G[X, Z]$ is at least $3|Z|$ and at most $2(|X| + |Z| + g - 2)$ by Euler's formula. Thus $|Z| < 2(|X| + g)$, implying $|X \cup Z| < 3002g$. Let $n := |V(H)| < 3002g$. Let v_1, \dots, v_n be an ordering of $V(H)$, where v_i has minimum degree in $H[\{v_i, \dots, v_n\}]$. Define $k := 7 + \lceil \sqrt{g/\log g} \rceil$. Let i be minimum such that v_i has degree at least k in $H[\{v_i, \dots, v_n\}]$. If i is defined, then let $A := \{v_1, \dots, v_{i-1}\}$ and $B := \{v_i, \dots, v_n\}$; otherwise let $A := V(H)$ and $B := \emptyset$.

Observe that $H = H\langle A \rangle \cap H\langle B \rangle \cap H\langle A, B \rangle$.

By construction, $H[A]$ is k -degenerate and has at most n vertices. By Lemma 3, $\text{box}(H[A]) \leq (k + 2)\lceil 2e \log n \rceil$. By Lemma 7, $H\langle A \rangle \leq (k + 2)\lceil 2e \log n \rceil$.

By construction, $H[B]$ has minimum degree at least k . The number of edges in $H[B]$ is at least $\frac{1}{2}k|B|$ and at most $3(|B| + g - 2)$, implying that $(\frac{k}{2} - 3)|B| < 3g$. By Lemma 2, $\text{box}(H[B]) \leq \frac{|B|}{2} < \frac{3g}{k-6}$. By Lemma 7, $H\langle B \rangle < \frac{3g}{k-6}$.

Now consider $H[A, B]$. By construction, every vertex in A has degree at most k in $H[A, B]$. A permutation σ of B catches a non-edge vw of $H[A, B]$ with $v \in A$ and $w \in B$ if there are edges vx, vy in $H[A, B]$ such that w is between x and y in σ . Let $t := \lceil \frac{3}{2}(k + 1) \log n \rceil$. Let $\sigma_1, \dots, \sigma_t$ be random permutations of B . For each non-edge vw of $H[A, B]$, the probability that σ_i catches vw equals $1 - \frac{2}{\deg(v)+1} \leq e^{-2/(k+1)}$. Thus, the probability that every σ_i catches vw is at most $e^{-2t/(k+1)} \leq n^{-3}$. Since the number of non-edges is at most n^2 , by the union bound, the probability that for some non-edge vw , every σ_i catches vw is at most $n^{-1} < 1$. Hence, with positive probability, for every non-edge vw , some σ_i does not catch vw . Therefore, there exist permutations $\sigma_1, \dots, \sigma_t$ of B such that for every non-edge vw , some σ_i does not catch vw .

For each permutation σ_i we introduce two dimensions. Represent each vertex $w \in B$ by the box with corners $(-\infty, +\infty)$ and $(2\sigma_i(w), 2\sigma_i(w))$. For each vertex $v \in A$, if v has no neighbours in B , then represent v by the point $(2|B|, -2|B|)$; otherwise, if x and y are respectively the leftmost and rightmost neighbours of v in σ_i , then represent v by the box with corners $(2\sigma_i(x) - 1, 2\sigma_i(y) + 1)$ and $(+\infty, -\infty)$. Observe that A and B are both cliques in this representation. If $vw \in E(G)$ and

$v \in A$ and $w \in B$, then the box representing v intersects the box representing w . For a non-edge vw with $v \in A$ and $w \in B$, the box representing v intersects the box representing w if and only if σ_i catches vw . Since for every non-edge vw , some σ_i does not catch vw , the boxes representing v and w do not intersect. Thus $\text{box}(H\langle A, B \rangle) \leq 2t \leq 2 + 3(k + 1) \log n$.

By Lemma 1,

$$\begin{aligned} \text{box}(H) &\leq \text{box}(H\langle A \rangle) + \text{box}(H\langle B \rangle) + \text{box}(H\langle A, B \rangle) \\ &\leq (k + 2) \lceil 2e \log n \rceil + \frac{3g}{k - 6} + 2 + 3(k + 1) \log n \\ &\leq (9k + 15) \log(3002g) + 3\sqrt{g \log g} \\ &\leq 12\sqrt{g \log g} + O(\sqrt{g / \log g}). \end{aligned}$$

Applying Lemma 1 again,

$$\begin{aligned} \text{box}(G) &\leq \text{box}(G_1) + \text{box}(G_2) + \text{box}(G_3) \\ &\leq 42 + 48 \log \log(1000g) + 12\sqrt{g \log g} + O(\sqrt{g / \log g}) \\ &\leq 12\sqrt{g \log g} + O(\sqrt{g / \log g}). \end{aligned} \quad \square$$

As noted in Section 1, when combined with the lower bound proved by [20], this shows that the maximum possible boxicity of a graph with Euler genus g is $\Theta(\sqrt{g \log g})$.

5. LAYERED TREewidth

A *tree decomposition* of a graph G is a set $(B_x : x \in V(T))$ of non-empty sets $B_x \subseteq V(G)$ (called *bags*) indexed by the nodes of a tree T , such that for each vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T , and for each edge $vw \in E(G)$ there is a node $x \in V(T)$ such that $v, w \in B_x$. The *width* of a tree decomposition $(B_x : x \in V(T))$ is $\max\{|B_x| - 1 : x \in V(T)\}$. The *treewidth* of a graph G , denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of G . Treewidth is a key parameter in algorithmic and structural graph theory (see [6, 26, 37] for surveys). Chandran and Sivadasan [10] proved:

Theorem 17 ([10]). *For every graph G ,*

$$\text{box}(G) \leq \text{tw}(G) + 2.$$

A *layering* of a graph G is a partition (V_1, V_2, \dots, V_n) of $V(G)$ such that for every edge $vw \in E(G)$, for some $i \in [n - 1]$, both v and w are in $V_i \cup V_{i+1}$. For example, if r is a vertex of a connected graph G and $V_i := \{v \in V(G) : \text{dist}(r, v) = i\}$ for $i \geq 0$, then (V_0, V_1, \dots) is a layering of G . The *layered treewidth* of a graph G is the minimum integer k such that there is a tree decomposition $(B_x : x \in V(T))$ and a layering (V_1, V_2, \dots, V_n) of G such that $|B_x \cap V_i| \leq k$ for each node $x \in V(T)$ and for each layer V_i . Of course, $\text{ltw}(G) \leq \text{tw}(G) + 1$, and often $\text{ltw}(G)$ is much less than $\text{tw}(G)$. For example, Dujmović et al. [15] proved that every planar graph has layered treewidth at most 3, whereas the $n \times n$ planar grid has treewidth n . Thus the following result provides a qualitative generalisation of Theorem 17.

Theorem 18. *For every graph G ,*

$$\text{box}(G) \leq 6 \text{ltw}(G) + 4.$$

Proof. Consider a tree decomposition $(B_x : x \in V(T))$ and a layering (V_1, V_2, \dots, V_n) of G such that $|B_x \cap V_i| \leq \text{ltw}(G)$ for each node $x \in V(T)$ and for each layer V_i . Note that $(B_x \cap (V_i \cup V_{i+1}) : x \in V(T))$ is a tree-decomposition of $G[V_i \cup V_{i+1}]$ with bags of size at most $2\text{ltw}(G)$. Thus $\text{tw}(G[V_i \cup V_{i+1}]) \leq 2\text{ltw}(G) - 1$. For $i \in \{0, 1, 2\}$, let

$$G_i := \bigcup_{j \equiv i \pmod{3}} G[V_j \cup V_{j+1}].$$

Each component of G_i is contained in $V_j \cup V_{j+1}$ for some $j \equiv i \pmod{3}$. The treewidth of a graph equals the maximum treewidth of its connected components. Thus $\text{tw}(G_i) \leq 2\text{ltw}(G) - 1$, and $\text{box}(G_i) \leq 2\text{ltw}(G) + 1$ by Theorem 17. Use three disjoint sets of $2\text{ltw}(G) + 1$ dimensions for each G_i , and add each vertex not in G_i to the dimensions used by G_i with interval \mathbb{R} . Finally, add one more dimension, where the interval for each vertex $v \in V_i$ is $[i, i + 1]$. For adjacent vertices in G , the corresponding boxes intersect in every dimension. Consider non-adjacent vertices v and w in G , say $v \in V_a$ and $w \in V_b$. If $|a - b| \geq 2$, then in the final dimension, the intervals for v and w are disjoint, as desired. If $|a - b| = 1$, then vw is a non-edge in some G_i , and thus the intervals for v and w are disjoint in some dimension corresponding to G_i . Hence we have a $3(2\text{ltw}(G) + 1) + 1$ -dimensional box representation of G . \square

The following two examples illustrate the generality of Theorem 18. A graph is (g, k) -planar if it has a drawing in a surface of Euler genus at most g with at most k crossings per edge; see [34, 36, 39] for example. Dujmović, Eppstein, and Wood [14] proved that every (g, k) -planar graph has layered treewidth at most $(4g + 6)(k + 1)$. Theorem 18 then implies that every (g, k) -planar graph has boxicity at most $6(4g + 6)(k + 1) + 4$. Map graphs provide a second example. Start with a graph G_0 embedded in a surface of Euler genus g , with each face labelled a “nation” or a “lake”, where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0 , where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a (g, d) -map graph; see [11, 12] for example. Dujmović et al. [14] proved that every (g, d) -map graph has layered treewidth at most $(2g + 3)(2d + 1)$. Theorem 18 then implies that every (g, d) -map graph has boxicity at most $6(2g + 3)(2d + 1) + 4$. By definition, a graph is $(g, 0)$ -planar if and only if it has Euler genus at most g . Similarly, it is easily seen that a graph is a $(g, 3)$ -map graph if and only if it has Euler genus at most g (see [14]). Thus these results provide qualitative generalisations of the fact that graphs with Euler genus g have boxicity $O(g)$, as proved by Esperet and Joret [22]. As discussed above, Esperet [20] improved this upper bound to $O(\sqrt{g} \log g)$, and Theorem 16 improves it further to $O(\sqrt{g \log g})$. On the other hand, Theorem 18 is within a constant factor of optimal, since Chandran and Sivadasan [10] constructed a family of graphs G with $\text{box}(G) \geq (1 - o(1)) \text{tw}(G)$. See [5] for more examples of graph classes with bounded layered treewidth for which Theorem 18 is applicable.

6. OPEN PROBLEMS

We conclude with a few open problems.

- What is the maximum boxicity of graphs with maximum degree 4?
- What is the maximum boxicity of k -degenerate graphs with maximum degree Δ ?
- What is the maximum boxicity of graphs with treewidth k ? Chandran and Sivadasan [10] proved lower and upper bounds of $k - 2\sqrt{k}$ and $k + 2$, respectively.
- What is the maximum boxicity of graphs with no K_t minor? The best known upper bound is $O(t^2 \log t)$ due to [23]. A lower bound of $\Omega(t\sqrt{\log t})$ follows from results of [20].

ACKNOWLEDGMENTS

Thanks to Louis Esperet and Tom Trotter for useful comments. The proofs of Propositions 8 and 9 are due to Tom Trotter.

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