

K_4 -MINOR-FREE INDUCED SUBGRAPHS OF SPARSE CONNECTED GRAPHS*

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Abstract. We prove that every connected graph G with m edges contains a set X of at most $\frac{3}{16}(m+1)$ vertices such that $G-X$ has no K_4 minor, or, equivalently, has treewidth at most 2. This bound is best possible. Connectivity is essential: If G is not connected, then only a bound of $\frac{1}{5}m$ can be guaranteed.

Key words. K_4 -minor, series-parallel, transversal

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1. Introduction. We consider the minimum size of a subset X of vertices to remove from a graph G to obtain a graph with no K_4 minor, or, equivalently, a graph of treewidth at most 2. Denote this quantity by $s(G)$. We are interested in bounding $s(G)$ from above by a function of the number m of edges of G . Several authors independently proved the following upper bound.

THEOREM 1.1 (see Scott and Sorkin [12]; Kneis et al. [10]; Edwards and Farr [4]; Borradaile, Eppstein, and Zhu [1]). *For every graph G with m edges,*

$$s(G) \leq \frac{1}{5}m.$$

The ratio $\frac{1}{5}$ is best possible: If G is the disjoint union of k copies of K_5 , then G has $m = 10k$ edges and $s(G) = 2k = \frac{1}{5}m$. While $s(G) \leq \frac{1}{5}m$ is tight for arbitrary graphs, this bound can be improved if we restrict our attention to *connected* graphs. The following is our main result.

THEOREM 1.2. *For every connected graph G with m edges,*

$$s(G) \leq \frac{3}{16}(m+1).$$

This bound is best possible: Take G to be the disjoint union of k copies of K_6 , and then add $k-1$ edges so that the resulting graph is connected. Then $s(G) = 3k$ and $m = 16k-1$, and thus $s(G) = \frac{3}{16}(m+1)$.

Edwards and Farr [5] previously studied $s(G)$ for connected graphs with maximum degree 6.

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THEOREM 1.3 (see Edwards and Farr [5]). *For every connected graph G with maximum degree at most 6 and n_d vertices of degree d ,*

$$s(G) \leq \frac{1}{4}n_3 + \frac{3}{8}n_4 + \frac{19}{40}n_5 + \frac{11}{20}n_6 + \frac{3}{20}.$$

The best upper bound in terms of m one could hope to prove from Theorem 1.3 is

$$s(G) \leq \frac{19}{100}m + \frac{3}{20}$$

(because of the 5-regular case). And with some work this is possible, even for graphs with unbounded degree. Nevertheless, Theorem 1.2 improves upon this bound. Also note that for connected graphs with maximum degree 4, Theorem 1.3 implies

$$s(G) \leq \frac{3}{16}m + \frac{3}{20},$$

matching the coefficient in Theorem 1.2. As is evident in our proof, the main difficulty in establishing the bound $\frac{3}{16}(m+1)$ in general is the delicate handling of vertices of degree 5.

We continue this introduction with an overview of related works, emphasizing the different motivations for studying the invariant $s(G)$ that appeared in the literature.

Max-2-CSP. In one line of research, bounding the invariant $s(G)$ for sparse graphs G was originally motivated by the design of exponential-time algorithms for a class of combinatorial optimization problems called Max-2-CSP. Informally, Max-2-CSP is the class of constraint satisfaction problems with at most two variables per clause (see, e.g., [13] for a precise definition). This includes such problems as Max Cut, Max-2-Sat, Max Independent Set, and many others. Problems in this class are naturally parameterized by the number r of different values a variable can take; for instance, $r = 2$ in the case of binary variables, which is the most common case. Several exact exponential-time algorithms have been developed for Max-2-CSP. When parameterized by the number m of edges of the constraint graph, the best one to date is a $O^*(r^{\frac{13}{75}m+o(m)})$ -time algorithm of Scott and Sorkin [13]. (The notation $O^*(\cdot)$ omits polynomial factors in m and r .) This algorithm is based on treewidth, its key component being a proof that the treewidth of G is at most $\frac{13}{75}m + o(m)$.

Unfortunately, this exponential-time treewidth-based algorithm also takes exponential space. This motivated a long line of research focusing on exponential-time but polynomial-space algorithms for Max-2-CSP [3, 6, 8, 7, 9, 10, 11, 13]. It is in this context that the invariant $s(G)$ attracted some attention: It is well known that a graph with no K_4 minor can be reduced to the empty graph by iteratively “eliminating” vertices of degree at most 2 (see Lemma 2.1 in section 2). As it turns out, these operations can also be realized on the constraint graph of a Max-2-CSP instance, in the sense that a vertex of degree at most 2 can be eliminated in polynomial time, giving a smaller equivalent instance (see, e.g., [13]). The interest of this observation is that if $X \subseteq V(G)$ is such that $G - X$ has no K_4 minor, then one can simply branch on the vertices in X and obtain a $O^*(r^{|X|})$ -time algorithm that uses only polynomial space. This observation motivated Scott and Sorkin [12] and independently Kneis et

al. [10]¹ to show that such a set X with $|X| \leq \frac{m}{5}$ always exists (c.f. Theorem 1.1), and that it can be found efficiently.

This approach was subsequently improved by Scott and Sorkin [13]. Informally, they showed that it is enough to bound the “reduction depth” of the recursion tree. This leads to the definition of the following graph invariant which is at most $s(G)$ (here we follow the terminology of Edwards [3]). First, let $r(G)$ denote the graph obtained from G by eliminating vertices of degree at most 2 (as in Lemma 2.1). It is known that $r(G)$ is uniquely defined [11, Theorem 3.8]. Then the *delete-reduction depth* $\text{drd}(G)$ of G is the least nonnegative integer k for which there is a sequence $G_1 = r(G), G_2, \dots, G_{k+1}$ of graphs and a sequence V_1, \dots, V_k of vertex sets such that

- $V_i \subseteq V(G_i)$ and V_i contains at most one vertex from each component of G_i , for each $i \in \{1, \dots, k\}$;
- $G_i = r(G_{i-1} - V_{i-1})$ for each $i \in \{2, \dots, k+1\}$;
- G_{k+1} is empty.

Note that if one also required $|V_i| = 1$ in the above definition, then the resulting invariant would be equal to $s(G)$ (we leave the proof to the reader). It follows that $\text{drd}(G) \leq s(G)$ for every graph G .

As noted by Edwards [3], it follows from the work of Scott and Sorkin [13] that Max-2-CSP can be solved in time $O^*(r^k)$ and polynomial space given a sequence V_1, \dots, V_k of vertex sets witnessing $\text{drd}(G) \leq k$ as in the above definition. Scott and Sorkin [13] obtained a $O^*(r^{\frac{19}{100}m})$ -time algorithm by showing that $\text{drd}(G) \leq \frac{19}{100}m + 2$ when G has m edges (with an algorithmic proof).

Note that if G is not connected, then, by definition, $\text{drd}(G)$ is the maximum of $\text{drd}(H)$ over all components H of G . Thus when bounding $\text{drd}(G)$, we may assume that G is connected, and hence the upper bound of Scott and Sorkin on $\text{drd}(G)$ can be viewed as a precursor of the bound $s(G) \leq \frac{19}{100}m + \frac{3}{20}$ of Edwards and Farr [5] for connected graphs G . As noted earlier, our upper bound of $\frac{3}{16}(m+1)$ on $s(G)$ when G is connected is tight. However, better upper bounds can be achieved for $\text{drd}(G)$: Edwards [3] proved that $\text{drd}(G) \leq \frac{9}{50}m + o(m)$. Again, that proof is algorithmic, and yields a $O^*(r^{\frac{9}{50}m + o(m)})$ -time, polynomial space, algorithm for Max-2-CSP, which is the best known to date. Another algorithm matching this time complexity was described recently by Gaspers and Sorkin [8].

Graph drawing. The invariant $s(G)$ is also among a set of closely related graph invariants studied by the graph drawing community. One general approach to drawing a graph G in the plane is to first identify an induced subgraph H of G that can be drawn nicely—e.g., a forest or a planar graph—and then add somehow the remaining vertices and edges to the drawing in a second phase. If the graph G is sparse enough, one might hope to be able to embed a large fraction of the graph in the first phase, which motivates extremal questions of the following type: Given a class of graphs \mathcal{C} that is considered to have nice drawings (say, planar graphs), what is the least integer

¹It should be noted that a better bound than $s(G) \leq \frac{1}{5}m$ was claimed by Kneis et al. in their conference paper from 2005 [10]: They claimed $s(G) \leq \frac{23}{120}m + 1$ for every graph G and $s(G) \leq \frac{3}{16}m + 1$ for every graph G with maximum degree at most 4. However, both proofs fail to address connectivity issues, and indeed the two statements are false, as seen by taking disjoint copies of K_5 . We note that other proofs in [10] are problematic and were later corrected by the authors in their journal version [11] and the above two results were dropped (also see the Acknowledgments section of [11] as well as the discussion of [10] by Scott and Sorkin [13, section 1.3]). However, the authors did not highlight the fact that the two results in [10] were wrong, resulting in an unfortunate confusion in the literature.

$f(m)$ such that every n -vertex m -edge graph G has an induced subgraph on at least $n - f(m)$ vertices in \mathcal{C} , or equivalently, has a set X of vertices with $|X| \leq f(m)$ such that $G - X$ is in \mathcal{C} ?

Borradaile, Eppstein, and Zhu [1] recently considered these questions with \mathcal{C} being the class of (1) pseudo-forests (at most one cycle in each component); (2) K_4 -minor-free graphs, and (3) planar graphs. They proved the following upper bounds on the corresponding function $f(m)$: (1) $f(m) \leq \frac{2}{9}m$, (2) $f(m) \leq \frac{1}{5}m$ (cf. Theorem 1.1), and (3) $f(m) \leq \frac{23}{120}m$. We note that their second result—i.e., $s(G) \leq \frac{1}{5}m$ —was also observed implicitly in an earlier work by Edwards and Farr [4].²

The result of Borradaile, Eppstein, and Zhu [1] for planar graphs improves on an earlier bound of Edwards and Farr [4] and is in fact slightly stronger than as stated above: It holds for the class \mathcal{C} of planar graphs with treewidth at most 3. Nevertheless, this remains the best known bound for arbitrary planar graphs. It was suggested by Rossmannith that an upper bound of the form $f(m) \leq \frac{1}{6}m + o(m)$ might hold for planar graphs (see [1]). Borradaile, Eppstein, and Zhu [1] showed that, if true, this would be best possible: They proved that if \mathcal{C} is any proper minor-closed class, then $f(m) \geq \frac{1}{6}m - o(m)$.

Note that, as in the context of Max-2-CSP, here one is also naturally interested in computing efficiently the small sets X such that $G - X$ is in \mathcal{C} guaranteed by these theorems, and indeed the results mentioned above come with efficient algorithms (see [1]).

This paper is organized as follows. We begin in section 2 with the necessary definitions and basic facts about the invariant $s(G)$. In section 3, we provide some general lemmas which are useful in our proofs, and in section 4 we prove our main result (Theorem 1.2).

2. Definitions and preliminaries. All graphs in this paper are finite, simple, and undirected. Let $N_G(v)$ denote the set of neighbors of vertex v in the graph G , and let $\deg_G(v) = |N_G(v)|$ denote its degree. Let G/uv denote the graph obtained from G by contracting the edge uv . (Since we restrict ourselves to simple graphs, loops and parallel edges resulting from edge contractions are deleted.) A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges.

A *k-cutset* of a connected graph G is a subset $X \subseteq V(G)$ with $|X| = k$ such that $G - X$ is not connected. A *separation* of G is a pair (G_1, G_2) of induced subgraphs of G such that $V(G_1) - V(G_2)$ and $V(G_2) - V(G_1)$ are not empty and there are no edges between these two sets. The *order* of the separation (G_1, G_2) is $|V(G_1) \cap V(G_2)|$.

A *stable set* in a graph is a set of pairwise nonadjacent vertices. The maximum size of a stable set in a graph G is denoted by $\alpha(G)$. A graph H is a *subdivision* of a graph G if H can be obtained from G by replacing edges of G with internally disjoint paths having the same endpoints. (The internal vertices of the paths are new vertices of the graph.)

Recall the basic observation that a graph G contains K_4 as a minor if and only if G contains a K_4 subdivision as a subgraph. Graphs with no K_4 minors can be characterized in different ways. For instance, these graphs are exactly the graphs of treewidth at most 2. The following characterization is useful for our purpose: G has no K_4 minor if and only if G can be reduced to the empty graph by iteratively applying the following three reductions (in any order):

- removing a vertex of degree at most 1,

²Although that bound is not stated explicitly in [4], it is a corollary of Lemma 3 in that paper.

- removing a vertex of degree 2 lying in a triangle, and
- contracting an edge incident to a vertex of degree 2 not in a triangle.

Observe in particular that every graph with no K_4 minor has a vertex of degree at most 2.

A subset X of vertices of a graph G such that $G - X$ has no K_4 minor is called a *transversal*. A *minimum transversal* of G is a transversal of minimum size; its size is denoted by $s(G)$. The following lemma summarizes some simple but essential properties of the invariant $s(G)$. A proof is included for completeness.

LEMMA 2.1. *Suppose that H is a graph obtained from a graph G using one of the following three operations:*

- removing a vertex of degree at most 1,
- removing a vertex of degree 2 lying in a triangle, or
- contracting an edge incident to a degree-2 vertex not in a triangle.

Then $s(H) = s(G)$.

Proof. First suppose that $H = G - v$ where v is a vertex of degree at most 1 in G . Clearly, $s(H) \leq s(G)$. Let S be a minimum transversal of H . If S is not a transversal of G , then $G - S$ contains a K_4 subdivision, and that subgraph includes v . However, this is not possible since v has degree at most 1. Hence S is a transversal of G , and $s(H) \geq s(G)$, implying $s(H) = s(G)$.

Next suppose that $H = G - v$ where v is a vertex of degree 2 in G lying in a triangle. Clearly, $s(H) \leq s(G)$ again. Let S be a minimum transversal of H . If S is not a transversal of G , then $G - S$ contains a K_4 subdivision, and that subgraph includes v . However, one can shorten the K_4 subdivision by removing v and adding the edge between its two neighbors (which exists in G), which is then a subgraph of $H - S$, a contradiction. Hence S is a transversal of G , and $s(H) \geq s(G)$, implying $s(H) = s(G)$.

Finally, suppose that $H = G/uv$ where v is a vertex of degree 2 in G not in a triangle, and let w denote the vertex resulting from the contraction of the edge uv . To show $s(H) \leq s(G)$, consider a minimum transversal S of G . If $u \in S$ or $v \in S$, then $(S - \{u, v\}) \cup \{w\}$ is a transversal of H of size at most $|S| = s(G)$. If, on the other hand, $u \notin S$ and $v \notin S$, then $H - S$ is a minor of $G - S$ (obtained by contracting the edge uv), and thus does not contain a K_4 minor. Hence S is a transversal of H .

It remains to show $s(H) \geq s(G)$. Let S be a minimum transversal of H . If $w \in S$, then $T := (S - w) \cup \{v\}$ is a transversal of G , of size $|S| = s(G)$. Indeed, v has degree at most 1 in $G - T$ and thus is not contained in a K_4 subdivision in $G - T$, implying that every K_4 subdivision in $G - T$ is also a subgraph of $H - S$.

If $w \notin S$, then we claim that S itself is a transversal of G . For, suppose not, and let J be subdivision of K_4 in $G - S$. Then J includes v , since otherwise H would exist in $J - S$ (replacing vertex u by w). But then v has degree 2 in J , and its two neighbors are not adjacent, implying that J/uv is a K_4 subdivision in $H - S$, a contradiction.

Therefore, $s(H) \geq s(G)$, implying $s(H) = s(G)$. □

Observe that Lemma 2.1 shows one direction of the aforementioned characterization of graphs without K_4 minors: If G can be reduced to the empty graph using the three operations, then $s(G) = 0$, i.e., G has no K_4 minor. To prove the other direction, namely “if G has no K_4 minor then G can be reduced to the empty graph,” by the lemma it is enough to show that G has a vertex of degree at most 2. This is a standard exercise; see, e.g., [2, Corollary 7.3.2].

Note that in the statement of Lemma 2.1 we could have merged the second and third operations simply by writing “contracting an edge incident to a degree-2 vertex”

(since we only consider simple graphs). We chose, however, to present it as above to emphasize the two possible situations for a vertex of degree 2, which will typically be treated separately in the proofs.

As mentioned in the introduction, $s(G) \leq \frac{1}{5}m$ for every graph G with m edges. This can be proved easily using “potential functions.” As these play an important role in this paper, we provide a quick sketch of the proof. (This is also a good warm-up for the more technical proofs to come.) The main idea is to show that there exists a function $\phi : \mathbb{N} \rightarrow \mathbb{R}$ (called a *potential function*) satisfying the following two properties:

- $\phi(d) \leq \frac{1}{2}d$ for all $d \geq 0$, and
- $s(G) \leq \frac{1}{5} \sum_{v \in V(G)} \phi(\deg(v))$ for every graph G .

Given such a function ϕ , the bound follows then by observing that

$$s(G) \leq \frac{1}{5} \sum_{v \in V(G)} \phi(\deg(v)) \leq \frac{1}{5} \sum_{v \in V(G)} \frac{\deg(v)}{2} = \frac{1}{5}m.$$

One possible choice for the potential function ϕ is the following:

$$\phi(d) := \begin{cases} 0 & \text{if } d \in \{0, 1, 2\}, \\ \frac{5}{4} & \text{if } d = 3, \\ \frac{1}{2}d & \text{if } d \geq 4. \end{cases}$$

It is clear that $\phi(d) \leq \frac{1}{2}d$ for all $d \geq 0$, and it thus remains to show that $s(G) \leq \frac{1}{5} \sum_{v \in V(G)} \phi(\deg(v))$ for every graph G , which we prove by induction on $|V(G)|$.

First, note that vertices of degree at most 2 can be eliminated from G without changing $s(G)$: If v has degree at most 2, then $s(G) = s(G')$, where G' is obtained by first removing v from G , and then adding an edge between the two neighbors of v if they were not already adjacent in G (see Lemma 2.1). Since the potential associated to each vertex of G' is at most its potential in G , we are done by induction in this case. Thus we may assume that G has no vertex of degree at most 2.

Now consider a vertex v of maximum degree. If $\deg(v) = 5$, then the potential $\phi(\deg(v))$ of v is at least $\frac{5}{2}$. Furthermore, if we remove v from G , then the potential of each neighbor of v drops by at least $\frac{1}{2}$ (since these neighbors have degree at least 3). Thus,

$$\sum_{v \in V(G)} \phi(\deg_G(v)) \geq \frac{5}{2} + 5\frac{1}{2} + \sum_{v \in V(G')} \phi(\deg_{G'}(v)).$$

Since $s(G) \leq s(G - v) + 1$ the result follows by induction.

Now, if $\deg(v) = 4$, then v has potential 2 and each of its neighbors has degree 3 or 4. Thus their potential drops by at least $\frac{3}{4}$ when removing v . (Here we see why it is useful to set $\phi(3) := \frac{5}{4}$ instead of simply $\frac{3}{2}$.) Hence, the sum of potentials decreases by at least $2 + 4\frac{3}{4} = 5$ when removing v from G , and we are done by induction as before.

Finally, if v has degree 3, then so do its three neighbors, and the drop in potential when removing v is $\frac{5}{4} + 3\frac{5}{4} = 5$. Once again, we are done by induction. This concludes the proof.

As can be seen in this proof, the main benefit of potential functions is that they allow a stronger bound for low-degree vertices, which then helps the induction go through.

3. Tools. In this section we introduce a few lemmas that are used in our main proof.

First we consider graphs G that are edge critical with respect to the invariant $s(G)$: An edge e of a graph G is said to be *critical* if $s(G - e) < s(G)$. The graph G is *critical* if all its edges are critical.

LEMMA 3.1. *Let G be a critical graph. Then, for every edge $uv \in E(G)$, there exists $S \subseteq V(G) - \{u, v\}$ such that $S \cup \{u\}$ and $S \cup \{v\}$ are both minimum transversals of G .*

Proof. Let S be a minimum transversal of $G - uv$. Then $|S| = s(G - uv) = s(G) - 1$, since uv is critical in G . Hence, $u \notin S$ and $v \notin S$ (otherwise, S would be a transversal of G). The graph $G - (S \cup \{u\})$ is a subgraph of $(G - uv) - S$. Thus, $S \cup \{u\}$ is a transversal of G , with size $|S| + 1 = s(G)$. By symmetry, the same is true for $S \cup \{v\}$. \square

LEMMA 3.2. *Let G be a connected critical graph with $|V(G)| \geq 2$. Then G is 2-connected.*

Proof. Arguing by contradiction, suppose that G is not 2-connected. Since K_2 is not critical, this implies $|V(G)| \geq 3$ and that G has a cutvertex u . Let G_1, G_2 be two subgraphs of G such that $G = G_1 \cup G_2$ with $|V(G_1)|, |V(G_2)| \geq 2$, and $V(G_1) \cap V(G_2) = \{u\}$. Let $V_i := V(G_i)$, and let v_i be an arbitrary neighbor of u in G_i for $i = 1, 2$.

Using Lemma 3.1, let S_i ($i = 1, 2$) be a subset of $V(G) - \{u, v_i\}$ such that $S_i \cup \{u\}$ and $S_i \cup \{v_i\}$ are both minimum transversals of G . Clearly, the set

$$(S_1 \cap V_1) \cup (S_2 \cap V_2) \cup \{u\}$$

is a transversal of G . Since its size is at least $s(G) = |S_2 \cap V_1| + |S_2 \cap V_2| + 1$, it follows that $|S_1 \cap V_1| \geq |S_2 \cap V_1|$. Similarly, since

$$(S_2 \cap V_1) \cup (S_1 \cap V_2) \cup \{u\}$$

is also a transversal of G , we deduce $|S_2 \cap V_1| \geq |S_1 \cap V_1|$, and thus

$$(1) \quad |S_1 \cap V_1| = |S_2 \cap V_1|.$$

If $S_2 \cap V_1$ is not a transversal of G_1 , then there exists a K_4 subdivision J in $G_1 - S_2$. But then $J \subseteq G - (S_2 \cup \{v_2\})$, contradicting the fact that $S_2 \cup \{v_2\}$ is a transversal of G . Thus $S_2 \cap V_1$ is a transversal of G_1 . By a symmetric argument, $S_1 \cap V_2$ is a transversal of G_2 . It follows that the set $T := (S_2 \cap V_1) \cup (S_1 \cap V_2)$ is a transversal of G . (Note that a K_4 subdivision in G cannot contain both a vertex from $G_1 - u$ and another from $G_2 - u$.) By (1),

$$|T| = |S_2 \cap V_1| + |S_1 \cap V_2| = |S_1 \cap V_1| + |S_1 \cap V_2| = s(G) - 1,$$

a contradiction. The lemma follows. \square

Now we turn our attention to cycles of even length that are (almost) induced. A cycle is said to be *even* or *odd* according to the parity of its length. An even cycle C of a graph G is said to be *almost induced* if either C has no chord, or C has exactly one chord e and e “splits” the cycle into two odd cycles. Observe that, if C is such a cycle, then there is a subset $S \subset V(C)$ with $|S| = \frac{|C|}{2}$ such that S is a stable set of G .

LEMMA 3.3. *Let G be a graph with minimum degree at least 3 and with no subgraph isomorphic to K_4 . Then G contains an even cycle. Moreover, every shortest even cycle of G is almost induced.*

Proof. It is well known that every graph with minimum degree at least 3 contains a K_4 minor, and thus a subdivision of K_4 . Let H be a subgraph of G isomorphic to a subdivision of K_4 . Consider two distinct vertices u, v of H with degree 3 in H . Since there are three internally disjoint uv -paths in H , two of them have the same parity. Thus the union of these two paths is an even cycle in G .

Let C be any shortest even cycle of G . It remains to show that C is almost induced. This is obviously true if C is induced. If C has exactly one chord e in G , then the two cycles obtained from C using e are odd, since otherwise there would be a shorter even cycle in G . Thus C is also almost induced in that case.

Enumerate the vertices of C in order as $C = v_1v_2 \dots v_p$, and suppose C has two distinct chords $e = v_iv_j$ ($i < j$) and $f = v_kv_\ell$ ($k < \ell$). Then $|C| \geq 6$, since otherwise the vertices of C would induce a subgraph isomorphic to K_4 in G . By changing the cyclic ordering of C if necessary, we may assume without loss of generality $i \leq k < j$. Also, exchanging e and f if necessary, we suppose that $\ell < j$ if $i = k$. By the observation above, each chord splits C into two odd cycles; that is, $j - i + 1$ and $\ell - k + 1$ are both odd. We distinguish two cases, depending whether the chords are crossing or not.

First assume $\ell \leq j$; that is, the two chords are not crossing. Then the cycle $v_iv_{i+1} \dots v_kv_\ell v_{\ell+1} \dots v_j$ has length $(k - i + 1) + (j - \ell + 1) = (j - i) - (\ell - k) + 2$, which is even. Since the latter cycle is shorter than C , this is a contradiction.

Now suppose $\ell > j$. Then, by assumption, we also have $k > i$. Recall also that $j > k$, and hence v_i, v_j, v_k , and v_ℓ are pairwise distinct, and the two chords cross. We can find two other cycles of G using C and these two edges, namely $C_1 = v_iv_{i+1} \dots v_kv_\ell v_{\ell-1} \dots v_j$ and $C_2 = v_jv_{j-1} \dots v_kv_\ell v_{\ell+1} \dots v_pv_1 \dots v_i$. Note that $j - k$ and $k - i$ have the same parity since $j - i$ is even, and similarly that $\ell - j$ and $j - k$ have the same parity since $\ell - k$ is even. Thus C_1 is an even cycle. A symmetric argument shows that C_2 is even as well. Since $|C_1| + |C_2| = |C| + 4$ and $|C| \geq 6$, one of these cycles is shorter than C , again a contradiction.

It follows that C is almost induced, as claimed. \square

4. Proof of main result. In order to prove Theorem 1.2, we use the following potential function: Let $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$ be defined as

$$\phi(d) := \begin{cases} 0 & \text{if } d \in \{0, 1, 2\}, \\ \frac{4}{3} & \text{if } d = 3, \\ \frac{1}{2}d & \text{otherwise.} \end{cases}$$

For a graph G , define the *potential* of G as

$$\phi(G) := \sum_{v \in V(G)} \phi(\deg(v)).$$

We often simply write $\phi_G(u)$ to denote the potential of vertex u in G , that is, $\phi(\deg_G(u))$. Also, we use the convention that $\phi_G(v) = 0$ whenever $v \notin V(G)$.

Observe that $\phi(G) \leq m$ for every graph G with m edges, since $\phi(d) \leq d/2$ for all $d \geq 0$. Thus the bound $s(G) \leq \frac{3}{16}m + \frac{3}{16}$ when G is connected follows from the following technical theorem. Here a graph G is said to be *reduced* if G is 2-connected, critical, and $\deg(v) \in \{3, 4, 5\}$ for all $v \in V(G)$.

THEOREM 4.1. For every connected graph G ,

- (a) $\phi(G) \geq \frac{16}{3}s(G) - 1$, and
- (b) $\phi(G) \geq \frac{16}{3}s(G)$ if G is reduced but not 3-connected.

The remainder of this section is devoted to the proof of Theorem 4.1. Arguing by contradiction, we let G be a counterexample to Theorem 4.1 with $|V(G)| + |E(G)|$ minimum. Our proof is in two steps: First, we prove in section 4.1 that G is not a counterexample to part (b) of Theorem 4.1. Then, building on this result, we show in section 4.2 that G is not a counterexample to part (a) either, giving the desired contradiction.

We begin with a useful lemma about the potential function ϕ .

LEMMA 4.2. Let a, b, c be integers with $a \geq 3$ and $b, c \geq 1$. Then

- (A) $\phi(a) \geq \phi(a - 1) + \frac{1}{2}$,
- (B) $\phi(a) \geq \phi(a - 2) + 1$,
- (C) $\phi(b + c - 1) \geq \phi(b) + \phi(c) - \frac{1}{2}$, and
- (D) $\phi(b + c) \geq \phi(b) + \phi(c) + 1$ if moreover $b, c \geq 2$ and $b + c \leq 5$.

Proof. (A) and (B) are direct consequences of the definition of ϕ . (C) is easily checked when $b + c \leq 4$, and for the case $b + c \geq 5$ it follows from the observation that

$$\phi(b) + \phi(c) - \frac{1}{2} \leq \frac{b}{2} + \frac{c}{2} - \frac{1}{2} = \phi(b + c - 1).$$

(D) is also straightforward to check. □

4.1. G satisfies Part (b) of Theorem 4.1. Recall that G is a counterexample to Theorem 4.1 with $|V(G)| + |E(G)|$ minimum. In this section we show that G is not a counterexample to part (b) of this theorem. To do so, we argue by contradiction and assume that G is reduced but not 3-connected. Recall that G being reduced implies in particular that G is 2-connected. Since G is also assumed not to be 3-connected, this will naturally lead us to consider 2-cutsets of G . Note that given a 2-cutset X of G , every separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) = X$ is such that G_1 and G_2 are both connected, since G is 2-connected. We first establish a technical lemma about 2-cutsets in G , Lemma 4.3 below, which we then use to derive the desired contradiction.

LEMMA 4.3. If G is a counterexample to Theorem 4.1(b), then every 2-cutset of G induces an edge.

Proof. Arguing by contradiction, assume that there exists a 2-cutset of G that does not induce an edge. Let $X = \{u, v\}$ be such a cutset and let (G_1, G_2) be a separation of G with $V(G_1) \cap V(G_2) = X$. Since G has minimum degree at least 3, we have $|V(G_1)|, |V(G_2)| \geq 4$. We may assume that X and (G_1, G_2) have been chosen so that

- (2) at most one vertex in X has degree 1 in G_i for each $i \in \{1, 2\}$.

Indeed, suppose this is not the case. Then without loss of generality both u and v have degree 1 in G_1 . Let u' and v' be the neighbors of respectively u and v in G_1 . If $u' = v'$, then u' would be a cutvertex of G (recall that $|V(G_1)| \geq 4$); thus, $u' \neq v'$. It follows that $X' := \{u', v\}$ is a cutset of G such that $u'v \notin E(G)$. Also, the two graphs $G'_1 := G[V(G_1) - \{u\}]$ and $G'_2 := G[V(G_2) \cup \{u'\}]$ define a corresponding separation

(G'_1, G'_2) of G with $V(G'_1) \cap V(G'_2) = X'$. Moreover, u' and v have degree at least 2 in G'_1 and G'_2 , respectively. Hence, replacing X with X' and (G_1, G_2) with (G'_1, G'_2) , we see that (2) holds.

Let S_i ($i = 1, 2$) be a minimum transversal of G_i . The set $S_1 \cup S_2 \cup \{u\}$ is a transversal of G , implying

$$(3) \quad s(G) \leq s(G_1) + s(G_2) + 1.$$

The remainder of the proof is split into two cases, depending on whether $s(G) \leq s(G_1) + s(G_2)$ or $s(G) = s(G_1) + s(G_2) + 1$.

Case 1. $s(G) \leq s(G_1) + s(G_2)$: By (2), without loss of generality, if u has degree 1 in G_1 or G_2 , then u has degree 1 in G_1 , and if v has degree 1 in G_1 or G_2 , then v has degree 1 in G_2 . Let u_1 be a neighbor of u in G_1 . Let v_2 be a neighbor of v in G_2 . Let G'_1 be obtained from G_1 by removing u if u has degree 1 in G_1 . Define G'_2 similarly with respect to G_2 and v . By our assumption on u and v ,

$$\begin{aligned} \phi(G) - \phi(G'_1) - \phi(G'_2) &= \phi_G(u) - \phi_{G'_1}(u) - \phi_{G'_2}(u) + \phi_G(u_1) - \phi_{G'_1}(u_1) \\ &\quad + \phi_G(v) - \phi_{G'_1}(v) - \phi_{G'_2}(v) + \phi_G(v_2) - \phi_{G'_2}(v_2). \end{aligned}$$

(Recall that $\phi_H(x) = 0$ if $x \notin V(H)$; thus $\phi_{G'_1}(u) = 0$ if $u \notin V(G'_1)$, for instance.) First, we show that

$$(4) \quad \phi_G(u) - \phi_{G'_1}(u) - \phi_{G'_2}(u) + \phi_G(u_1) - \phi_{G'_1}(u_1) \geq 1.$$

If u has degree 1 in G_1 , then Lemma 4.2(A) yields

$$\phi_G(u) - \phi_{G'_1}(u) - \phi_{G'_2}(u) = \phi_G(u) - \phi_{G'_2}(u) \geq \frac{1}{2}$$

and

$$\phi_G(u_1) - \phi_{G'_1}(u_1) \geq \frac{1}{2},$$

implying (4). If, on the other hand, u has degree at least 2 in G_1 , then $\phi_G(u_1) = \phi_{G'_1}(u_1)$, and

$$\phi_G(u) - \phi_{G'_1}(u) - \phi_{G'_2}(u) \geq 1$$

by Lemma 4.2(D). Thus, (4) holds in this case too.

By a symmetric argument,

$$(5) \quad \phi_G(v) - \phi_{G'_1}(v) - \phi_{G'_2}(v) + \phi_G(v_2) - \phi_{G'_2}(v_2) \geq 1.$$

Combining (4) and (5),

$$(6) \quad \phi(G) - \phi(G'_1) - \phi(G'_2) \geq 2.$$

Now, G'_1 and G'_2 are obviously connected. Moreover, $s(G'_i) = s(G_i)$ for $i \in \{1, 2\}$ by Lemma 2.1. Since both G'_1 and G'_2 are smaller than G , each of them satisfies Theorem 4.1. Using part (a) of the latter theorem on these two graphs in combination with (6),

$$\phi(G) \geq \phi(G'_1) + \phi(G'_2) + 2 \geq \left(\frac{16}{3}s(G_1) - 1 \right) + \left(\frac{16}{3}s(G_2) - 1 \right) + 2 \geq \frac{16}{3}s(G),$$

which contradicts the fact that G is a counterexample.

Case 2. $s(G) = s(G_1) + s(G_2) + 1$: Let $G'_i := G_i + uv$ for $i = 1, 2$. We claim that $s(G'_i) > s(G_i)$ for some $i \in \{1, 2\}$. Indeed, assume otherwise, let S_i ($i = 1, 2$) be a minimum transversal of G'_i , and let $S := S_1 \cup S_2$. Then $G - S$ contains a subgraph H isomorphic to a subdivision of K_4 since $|S| \leq s(G_1) + s(G_2) < s(G)$. Now, thanks to the edge uv , one can easily find from H a subdivision of K_4 in one of $G'_1 - S_1$ and $G'_2 - S_2$, a contradiction. Hence, $s(G'_i) > s(G_i)$ for some $i \in \{1, 2\}$; without loss of generality, $s(G'_2) > s(G_2)$.

Let S_i ($i = 1, 2$) be a minimum transversal of G_i . The sets S_1 and S_2 cannot include any vertex of the cutset X . Indeed, otherwise $S_1 \cup S_2$ would be a transversal of G of size $|S_1 \cup S_2| \leq s(G_1) + s(G_2) < s(G)$, a contradiction. (Here we use that every K_4 subdivision in G that meets both $V(G_1) - X$ and $V(G_2) - X$ contains both vertices in X .) Now, if $s(G_1 - u) < s(G_1)$, then any minimum transversal of $G_1 - u$ can be extended to one of G of size at most $s(G)$ by adding u , which is not possible as we have seen. Since the same holds for $G_1 - v_1$,

$$(7) \quad s(G_1 - u) = s(G_1 - v) = s(G_1).$$

(Note that this is also true for G_2 but we will not need this fact.) Exchanging u and v if necessary, we may assume

$$(8) \quad \deg_{G_1}(u) \leq \deg_{G_1}(v).$$

Let H_1 be the graph $G_1 - v$ where the vertex u is removed if u has degree 1 in G_1 . Note that the graph $G_1 - v$ is connected (otherwise, v would be a cutvertex of G). Thus, H_1 is connected as well. Also,

$$s(H_1) = s(G_1 - v) = s(G_1),$$

by Lemma 2.1 and (7). Since both H_1 and G'_2 are connected and smaller than G , they satisfy Theorem 4.1. It follows that

$$\phi(H_1) + \phi(G'_2) \geq \left(\frac{16}{3}s(G_1) - 1\right) + \left(\frac{16}{3}(s(G_2) + 1) - 1\right) = \frac{16}{3}s(G) - 2.$$

(Recall that $s(G'_2) > s(G_2)$ and $s(G) = s(G_1) + s(G_2) + 1$.) In order to conclude the proof it is enough to show that $\phi(G) \geq \phi(H_1) + \phi(G'_2) + 2$, since this implies that $\phi(G) \geq \frac{16}{3}s(G)$ and thus that G is not a counterexample.

Case 2a. Vertex u has degree 1 in G_1 : Then $H_1 = G_1 - \{u, v\}$. Also, by (2), the vertex v has degree at least 2 in G_1 . Using Lemma 4.2(A) and (B), and considering the neighbors of u and v in G_1 , we obtain

$$\phi(G_1) \geq \phi(H_1) + \phi_{G_1}(u) + \phi_{G_1}(v) + \frac{3}{2}.$$

(Note that there are two cases to consider: either u and v have one common neighbor or they have none.) Also,

$$\phi_G(v) \geq \phi_{G'_2}(v) + \frac{1}{2},$$

since $\deg_{G_1}(v) \geq 1$ and thus $\deg_{G'_2}(v) \leq \deg_G(v) - 1$. Using these observations and the fact that $\phi_G(u) = \phi_{G'_2}(u)$,

$$\begin{aligned} \phi(G) &= \left(\phi(G_1) - \phi_{G_1}(u) - \phi_{G_1}(v)\right) + \phi(G'_2) + \left(\phi_G(u) - \phi_{G'_2}(u)\right) + \left(\phi_G(v) - \phi_{G'_2}(v)\right) \\ &\geq \left(\phi(H_1) + \frac{3}{2}\right) + \phi(G'_2) + \frac{1}{2} = \phi(H_1) + \phi(G'_2) + 2, \end{aligned}$$

as desired.

Case 2b. Vertex u has degree at least 2 in G_1 : Then $H_1 = G_1 - v$, and v has degree at least 2 in G_1 by (8). Considering the neighbors of v in G_1 ,

$$(9) \quad \phi(G_1) \geq \phi(H_1) + \phi_{G_1}(v) + \frac{1}{2} \deg_{G_1}(v),$$

from Lemma 4.2(A). Similarly,

$$(10) \quad \phi_G(v) \geq \begin{cases} \phi_{G'_2}(v) + \frac{1}{2} & \text{if } \deg_{G_1}(v) = 2, \\ \phi_{G'_2}(v) + 1 & \text{if } \deg_{G_1}(v) \geq 3. \end{cases}$$

Also,

$$(11) \quad \phi_G(u) \geq \begin{cases} \phi_{G_1}(u) + \phi_{G'_2}(u) + \frac{1}{2} & \text{if } \deg_{G_1}(u) = 2, \\ \phi_{G_1}(u) + \phi_{G'_2}(u) - \frac{1}{2} & \text{if } \deg_{G_1}(u) \geq 3. \end{cases}$$

(The bound for the $\deg_{G_1}(u) \geq 3$ case is derived using Lemma 4.2(C).) By (9),

$$\begin{aligned} \phi(G) &= (\phi(G_1) - \phi_{G_1}(v)) + \phi(G'_2) + (\phi_G(v) - \phi_{G'_2}(v)) + (\phi_G(u) - \phi_{G_1}(u) - \phi_{G'_2}(u)) \\ &\geq \phi(H_1) + \frac{1}{2} \deg_{G_1}(v) + \phi(G'_2) + \phi_G(v) - \phi_{G'_2}(v) + \phi_G(u) - \phi_{G_1}(u) - \phi_{G'_2}(u). \end{aligned}$$

Combining this with (10), (11), and the inequality $\deg_{G_1}(u) \leq \deg_{G_1}(v)$ (cf. (8)), we deduce that $\phi(G) \geq \phi(H_1) + \phi(G'_2) + 2$, both in the case $\deg_{G_1}(u) = 2$ and in the case $\deg_{G_1}(u) \geq 3$. This concludes the proof. \square

LEMMA 4.4. G satisfies part (b) of Theorem 4.1.

Proof. Assume that the lemma is false. Then, by Lemma 4.3, G has a cutset $X = \{u, v\}$ with $uv \in E(G)$. Let (G_1, G_2) be a separation of G with $V(G_1) \cap V(G_2) = X$. Notice that $|V(G_i)| \geq 4$ for $i = 1, 2$ (as in Lemma 4.3).

Let S_i ($i = 1, 2$) be a minimum transversal of G_i , and let $S := S_1 \cup S_2$. If $G - S$ contains a subdivision of K_4 , then one can find a subdivision of K_4 in one of $G_1 - S_1$ and $G_2 - S_2$ thanks to the existence of the edge uv . Thus, S is a transversal of G . This implies

$$s(G) \leq s(G_1) + s(G_2).$$

First, suppose $s(G) < s(G_1) + s(G_2)$. Since the graph G_i ($i = 1, 2$) is connected and smaller than G , Theorem 4.1 gives

$$\phi(G_1) + \phi(G_2) \geq \frac{16}{3}(s(G_1) + s(G_2)) - 2 \geq \frac{16}{3}s(G) + \frac{16}{3} - 2.$$

Also,

$$\phi_G(x) \geq \phi_{G_1}(x) + \phi_{G_2}(x) - \frac{1}{2}$$

for each $x \in X$ by Lemma 4.2(C). It follows that

$$\begin{aligned} \phi(G) &= \phi(G_1) + \phi(G_2) + (\phi_G(u) - \phi_{G_1}(u) - \phi_{G_2}(u)) + (\phi_G(v) - \phi_{G_1}(v) - \phi_{G_2}(v)) \\ &\geq \left(\frac{16}{3}s(G) + \frac{16}{3} - 2\right) - \frac{1}{2} - \frac{1}{2} \geq \frac{16}{3}s(G), \end{aligned}$$

contradicting the fact that G is a counterexample to Theorem 4.1(b). Therefore, $s(G) = s(G_1) + s(G_2)$.

If the edge uv is critical in both G_1 and G_2 , then $S_i \cup \{u\}$ ($i = 1, 2$) is a transversal of G_i , where S_i ($i = 1, 2$) is a minimum transversal of $G_i - uv$ of size $s(G_i) - 1$. But then $S_1 \cup S_2 \cup \{u\}$ is a transversal of G of size $s(G) - 1$, a contradiction. Hence, uv is not critical in at least one of G_1, G_2 , say G_1 .

Let G'_1 be the graph $G_1 - uv$ where each vertex of X of degree 1 in $G_1 - uv$ is removed. The graph G'_1 is connected and $s(G'_1) = s(G_1 - uv) = s(G_1)$. Also, G'_1 and G_2 are both smaller than G , and thus satisfy Theorem 4.1. Hence,

$$\phi(G'_1) + \phi(G_2) \geq \frac{16}{3}(s(G_1) + s(G_2)) - 2 = \frac{16}{3}s(G) - 2.$$

Our aim now is to reach a contradiction by showing $\phi(G) \geq \phi(G'_1) + \phi(G_2) + 2$.

Consider the vertex u and let u_1 be a neighbor of u in $G_1 - uv$. Then, by Lemma 4.2(A),

$$(12) \quad \phi_G(u_1) - \phi_{G'_1}(u_1) \geq \begin{cases} \frac{1}{2} & \text{if } \deg_{G_1-uv}(u) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$(13) \quad \phi_G(u) - \phi_{G'_1}(u) - \phi_{G_2}(u) \geq \begin{cases} \frac{1}{2} & \text{if } \deg_{G_1-uv}(u) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

(The lower bound for the case $\deg_{G_1-uv}(u) > 1$ is derived using Lemma 4.2(D).)

Let v_1 be a neighbor of v in $G_1 - uv$. Clearly, (12) and (13) are also true if we replace u and u_1 with v and v_1 , respectively. Noticing $u_1 \neq v_1$ if $\deg_{G_1-uv}(u) = \deg_{G_1-uv}(v) = 1$ (otherwise, u_1 would be a cutvertex of G), we derive from the previous inequalities that $\phi(G) \geq \phi(G'_1) + \phi(G_2) + 2$ in every case; that is, when the number of vertices of X having degree 1 in $G_1 - uv$ is 0, 1, and 2. \square

By Lemma 4.4, the graph G is not a counterexample to part (b) of Theorem 4.1. Thus, G is a counterexample to part (a) of the theorem.

4.2. G satisfies part (a) of Theorem 4.1. Building on the fact that G is not a counterexample to part (b) of Theorem 4.1, we show in this section that G is not a counterexample to part (a) either, giving the desired contradiction.

Properties (A) and (B) from Lemma 4.2 have already been used several times in the proofs and are used many more times in what follows. For the sake of brevity, we will take them as granted from now on, no longer making explicit reference to the lemma.

LEMMA 4.5. *The graph G is reduced.*

Proof. First, suppose that G has a vertex v of degree at most 2, and let G' be obtained from G by applying the appropriate operation from Lemma 2.1 on v . The graph G' is connected with $s(G') = s(G)$ and $\phi(G') \leq \phi(G)$. This implies that G' is a smaller counterexample, a contradiction. Hence, G has minimum degree at least 3.

Next, assume that G has a bridge $uv \in E(G)$, and denote by G_1 and G_2 the two components of $G - uv$, with $u \in V(G_1)$ and $v \in V(G_2)$. Since u and v each have degree at least 3 in G , we have $\phi_G(u) \geq \phi_{G_1}(u) + \frac{1}{2}$ and $\phi_G(v) \geq \phi_{G_2}(v) + \frac{1}{2}$. Also, $s(G) = s(G_1) + s(G_2)$, which follows from the fact that no K_4 subdivision in G

contains the edge uv . Moreover, G_1 and G_2 are both connected and smaller than G . Applying Theorem 4.1 on these two graphs gives

$$\begin{aligned}\phi(G) &= \phi(G_1) + \phi(G_2) + \left(\phi_G(u) - \phi_{G_1}(u)\right) + \left(\phi_G(v) - \phi_{G_2}(v)\right) \\ &\geq \phi(G_1) + \phi(G_2) + \frac{1}{2} + \frac{1}{2} \geq \frac{16}{3}(s(G_1) + s(G_2)) - 2 + 1 = \frac{16}{3}s(G) - 1,\end{aligned}$$

a contradiction. Hence, $G - e$ is connected for every edge $e \in E(G)$. This implies that every edge $e \in E(G)$ is critical, since otherwise $G - e$ would be a smaller counterexample. It follows then from Lemma 3.2 that G is 2-connected.

Now, suppose that G has a vertex v of degree at least 6. Then $G - v$ is connected and $s(G - v) \geq s(G) - 1$. Moreover, $\phi(G) - \phi(G - v) \geq 3 + 6 \cdot \frac{1}{2} = 6$, since every neighbor of v has degree at least 3 in G . It follows that

$$\phi(G) \geq \phi(G - v) + 6 \geq \frac{16}{3}s(G - v) - 1 + 6 \geq \frac{16}{3}s(G) - 1 - 16/3 + 6 \geq \frac{16}{3}s(G) - 1,$$

which is again a contradiction. Hence, G has maximum degree at most 5, and therefore G is reduced. \square

Vertices of degree 5 in G are the focus of our attention in the next lemmas. We study the local structure of G around such vertices, eventually concluding that there is no degree-5 vertex in G .

LEMMA 4.6. *If $v \in V(G)$ has degree 5, then v has no neighbor of degree 3, and at most one of degree 4.*

Proof. Since $G - v$ is connected and $s(G - v) \geq s(G) - 1$,

$$\phi(G) = \phi(G - v) + \lambda \geq \frac{16}{3}s(G - v) - 1 + \lambda \geq \frac{16}{3}s(G) - 1 - \frac{16}{3} + \lambda,$$

where

$$\lambda := \frac{5}{2} + \sum_{x \in N_G(v)} \left(\phi_G(x) - \phi_{G-v}(x)\right).$$

For every integer $d \geq 3$,

$$\phi(d) - \phi(d - 1) = \begin{cases} \frac{4}{3} & \text{if } d = 3, \\ \frac{2}{3} & \text{if } d = 4, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Hence, if $N_G(v)$ contains a vertex of degree 3 or two vertices of degree 4, then $\lambda \geq \frac{16}{3}$, implying that G is not a counterexample to Theorem 4.1(a). \square

Thus there are only two possibilities for a vertex v of degree 5 in G : either all neighbors of v also have degree 5, or four of them have degree 5 and one has degree 4. In the former case, we call v a *pure degree-5* vertex, while in the latter case we call v a *mixed degree-5* vertex.

LEMMA 4.7. *The graph G is 3-connected. Moreover, if $X \subseteq V(G)$ is a 3-cutset of G , then $\deg_G(v) \in \{3, 4\}$ for every vertex $v \in X$.*

Proof. We already know that G is reduced by Lemma 4.5, and thus in particular that G is 2-connected. The graph G is also 3-connected, as otherwise G would also be a counterexample to Theorem 4.1(b), contradicting Lemma 4.4.

Thus it remains to show the second part of the statement. Arguing by contradiction, suppose that $X \subseteq V(G)$ is a 3-cutset of G containing a vertex v of degree 5. Then $G - v$ is 2-connected but not 3-connected. Moreover, by Lemma 4.6, $G - v$ has minimum degree at least 3.

First, assume that some edge $e \in E(G - v)$ is not critical in $G - v$. Let $G' := (G - v) - e$. Observe that $\phi(G - v) \geq \phi(G') + 1$. Since $s(G') = s(G - v) \geq s(G) - 1$ and the graph G' is connected and smaller than G , Theorem 4.1 gives

$$\begin{aligned} \phi(G) &= \phi(G - v) + \frac{5}{2} + \sum_{x \in N_G(v)} (\phi_G(x) - \phi_{G-v}(x)) \\ &\geq \phi(G - v) + 5 \\ &\geq \phi(G') + 6 \geq \frac{16}{3}s(G') - 1 + 6 \geq \frac{16}{3}s(G) - \frac{16}{3} - 1 + 6 > \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Hence, $G - v$ is critical. In particular, $G - v$ is reduced. Using Theorem 4.1(b) on $G - v$,

$$\begin{aligned} \phi(G) &= \phi(G - v) + \frac{5}{2} + \sum_{x \in N_G(v)} (\phi_G(x) - \phi_{G-v}(x)) \\ &\geq \phi(G - v) + 5 \geq \frac{16}{3}s(G - v) + 5 \geq \frac{16}{3}s(G) - \frac{16}{3} + 5 > \frac{16}{3}s(G) - 1, \end{aligned}$$

which is again a contradiction. □

The above proof shows why it is useful to prove a stronger bound in the case that the graph is reduced but not 3-connected. (Indeed, it is when we were trying to prove Lemma 4.7 that we were led to add part (b) to Theorem 4.1.)

For $v \in V(G)$, denote by G_v the subgraph of G induced by the neighborhood of v . (Note that v is not included in G_v .)

LEMMA 4.8. *Suppose that $v \in V(G)$ is a pure degree-5 vertex. Then $\alpha(G_v) = 2$.*

Proof. First, suppose that G_v has a stable set S of size 3. By Lemma 4.7, the graph $G - S$ is connected. Since S is a stable set, each vertex $u \in S$ has exactly four neighbors outside $S \cup \{v\}$. Let T be the set of all such neighbors. Note that each vertex in T has degree at least 4 by Lemma 4.6, and has between one and three neighbors in S . Using $s(G - S) \geq s(G) - 3$ and Theorem 4.1 on $G - S$,

$$\begin{aligned} \phi(G) &= \phi(G - S) + (\phi_G(v) - \phi_{G-S}(v)) + \sum_{u \in S} \phi_G(u) + \sum_{u \in T} (\phi_G(u) - \phi_{G-S}(u)) \\ &\geq \phi(G - S) + \frac{5}{2} + 3 \frac{5}{2} + 6 \\ &= \phi(G - S) + 16 \\ &\geq \frac{16}{3}s(G - S) - 1 + 16 \geq \frac{16}{3}s(G) - 3 \frac{16}{3} - 1 + 16 = \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Hence, $\alpha(G_v) \leq 2$.

On the other hand, if $\alpha(G_v) = 1$, then G is isomorphic to K_6 , because G has maximum degree at most 5. Since K_6 is not a counterexample to Theorem 4.1, we deduce $\alpha(G_v) = 2$, as claimed. □

For a subset S of vertices of G , let $e_G(S)$ denote the number of edges between S and $V(G) - S$. Observe that, if $|S| \leq 4$ and all vertices in $V(G) - S$ having a neighbor

in S have degree at least 4, then

$$\phi(G) \geq \phi(G - S) + \sum_{u \in S} \phi_G(u) + \frac{1}{2}e_G(S).$$

This observation is used in the subsequent proofs.

A *sparse bipartition* of a 5-vertex graph H is a partition (X, Y) of $V(H)$ such that $|X| = 3$, $|Y| = 2$, and $|E(H[X])| = |E(H[Y])| = 1$.

LEMMA 4.9. *Suppose that $v \in V(G)$ is a pure degree-5 vertex. Then G_v does not admit a sparse bipartition.*

Proof. Arguing by contradiction, assume G_v has a sparse bipartition (X, Y) . Let $S := X \cup \{v\}$. In $G - X$, the two neighbors of v are adjacent, implying $s(G - S) = s(G - X) \geq s(G) - 3$. By Lemma 4.7, $G - X$ is connected, and hence $G - S$ is also connected. It follows that

$$\begin{aligned} \phi(G) &\geq \phi(G - S) + \sum_{u \in S} \phi_G(u) + \frac{1}{2}e_G(S) = \phi(G - S) + 10 + 6 \\ &\geq \frac{16}{3}s(G - S) - 1 + 16 \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. \square

LEMMA 4.10. *Suppose that $v \in V(G)$ is a pure degree-5 vertex. Then G_v is isomorphic to C_5 , the cycle on five vertices.*

Proof. As shown in the previous two lemmas, $\alpha(G_v) = 2$ and G_v does not admit a sparse bipartition. Observe furthermore that the subset X of vertices of G_v having degree at most 3 in G_v is a cutset of G separating $\{v\} \cup (V(G_v) \setminus X)$ from the rest of the graph. It follows from Lemma 4.7 that $|X| \geq 4$, and thus that G_v contains at most one vertex of degree 4 in G_v .

We leave it to the reader to check that, up to isomorphism, there are exactly three graphs H on five vertices with $\alpha(H) = 2$, having no sparse bipartition, and with at most one vertex of degree 4: the wheel W_5 , the graph K_4 with an edge subdivided once (which we denote by W_5^-), and the cycle C_5 . Thus G_v is isomorphic to one of these three graphs, illustrated in Figure 1.

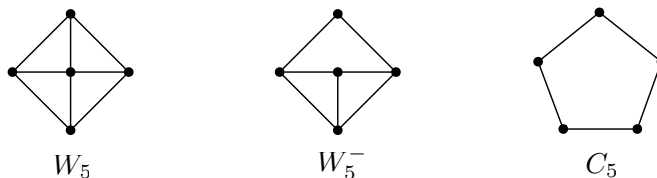


FIG. 1. Three possibilities for the graph G_v .

First, suppose $G_v \cong W_5$, and let $X \subseteq V(G_v)$ be such that $|X| = 3$ and every vertex in X has degree 3 in G_v . Let also $S := X \cup \{v\}$. Then $s(G - S) = s(G - X) \geq s(G) - 3$. Also, $G - X$ is connected by Lemma 4.7, and hence so is $G - S$. It follows

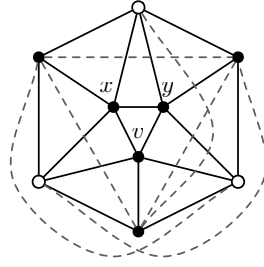


FIG. 2. The subgraph of G induced by Z . Dashed edges indicate edges that may or may not be present in $G[Z]$; white vertices are the vertices in S .

that

$$\begin{aligned} \phi(G) &\geq \phi(G - S) + 4\phi(5) + (\phi(5) - \phi(1)) + (\phi(5) - \phi(2)) + 3\frac{1}{2} \\ &= \phi(G - S) + 16 + \frac{1}{2} \geq \frac{16}{3}s(G - S) - 1 + 16 + \frac{1}{2} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Hence, G_v is not isomorphic to W_5 .

Now, assume $G_v \cong W_5^-$, and let $X \subseteq V(G_v)$ consist of the unique vertex of G_v with degree 2 together with its two neighbors. Let $S := X \cup \{v\}$. Similarly as above, the graph $G - S$ is connected, and we can check that $s(G - S) \geq s(G) - 3$ and $\phi(G) \geq \phi(G - S) + 17$, implying again $\phi(G) \geq \frac{16}{3}s(G) - 1$. Thus, G_v is not isomorphic to W_5^- either.

It follows that G_v is isomorphic to C_5 . □

Next, we deal with the case where $v \in V(G)$ is a pure degree-5 vertex and G_v is isomorphic to C_5 .

LEMMA 4.11. *There is no pure degree-5 vertex in G .*

Proof. Suppose that $v \in V(G)$ is a pure degree-5 vertex. Thus, $G_v \cong C_5$ by Lemma 4.10. First, assume that G_v contains at least three vertices which are mixed degree-5 vertices in G . Among them, there are two vertices x and y which are not adjacent (since G_v has no triangle). Moreover, x and y have at least two common neighbors with degree 5 in G , namely v and some vertex in $V(G_v) - \{x, y\}$. Combining these observations with the fact that x and y each have a degree-4 neighbor (possibly the same vertex), one obtains

$$\begin{aligned} \phi(G) &\geq \phi(G - \{x, y\}) + 2\phi(5) + 2(\phi(5) - \phi(3)) + 2\frac{2}{3} + 4\frac{1}{2} \\ &= \phi(G - \{x, y\}) + 10 + \frac{2}{3} \geq \frac{16}{3}s(G - \{x, y\}) - 1 + 10 + \frac{2}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Hence, there are at most two mixed degree-5 vertices in G_v .

Now, there are two pure degree-5 vertices x and y which are adjacent in G_v . Since G_v, G_x , and G_y are all isomorphic to C_5 by Lemma 4.10, the graph $G[Z]$ induced by the set $Z := V(G_v) \cup V(G_x) \cup V(G_y)$ is as depicted in Figure 2.

Let S be the set consisting of the three vertices in $Z - \{v, x, y\}$ that have exactly two neighbors in $\{v, x, y\}$. The set S is a stable set of G and the graph $G - S$ is connected (as follows from Lemma 4.7). Let a and b be the number of vertices with

respectively two and three neighbors in S . (Thus there are exactly a edges between S and $V(G) - Z$.) It follows that

$$\begin{aligned} \phi(G) &\geq \phi(G - S) + 3\phi(5) + (3 + a)(\phi(5) - \phi(3)) + b(\phi(5) - \phi(2)) + a\frac{1}{2} \\ &\geq \phi(G - S) + 3\phi(5) + 6(\phi(5) - \phi(3)) + 3\frac{1}{2} \\ &= \phi(G - S) + 16 \geq \frac{16}{3}s(G - S) - 1 + 16 \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

which is again a contradiction. \square

Our next aim is to prove that G does not have mixed degree-5 vertices either. We follow an approach similar to the one used for pure degree-5 vertices.

A *diamond* in G is an induced subgraph H of G isomorphic to $K_4 - e$ (the graph K_4 minus an edge) with the property that the two vertices with degree 2 in H both have degree 5 in G .

LEMMA 4.12. *There is no diamond in G .*

Proof. Suppose the contrary—that G contains a diamond H . Let x, y be the two vertices with degree 2 in H . If some vertex of $V(H) \setminus \{x, y\}$ has degree 4 in G , then

$$\phi(G) \geq \phi(G - \{x, y\}) + 2\phi(5) + (\phi(4) - \phi(2)) + (\phi(5) - \phi(3)) + 6\frac{1}{2} \geq \phi(G - \{x, y\}) + 10 + \frac{2}{3}.$$

If, on the other hand, both vertices in $V(H) - \{x, y\}$ have degree 5 in G , then

$$\phi(G) \geq \phi(G - \{x, y\}) + 2\phi(5) + 2(\phi(5) - \phi(3)) + 2\frac{2}{3} + 4\frac{1}{2} = \phi(G - \{x, y\}) + 10 + \frac{2}{3}.$$

(Here, we use that x and y each have a degree-4 neighbor in G .) Thus, in both cases,

$$\phi(G) \geq \phi(G - \{x, y\}) + 10 + \frac{2}{3} \geq \frac{16}{3}s(G - \{x, y\}) - 1 + 10 + \frac{2}{3} \geq \frac{16}{3}s(G) - 1,$$

contradicting the fact that G is a counterexample. \square

LEMMA 4.13. *Suppose $v \in V(G)$ is a mixed degree-5 vertex and let w be the unique neighbor of v of degree 4. Then $w \in S$ for every stable set S of G_v of size 3.*

Proof. If there is a stable set of size 3 in G_v avoiding w , then we can reach a contradiction exactly as in the first part of the proof of Lemma 4.8. \square

Suppose that $v \in V(G)$ is a mixed degree-5 vertex. Here, we define a sparse bipartition of G_v as a partition (X, Y) of $V(G_v)$ such that $|X| = 3, |Y| = 2$, and $|E(G[X])| = |E(G[Y])| = 1$, as for pure degree-5 vertices, *but* with the additional requirement that Y includes the unique neighbor of v having degree 4 in G .

LEMMA 4.14. *Suppose that $v \in V(G)$ is a mixed degree-5 vertex. Then G_v does not admit a sparse bipartition.*

Proof. Assume (X, Y) is a sparse bipartition of G_v . Since the two vertices in Y are adjacent, $s(G - S) = s(G - X) \geq s(G) - 3$, where $S := X \cup \{v\}$. Let w be the vertex in Y of degree 4. Observe that w sees at most three vertices in S , since w is adjacent to the other vertex in Y . Thus every degree-4 vertex in G sees at most three vertices from S . Since $G - S$ is connected,

$$\phi(G) \geq \phi(G - S) + 4\phi(5) + 4\frac{2}{3} + 8\frac{1}{2} \geq \phi(G - S) + 16 \geq \frac{16}{3}s(G) - 1,$$

a contradiction. \square

LEMMA 4.15. *Suppose that $v \in V(G)$ is a mixed degree-5 vertex and let w be the unique neighbor of v of degree 4. Then w has degree at most 1 in G_v .*

Proof. Arguing by contradiction, assume x, y are two distinct neighbors of w in G_v . If $xy \notin E(G)$, then we obtain

$$\begin{aligned} \phi(G) &\geq \phi(G - \{x, y\}) + 2\phi(5) + (\phi(5) - \phi(3)) + (\phi(4) - \phi(2)) + 6\frac{1}{2} \\ &\geq \phi(G - \{x, y\}) + 10 + \frac{2}{3} \geq \frac{16}{3}s(G - \{x, y\}) - 1 + 10 + \frac{2}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Thus, we may assume $xy \in E(G)$.

Let $Q := N_G(v) - \{w\}$. We have $\alpha(G[Q]) \leq 2$ by Lemma 4.13. Also, if $|E(G[X])| = 2$ for some $X \subseteq Q$ with $|X| = 3$, then the subgraph of G induced by $X \cup \{v\}$ is a diamond of G , which Lemma 4.12 forbids. On the other hand, if we had $|E(G[X])| = 1$ for some $X \in \{Q - \{x\}, Q - \{y\}\}$, then $(X, N_G(v) - X)$ would be a sparse bipartition of G_v , which would contradict Lemma 4.14.

Since $xy \in E(G)$, it follows from the previous observations that $G[Q]$ is a complete graph. In particular, x and y have no neighbor outside $N_G(v) \cup \{v\}$. Notice also that some vertex in Q is not adjacent to w , and thus has a neighbor outside $N_G(v) \cup \{v\}$. It follows that $Z := N_G(v) - \{x, y\}$ separates $\{v, x, y\}$ from the rest of the graph. Since $|Z| = 3$ and Z includes a vertex of degree 5, this contradicts Lemma 4.7. \square

LEMMA 4.16. *Suppose that $v \in V(G)$ is a mixed degree-5 vertex and let w be the unique neighbor of v with degree 4. Then G_v is one of the two graphs depicted in Figure 3.*

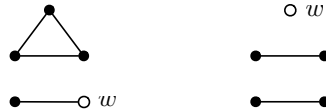


FIG. 3. *Two possibilities for the graph G_v when v is a mixed degree-5 vertex.*

Proof. Let $Q := N_G(v) - \{w\}$. Lemmas 4.12 and 4.13 imply that $G[Q]$ has at most two components and that each component is a complete graph. Hence, $G[Q]$ is isomorphic to K_4 , $K_3 \cup K_1$, or $K_2 \cup K_2$. (As expected, $K_t \cup K_\ell$ denotes the disjoint union of K_t and K_ℓ .)

First, assume $G[Q] \cong K_4$, and let x, y be two distinct vertices of Q such that $wx \notin E(G)$ and $wy \notin E(G)$ (such vertices exist by Lemma 4.15). Let $S := \{v, w, x, y\}$. Then $s(G - S) = s(G - \{w, x, y\}) \geq s(G) - 3$. Using that each of x and y has a neighbor outside $N_G(v) \cup \{v\}$, that w has at least two neighbors outside $N_G(v) \cup \{v\}$ (cf. Lemma 4.15), and, as usual, that $G - S$ is connected,

$$\begin{aligned} \phi(G) &\geq \phi(G - S) + 3\phi(5) + \phi(4) + 2(\phi(5) - \phi(2)) + 4\frac{1}{2} \\ &\geq \phi(G - S) + 16 \geq \frac{16}{3}s(G - S) - 1 + 16 \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Thus $G[Q]$ is not isomorphic to K_4 .

Recall that w has degree at most 1 in G_v , by Lemma 4.15. Suppose that w has degree 1 in G_v , and let z be its unique neighbor. We cannot have $|E(G[N_G(v) -$

$\{w, z\}\}) = 1$, since otherwise $(N_G(v) - \{w, z\}, \{w, z\})$ would be a sparse bipartition of G_v . It follows that $G[Q]$ is isomorphic to $K_3 \cup K_1$ and that z is the isolated vertex of that graph. Hence, G_v is isomorphic to the left graph in Figure 3.

Now, assume w has degree 0 in G_v . Suppose $G[Q] \cong K_3 \cup K_1$, and let X be a stable set of G_v with $|X| = 3$ (thus $w \in X$). Let $S := X \cup \{v\}$. Similarly as before, we deduce

$$\begin{aligned} \phi(G) &\geq \phi(G - S) + 3\phi(5) + \phi(4) + 2(\phi(5) - \phi(3)) + 9\frac{1}{2} \\ &\geq \phi(G - S) + 16 \geq \frac{16}{3}s(G - S) - 1 + 16 \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Thus $G[Q] \cong K_2 \cup K_2$, and G_v is isomorphic to the graph on the right in Figure 3. \square

LEMMA 4.17. *If G has maximum degree 5, then there exists an almost induced even cycle C in G such that every vertex in C has degree 5.*

Proof. Following Lemma 4.16, a vertex v of G is said to be of *type 1* (*type 2*) if v is a mixed degree-5 vertex and G_v is isomorphic to the left (resp., right) graph in Figure 3.

Let $H_i \subseteq G$ be the subgraph of G induced by the set of vertices of type i for $i = 1, 2$. Notice that, if $v \in V(G)$ is of type i , then so are the four neighbors of v that have degree 5 in G . Thus H_i is either 4-regular or empty for $i = 1, 2$.

If there is a type 2 vertex in G , then H_2 is 4-regular and has no subgraph isomorphic to K_4 . Using Lemma 3.3 on H_2 , we obtain a cycle C as desired. Thus, we may assume that every degree-5 vertex in G is of type 1. In particular, there is at least one such vertex.

Now, the graph H_1 is 4-regular, and every vertex of H_1 is contained in exactly one copy of K_4 . More precisely, all copies of K_4 in H_1 are pairwise vertex-disjoint, they cover all of $V(H_1)$, and each copy sends exactly four edges to other copies of K_4 in H_1 . Observe also that the set of edges of H_1 that link two distinct copies of K_4 form a perfect matching of H_1 .

Let \tilde{H}_1 be the multigraph obtained by contracting each copy of K_4 into one vertex (parallel edges between distinct vertices are kept, but loops are removed). It follows from the previous observation that \tilde{H}_1 is 4-regular. Let \tilde{C} be any induced cycle of \tilde{H}_1 (note that a cycle of length 2 is allowed). The cycle \tilde{C} naturally corresponds to an induced cycle C of H_1 having length $2|\tilde{C}|$. This latter cycle is as desired: C has even length, is induced (and thus almost induced) in G , and contains only vertices of degree 5. \square

LEMMA 4.18. *G has maximum degree at most 4.*

Proof. Arguing by contradiction, we assume that G has maximum degree 5. Let C be a cycle of G as in Lemma 4.17. Enumerate the vertices of C in order as v_1, v_2, \dots, v_{2k} so that $S := \{v_1, v_3, \dots, v_{2k-1}\}$ is a stable set of G . For $i \in \{1, 2, \dots, k\}$, let $S_i := \{v_1, v_3, \dots, v_{2i-1}\}$, and define X_i^4 (X_i^5) as the set of vertices in $V(G) - V(C)$ of degree 4 (resp., degree 5) that have a neighbor in S_i . Let j be the largest index such that the following three properties hold:

- $G - S_j$ is 3-connected;
- $|X_j^4| = j$; and
- $|X_j^5| = 2j$.

Note that $|X_1^4| = 1$, $|X_1^5| = 2$, and $G - S_1$ is 3-connected by Lemma 4.7; thus, j is well defined. We distinguish two cases, depending on whether $j = k$ or $j < k$.

Case 1. $j = k$: We have $s(G - S) \geq s(G) - k$. Since $|X_k^4| = k$ and $|X_j^5| = 2k$, no two vertices in $S (= S_k)$ have a common neighbor in $V(G) - V(C)$. Thus, every vertex in $V(G) - V(C)$ has at most one neighbor in S . Also, every vertex in $V(C) - S$ has exactly two neighbors in S , because C is almost induced. Since $G - S$ is connected,

$$\begin{aligned} \phi(G) &= \phi(G - S) + k\phi(5) + k(\phi(5) - \phi(3)) + k(\phi(4) - \phi(3)) + 2k(\phi(5) - \phi(4)) \\ &= \phi(G - S) + k\frac{16}{3} \geq \frac{16}{3}s(G - S) - 1 + k\frac{16}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

contradicting the fact that G is a counterexample.

Case 2. $j < k$: Here, we consider the set S_{j+1} . Exactly j vertices in $V(C) - S_{j+1}$ have two neighbors in S_{j+1} , and exactly two have one. Let $G' := G - S_{j+1}$. For $\ell \in \{4, 5\}$, let

$$\Delta_{j+1}^\ell := \sum_{v \in X_{j+1}^\ell} (\phi_G(v) - \phi_{G'}(v)).$$

We have

$$\begin{aligned} \phi(G) &= \phi(G') + (j + 1)\phi(5) + j(\phi(5) - \phi(3)) + 2(\phi(5) - \phi(4)) + \Delta_{j+1}^4 + \Delta_{j+1}^5 \\ &= \phi(G') + (j + 1)\frac{16}{3} - \frac{1}{6} + \left(\Delta_{j+1}^4 - (j + 1)\frac{2}{3}\right) + \left(\Delta_{j+1}^5 - 2(j + 1)\frac{1}{2}\right). \end{aligned}$$

Observe that, by our choice of j , every vertex in X_{j+1}^4 and X_{j+1}^5 has at most two neighbors in S_{j+1} . Since

$$\phi(4) - \phi(2) \geq 2(\phi(4) - \phi(3)) + \frac{1}{6}$$

and

$$\phi(5) - \phi(3) = 2(\phi(5) - \phi(4)) + \frac{1}{6},$$

we have

$$\Delta_{j+1}^4 \geq \begin{cases} (j + 1)\frac{2}{3} + \frac{1}{6} & \text{if } |X_{j+1}^4| < j + 1, \\ (j + 1)\frac{2}{3} & \text{otherwise,} \end{cases}$$

and

$$\Delta_{j+1}^5 \geq \begin{cases} 2(j + 1)\frac{1}{2} + \frac{1}{6} & \text{if } |X_{j+1}^5| < 2(j + 1), \\ 2(j + 1)\frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus, if $|X_{j+1}^4| < j + 1$ or $|X_{j+1}^5| < 2(j + 1)$, since $s(G') \geq s(G) - (j + 1)$ and that G' is connected,

$$\phi(G) \geq \phi(G') + (j + 1)\frac{16}{3} \geq \frac{16}{3}s(G') - 1 + (j + 1)\frac{16}{3} \geq \frac{16}{3}s(G) - 1,$$

a contradiction. Therefore, $|X_{j+1}^4| = j + 1$ and $|X_{j+1}^5| = 2(j + 1)$. By definition of j , this implies that G' is 2-connected but not 3-connected. Note also that $\Delta_{j+1}^4 = (j + 1)\frac{2}{3}$ and $\Delta_{j+1}^5 = 2(j + 1)\frac{1}{2}$, implying

$$\phi(G) = \phi(G') + (j + 1)\frac{16}{3} - \frac{1}{6}.$$

Every vertex of C has degree 5 in G and every vertex in $V(G) - V(C)$ has at most one neighbor in S_{j+1} . Moreover, no degree-3 vertex in $V(G) - V(C)$ has a neighbor in C . It follows that G' has minimum degree 3.

Suppose $s(G') = s(G' - e)$ for some edge $e \in E(G')$. Then $\phi(G') \geq \phi(G' - e) + 1$ (since G' has minimum degree at least 3), and since $G' - e$ is connected,

$$\phi(G) = \phi(G') + (j+1)\frac{16}{3} - \frac{1}{6} \geq \phi(G' - e) + (j+1)\frac{16}{3} - \frac{1}{6} + 1 \geq \frac{16}{3}s(G) - 1,$$

a contradiction. Hence, the graph G' is critical, which in turn implies that G' is reduced.

Now, since G' is reduced but not 3-connected, we may apply Theorem 4.1(b) on G' , yielding

$$\phi(G) \geq \phi(G') + (j+1)\frac{16}{3} - \frac{1}{6} \geq \frac{16}{3}s(G') + (j+1)\frac{16}{3} - \frac{1}{6} \geq \frac{16}{3}s(G) - 1,$$

which is again a contradiction. \square

Note that the last paragraph of the above proof relies crucially on the stronger hypothesis in the reduced but not 3-connected case.

Lemma 4.18 concludes the heart of the proof, namely showing that there is no vertex of degree 5 in G . Now, it only remains to deal with vertices of degree 3 and 4, which is fairly easy in comparison.

LEMMA 4.19. *G is 4-regular and has no subgraph isomorphic to K_4 .*

Proof. First, suppose that G contains some vertex v of degree 4 having a degree-3 neighbor. Since $G - v$ is connected,

$$\begin{aligned} \phi(G) &\geq \phi(G - v) + \phi(4) + 3(\phi(4) - \phi(3)) + (\phi(3) - \phi(2)) \\ &= \phi(G - v) + \frac{16}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Thus, G is either cubic (3-regular) or 4-regular. If G is cubic, then, letting $v \in V(G)$ be an arbitrary vertex of G ,

$$\phi(G) \geq \phi(G - v) + \phi(3) + 3(\phi(3) - \phi(2)) = \phi(G - v) + \frac{16}{3} \geq \frac{16}{3}s(G) - 1,$$

again a contradiction. Hence, G is 4-regular. Also, $|V(G)| \geq 6$, since K_5 is not a counterexample.

Now, assume $X \subset V(G)$ induces a subgraph of G isomorphic to K_4 . Let $u \in X$ and let v be the unique neighbor of u in $V(G) - X$. Since $G \not\cong K_5$, there is some vertex $w \in X$ that is not adjacent to v .

The two neighbors x and y of u in $G - \{v, w\}$ are adjacent; hence

$$s(G - \{u, v, w\}) = s(G - \{v, w\}) \geq s(G) - 2.$$

Let z be the unique neighbor of w in $V(G) - (X \cup \{v\})$. The vertices x and y have degree at most 2 in $G - \{u, v, w\}$, and z has degree at most 3 in that graph. Combining these observations with the fact that $G - \{u, v, w\}$ is connected (since $G - \{v, w\}$ is), we deduce

$$\begin{aligned} \phi(G) &\geq \phi(G - \{u, v, w\}) + 3\phi(4) + 2(\phi(4) - \phi(2)) + (\phi(4) - \phi(3)) \\ &= \phi(G - \{u, v, w\}) + 2\frac{16}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

a contradiction. Therefore, G contains no subgraph isomorphic to K_4 . \square

We are now in a position to complete the proof of Theorem 4.1 by showing that $\phi(G) \geq \frac{16}{3}s(G) - 1$, and thus that G is not a counterexample to part (a) of Theorem 4.1, a final contradiction.

LEMMA 4.20. $\phi(G) \geq \frac{16}{3}s(G) - 1$.

Proof. The proof is similar to that of Lemma 4.18. The main difference is that, in the latter proof, a vertex v of the cycle C with two neighbors in the stable set S_i has degree 3 in $G - S_i$, while here it will have degree 2 in $G - S_i$. In some cases, we will need to eliminate these degree-2 vertices using the relevant operations (cf. Lemma 2.1). A second difference is that here we are able to choose C simply as a *shortest* even cycle in G , which will ensure that no two vertices in S have a common neighbor outside $V(C)$ when $|C| \geq 6$, simplifying the case analysis somewhat. (We could not have done that in the proof of Lemma 4.18 because the graph might have contained K_4 as a subgraph.)

Let C be a shortest even cycle in G . Since G is 4-regular and has no subgraph isomorphic to K_4 , by Lemma 3.3 such a cycle exists, and it is almost induced. Enumerate the vertices of C in order as v_1, v_2, \dots, v_{2k} so that $S := \{v_1, v_3, \dots, v_{2k-1}\}$ is a stable set of G . We may further assume that if C is not induced, then the unique chord of C is incident to v_{2k} .

First suppose that $|C| = 4$. Then $G - S$ is connected. We have $s(G - S) \geq s(G) - 2$, and

$$\phi(G) \geq \phi(G - S) + 4\phi(4) + 4(\phi(4) - \phi(3)) = \phi(G - S) + 10 + \frac{2}{3} \geq \frac{16}{3}s(G) - 1.$$

Next, assume $|C| \geq 6$. Here we cannot simply remove S from G , since $G - S$ might no longer be connected. For each $i \in \{1, 2, \dots, k\}$, let $S_i := \{v_1, v_3, \dots, v_{2i-1}\}$, let x_i be a neighbor of v_{2i} outside $V(C)$, and let $M_i := \{v_2x_1, v_4x_2, \dots, v_{2i}x_i\}$. Observe that no two vertices of C which are at even distance on C share a common neighbor outside C , for otherwise there would be an even cycle shorter than C . (Here we use that $|C| \geq 6$.) Thus, no two vertices in S have a common neighbor outside $V(C)$, and no two vertices in $V(C) - S$ have a common neighbor outside $V(C)$. In particular, M_i is a matching of G for each $i \in \{1, 2, \dots, k\}$.

Let j be the largest index in $\{1, 2, \dots, k\}$ such that

- $G - S_j$ is 2-connected, and
- none of $v_2, v_4, \dots, v_{2j-2}$ lies in a triangle in $G - S_j$.

Since $G - S_1$ is 2-connected and the second condition is vacuous for $j = 1$, the two properties hold for $j = 1$, and thus the index j is well defined. We distinguish two cases, depending on whether $j = k$ or $j < k$.

Case 1. $j = k$: We have $s(G - S) \geq s(G) - k$ and

$$\begin{aligned} \phi(G) &= \phi(G - S) + k\phi(4) + k(\phi(4) - \phi(2)) + 2k(\phi(4) - \phi(3)) \\ &= \phi(G - S) + k\frac{16}{3} \geq \frac{16}{3}s(G - S) - 1 + k\frac{16}{3} \geq \frac{16}{3}s(G) - 1, \end{aligned}$$

as desired.

Case 2. $j < k$: Here we know that $G - S_{j+1}$ is connected (but perhaps not 2-connected).

First suppose that v_{2j} is in a triangle of $G - S_{j+1}$, and let $G' := G - (S_{j+1} \cup \{v_{2j}\})$. Then $s(G') = s(G - S_{j+1}) \geq s(G) - (j + 1)$. Also, x_j has degree at least 3 in $G - S_{j+1}$, since each vertex of $V(G) - V(C)$ sees at most one vertex from S in G . (This is also true for the other neighbor of v_{2j} in $G - S_{j+1}$ if it is outside $V(C)$;

note, however, that this second neighbor could be the vertex v_{2k} in case C has a chord.) Since $\phi(3) - \phi(2) \geq \phi(4) - \phi(3)$, considering the vertex x_j we deduce that $\phi(G - S_{j+1}) \geq \phi(G') + (\phi(4) - \phi(3)) = \phi(G') + \frac{2}{3}$.

Combining the previous observations,

$$\begin{aligned} \phi(G) &= \phi(G - S_{j+1}) + (j+1)\phi(4) + j(\phi(4) - \phi(2)) + 2(j+1)(\phi(4) - \phi(3)) + 2(\phi(4) - \phi(3)) \\ &= \phi(G - S_{j+1}) + (j+1)\frac{16}{3} - \frac{2}{3} \\ &\geq \phi(G') + (j+1)\frac{16}{3} \geq \frac{16}{3}(s(G) - (j+1)) - 1 + (j+1)\frac{16}{3} = \frac{16}{3}(s(G) - 1). \end{aligned}$$

Hence we may assume that v_{2j} is not in a triangle in $G - S_{j+1}$, and it follows that the latter graph is connected but not 2-connected.

Let z be a cutvertex of $G - S_{j+1}$. We may assume that z has been chosen so that it is distinct from v_2, v_4, \dots, v_{2j} (if not, simply replace z by one of its two neighbors). Now we will focus on the graph $G - S_j$. Clearly, $\{v_{2j+1}, z\}$ is a 2-cutset of that graph. More importantly, $\{v_{2j+1}, z\}$ is also a 2-cutset of $G' := (G - S_j)/M_j$, the graph obtained from $G - S_j$ by contracting each edge of the matching M_j . Thus G' is not 3-connected. On the other hand, G' is 2-connected, since any cutvertex of G' would also be a cutvertex of $G - S_j$ (which is 2-connected).

We claim that v_2, v_4, \dots, v_{2j} are the only vertices of degree 2 in $G - S_j$, and that every other vertex has degree at least 3. This is clear if none of v_2, v_4, \dots, v_{2j} is incident to a chord of C , since no vertex from $V(G) - V(C)$ sees two vertices from S in G . If, on the other hand, there is a chord of C incident to one of these vertices, then it is of the form $v_{2\ell}v_{2k}$ for some $\ell \in \{1, 2, \dots, j\}$, and we observe that v_{2k} has degree 3 in $G - S_j$ since $j < k$. Again this shows that v_2, v_4, \dots, v_{2j} are the only vertices of degree 2 in $G - S_j$.

By the previous observation, it follows that G' has minimum degree 3. If some edge e of G' is not critical, then, since $G' - e$ is connected and $s(G' - e) = s(G') = s(G - S_j)$,

$$\begin{aligned} \phi(G) &= \phi(G - S_j) + j\phi(4) + (j-1)(\phi(4) - \phi(2)) + 2j(\phi(4) - \phi(3)) + 2(\phi(4) - \phi(3)) \\ &= \phi(G - S_j) + j\frac{16}{3} - \frac{2}{3} \\ &= \phi(G') + j\frac{16}{3} - \frac{2}{3} \\ &\geq \phi(G' - e) + j\frac{16}{3} \geq \frac{16}{3}(s(G) - j) - 1 + j\frac{16}{3} = \frac{16}{3}(s(G) - 1). \end{aligned}$$

Thus we may assume that G' is critical, implying that G' is reduced. Since G' is not 3-connected, applying part (b) of Theorem 4.1 on G' yields

$$\begin{aligned} \phi(G) &= \phi(G - S_j) + j\phi(4) + (j-1)(\phi(4) - \phi(2)) + 2j(\phi(4) - \phi(3)) + 2(\phi(4) - \phi(3)) \\ &= \phi(G - S_j) + j\frac{16}{3} - \frac{2}{3} \\ &= \phi(G') + j\frac{16}{3} - \frac{2}{3} \\ &\geq \frac{16}{3}s(G') + j\frac{16}{3} - \frac{2}{3} \geq \frac{16}{3}(s(G) - j) + j\frac{16}{3} - \frac{2}{3} \geq \frac{16}{3}(s(G) - 1), \end{aligned}$$

which concludes the proof. \square

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