European Journal of Combinatorics 33 (2012) 1226–1245

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Small minors in dense graphs

Samuel Fiorini [a], Gwenaël Joret [b,1], Dirk Oliver Theis [c], David R. Wood [d]

[a] Département de Mathématique, Université Libre de Bruxelles, Belgium

[b] Département d'Informatique, Université Libre de Bruxelles, Belgium

[c] Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, Germany

[d] Department of Mathematics and Statistics, The University of Melbourne, Australia

ARTICLE INFO

Article history:
Received 12 July 2011
Received in revised form
23 February 2012
Accepted 23 February 2012
Available online 17 March 2012

ABSTRACT

A fundamental result in structural graph theory states that every graph with large average degree contains a large complete graph as a minor. We prove this result with the extra property that the minor is small with respect to the order of the whole graph. More precisely, we describe functions f and h such that every graph with n vertices and average degree at least $f(t)$ contains a K_t-model with at most $h(t) \cdot \log n$ vertices. The logarithmic dependence on n is best possible (for fixed t). In general, we prove that $f(t) \leq 2^{t-1} + \varepsilon$. For $t \leq 4$, we determine the least value of $f(t)$; in particular, $f(3) = 2 + \varepsilon$ and $f(4) = 4 + \varepsilon$. For $t \leq 4$, we establish similar results for graphs embedded on surfaces, where the size of the K_t-model is bounded (for fixed t).

1. Introduction

A fundamental result in structural graph theory states that every sufficiently dense graph contains a large complete graph as a minor.[2] More precisely, there is a minimum function $f(t)$ such that every graph with average degree at least $f(t)$ contains a K_t-minor. Mader [17] first proved that $f(t) \leq 2^{t-2}$, and later proved that $f(t) \in O(t \log t)$ [18]. Kostochka [8,9] and Thomason [23,24] proved that $f(t) \in \Theta\left(t\sqrt{\log t}\right)$; see [25] for a survey of related results.

E-mail addresses: sfiorini@ulb.ac.be (S. Fiorini), gjoret@ulb.ac.be (G. Joret), dirk.theis@ovgu.de (D.O. Theis), woodd@unimelb.edu.au (D.R. Wood).

1 Postdoctoral Researcher of the Fonds National de la Recherche Scientifique (F.R.S.–FNRS).

2 We consider simple, finite, undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. Let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges.

doi:10.1016/j.ejc.2012.02.003

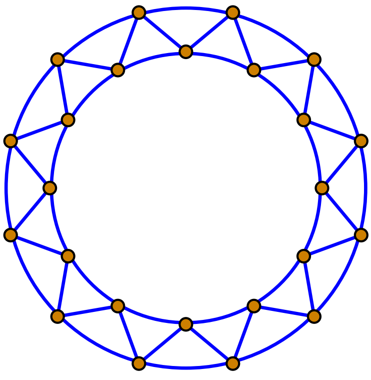

Fig. 1. C^2_{24}.

Here we prove similar results with the extra property that the K_t-minor is 'small' with respect to the order of the graph. This idea is evident when $t = 3$. A graph contains a K_3-minor if and only if it contains a cycle. Every graph with average degree at least 2 contains a cycle, whereas every graph G with average degree at least 3 contains a cycle of length $O(\log |G|)$. That is, high average degree forces a short cycle, which can be thought of as a small K_3-minor.

In general, we measure the size of a K_t-minor via the following definition. A K_t-*model* in a graph G consists of t connected subgraphs $B_1, \ldots, B_t$ of G, such that $V(B_i) \cap V(B_j) = \varnothing$ and some vertex in B_i is adjacent to some vertex in B_j for all distinct $i, j \in \{1, \ldots, t\}$. The B_i are called *branch sets*. Clearly a graph contains a K_t-minor if and only if it contains a K_t-model. We measure the size of a K_t-model by the total number of vertices, $\sum_{i=1}^{t} |B_i|$. Our main result states that every sufficiently dense graph contains a small model of a complete graph.

Theorem 1.1. *There are functions f and h such that every graph G with average degree at least $f(t)$ contains a K_t-model with at most $h(t) \cdot \log |G|$ vertices.*

For fixed t, the logarithmic upper bound in Theorem 1.1 is within a constant factor of being optimal, since every K_t-model contains a cycle, and for all $d \geq 3$ and $n > 3d$ such that nd is even, Chandran [2] constructed a graph with n vertices, average degree d, and girth at least $(\log_d n) - 1$. (The *girth* of a graph is the length of a shortest cycle.)

In this paper we focus on minimising the function f in Theorem 1.1 and do not calculate h explicitly. In particular, Theorem 4.3 proves Theorem 1.1 with $f(t) \leq 2^{t-1} + \varepsilon$ for any $\varepsilon > 0$ (where the function h also depends on ε). Note that for Theorem 1.1 and all our results, the proofs can be easily adapted to give polynomial algorithms that compute the small K_t-model.

For $t \leq 4$, we determine the least possible value of $f(t)$ in Theorem 1.1. The $t = 2$ case is trivial—one edge is a small K_2-minor. To force a small K_3-model, average degree 2 is not enough, since every K_3-model in a large cycle uses every vertex. On the other hand, we prove that average degree $2 + \varepsilon$ forces a cycle of length $O_\varepsilon(\log |G|)$; see Lemma 3.2. For $t = 4$ we prove that average degree $4 + \varepsilon$ forces a K_4-model with $O_\varepsilon(\log |G|)$ vertices; see Theorem 3.3. This result is also best possible. Consider the square of an even cycle C^2_{2n}, which is a 4-regular graph illustrated in Fig. 1. If the base cycle is $(v_1, \ldots, v_{2n})$ then $C^2_{2n} - \{v_i, v_{i+1}\}$ is outerplanar for each i. Since outerplanar graphs contain no K_4-minor, every K_4-model in C^2_{2n} contains v_i or v_{i+1} for each i, and thus contains at least n vertices.

Motivated by Theorem 1.1, we then consider graphs that contain K_3-models and K_4-models of bounded size (not just small with respect to $|G|$). First, we prove that planar graphs satisfy this property. In particular, every planar graph with average degree at least $2 + \varepsilon$ contains a K_3-model with $O\left(\frac{1}{\varepsilon}\right)$ vertices (Theorem 5.1). This bound on the average degree is best possible since a cycle is planar and has average degree 2. Similarly, every planar graph with average degree at least $4 + \varepsilon$ contains a K_4-model with $O\left(\frac{1}{\varepsilon}\right)$ vertices (Theorem 5.8). Again, this bound on the average degree is best possible since C_{2n}^2 is planar and has average degree 4. These results generalise for graphs embedded on other surfaces (Theorems 6.2 and 6.4).

Finally, we mention three other results in the literature that force a model of a complete graph of bounded size.

- Kostochka and Pyber [11] proved that for every integer t and $\varepsilon > 0$, every n-vertex graph with at least $4^{t^2} n^{1+\varepsilon}$ edges contains a subdivision of K_t with at most $\frac{7}{\varepsilon} t^2 \log t$ vertices; see [7] for recent related results. We emphasise that, for fixed t, the results in [11,7] prove that a super-linear lower bound on the number of edges (in terms of the number of vertices) forces a K_t-minor (in fact, a subdivision) of bounded size, whereas Theorem 1.1 proves that a linear lower bound on the number of edges forces a small K_t-minor (of size logarithmic in the order of the graph). Also note that Theorem 1.1 can be proved by adapting the proof of Kostochka and Pyber [11]. As far as we can tell, this method does not give a bound better than $f(t) \le 16^t + \varepsilon$ (ignoring lower order terms). This bound is inferior to our Theorem 4.3, which proves $f(t) \le 2^{t-1} + \varepsilon$. Also note that the method of Kostochka and Pyber [11] can be adapted to prove the following result about forcing a small subdivision.

  Theorem 1.2. *There is a function h such that for every integer $t \ge 2$ and real $\varepsilon > 0$, every graph G with average degree at least $4^{t^2} + \varepsilon$ contains a subdivision of K_t with at most $h(t, \varepsilon) \cdot \log |G|$ division vertices per edge.*

- Kühn and Osthus [16] proved that every graph with minimum degree at least t and girth at least 27 contains a K_{t+1}-subdivision. Every graph with average degree at least $2t$ contains a subgraph with minimum degree at least t. Thus every graph with average degree at least $2t$ contains a K_{t+1}-subdivision or a K_3-model with at most 26 vertices.
- Krivelevich and Sudakov [13] proved that for all integers $s' \ge s \ge 2$, there is a constant $c > 0$, such that every $K_{s,s'}$-free graph with average degree r contains a minor with average degree at least $cr^{1+1/(2s-2)}$. Applying the result of Kostochka [8,9] and Thomason [23] mentioned above, for every integer $s \ge 2$ there is a constant c such that every graph with average degree at least $c\left(t\sqrt{\log t}\right)^{1-1/(2s-1)}$ contains a K_t-minor or a $K_{s,s}$-subgraph, in which case there is a K_{s+1}-model with $2s$ vertices.

2. Definitions and notations

See [3] for undefined graph-theoretic terminology and notation. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. Let $e(S) := \|G[S]\|$. For disjoint sets $S, T \subseteq V(G)$, let $e(S, T)$ be the number of edges between S and T in G.

A *separation* in a graph G is a pair of subgraphs $\{G_1, G_2\}$, such that $G = G_1 \cup G_2$ and $V(G_1) \setminus V(G_2) \neq \emptyset$ and $V(G_2) \setminus V(G_1) \neq \emptyset$. The *order* of the separation is $|V(G_1) \cap V(G_2)|$. A separation of order 1 corresponds to a cut-vertex v, where $V(G_1) \cap V(G_2) = \{v\}$. A separation of order 2 corresponds to a cut-pair v, w, where $V(G_1) \cap V(G_2) = \{v, w\}$.

See [20] for background on graphs embedded in surfaces. Let $\mathbb{S}_h$ be the orientable surface obtained from the sphere by adding h handles. The *Euler genus* of $\mathbb{S}_h$ is $2h$. Let $\mathbb{N}_c$ be the non-orientable surface obtained from the sphere by adding c cross-caps. The *Euler genus* of $\mathbb{N}_c$ is c.

An *embedded graph* means a connected graph that is 2-cell embedded in $\mathbb{S}_h$ or $\mathbb{N}_c$. A *plane graph* is a planar graph embedded in the plane. Let $F(G)$ denote the set of faces in an embedded graph G. For a face $f \in F(G)$, let $|f|$ be the length of the facial walk around f. For a vertex v of G, let $F(G, v)$ be the multiset of faces incident to v, where the multiplicity of a face f in $F(G, v)$ equals the multiplicity of v in the facial walk around f. Thus $|F(G, v)| = \deg(v)$.

Euler's formula states that $|G| - \|G\| + |F(G)| = 2 - g$ for a connected graph G embedded in a surface with Euler genus g. Note that $g \le \|G\| - |G| + 1$ since $|F(G)| \ge 1$. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds.

We now review some well-known results that will be used implicitly (see [3, Section 7.3]). If a graph G contains no K_4-minor then $\|G\| \le 2|G| - 3$, and if $|G| \ge 2$ then G contains at least two vertices with degree at most 2. Hence, if $\|G\| > 2|G| - 3$ then G contains a K_4-minor. Similarly, if $|G| \geq 2$ and at most one vertex in G has degree at most 2, then G contains a K_4-minor.

Throughout this paper, logarithms are binary unless stated otherwise.

3. Small K_3-models and K_4-models

In this section we prove tight bounds on the average degree that forces a small K_3-model or K_4-model. The following lemma is at the heart of many of our results. It is analogous to Lemma 1.1 in [11]

Lemma 3.1. *There is a function p such that for every two reals $d > d' \geq 2$, every graph G with average degree at least d contains a subgraph with average degree at least d' and diameter at most $p(d, d') \cdot \log |G|$.*

Proof. We may assume that every proper subgraph of G has average degree strictly less than d (otherwise, simply consider a minimal subgraph with that property). Let

$$\beta := \frac{d}{d'} > 1 \quad \text{and} \quad p(d, d') := \frac{2}{\log \beta} + 2.$$

Let v be an arbitrary vertex of G. Let $B_k(v)$ be the subgraph of G induced by the set of vertices at distance at most k from v. Let $k \geq 1$ be the minimum integer such that $|B_k(v)| < \beta \cdot |B_{k-1}(v)|$. (There exists such a k, since $\beta > 1$ and G is finite.) It follows that $\beta^{k-1} \leq |B_{k-1}(v)| \leq |G|$, and $B_k(v)$ has diameter at most $2k \leq 2(\log_\beta |G| + 1) \leq p(d, d') \cdot \log |G|$.

We now show that $B_k(v)$ also has average degree at least d'. Let

$$A := V(B_{k-1}(v)),$$
$$B := V(B_k(v)) \setminus V(B_{k-1}(v)),$$
$$C := V(G) \setminus (A \cup B).$$

If $C = \varnothing$, then $B_k(v) = G[A \cup B] = G$, and hence $B_k(v)$ has average degree at least $d \geq d'$. Thus, we may assume that $C \neq \varnothing$. Let d'' be the average degree of $B_k(v)$. Thus,

$$2\big(e(A) + e(B) + e(A, B)\big) = d'' \cdot (|A| + |B|). \tag{1}$$

Since C is non-empty, $G - A$ is a proper non-empty subgraph of G. By our hypothesis on G, this subgraph has average degree strictly less than d; that is,

$$2\big(e(B) + e(C) + e(B, C)\big) < d \cdot (|B| + |C|). \tag{2}$$

By (1) and (2) and since $e(A, C) = 0$,

$$\begin{aligned} 2\|G\| &= 2\big(e(A) + e(B) + e(C) + e(A, B) + e(B, C)\big) \\ &= d''(|A| + |B|) + 2e(C) + 2e(B, C) \\ &< d''(|A| + |B|) + d(|B| + |C|) - 2e(B) \\ &\leq d|G| - d|A| + d''(|A| + |B|). \end{aligned}$$

Thus $d''(|A| + |B|) > d|A|$ (since $2\|G\| \geq d\,|G|$). On the other hand, by the choice of k,

$$\frac{|A|}{|A| + |B|} > \frac{1}{\beta}.$$

Hence

$$d'' > d\frac{|A|}{|A| + |B|} > \frac{d}{\beta} = d',$$

as desired. □

Lemma 3.2. *There is a function g such that for every real $\varepsilon > 0$, every graph G with average degree at least $2 + \varepsilon$ has girth at most $g(\varepsilon) \cdot \log |G|$,*

Proof. By Lemma 3.1, G contains a subgraph G' with average degree at least 2 and diameter at most $p(2+\varepsilon, 2) \cdot \log|G|$. Let T be a breadth-first search tree in G'. Thus T has diameter at most $2p(2+\varepsilon, 2) \cdot \log|G|$. Since G' has average degree at least 2, G' is not a tree, and there is an edge $e \in E(G') \setminus E(T)$. Thus T plus e contains a cycle of length at most $2p(2+\varepsilon, 2) \cdot \log|G| + 1$. □

Theorem 3.3. *There is a function h such that for every real $\varepsilon > 0$, every graph G with average degree at least $4+\varepsilon$ contains a K_4-model with at most $h(\varepsilon) \cdot \log|G|$ vertices.*

Proof. By Lemma 3.1, G contains a subgraph G' with average degree at least $4+\frac{\varepsilon}{2}$ and diameter at most $p\left(4+\varepsilon, 4+\frac{\varepsilon}{2}\right) \cdot \log|G|$. Let v be an arbitrary vertex of G'. Let T be a breadth-first search tree from v in G'. Let k be the depth of T. Thus $k \le p\left(4+\varepsilon, 4+\frac{\varepsilon}{2}\right) \cdot \log|G|$.

Let $H := G' - E(T)$. Since $\|T\| = |G| - 1$, the graph H has average degree at least $2+\frac{\varepsilon}{2}$. By Lemma 3.2, H contains a cycle C of length at most $g\left(\frac{\varepsilon}{2}\right) \cdot \log|G|$. We will prove the theorem with $h(\varepsilon) := g\left(\frac{\varepsilon}{2}\right) + 3p\left(4+\varepsilon, 4+\frac{\varepsilon}{2}\right)$.

Observe that $v \notin V(C)$, since v is isolated in H. A vertex w of C is said to be *maximal* if, in the tree T rooted at v, no other vertex of C is an ancestor of w. Let $\operatorname{dist}(x)$ be the distance between v and each vertex x in T.

Consider an edge xx' in C where x is maximal and x' is not. Since T is a breadth-first search tree, $\operatorname{dist}(x') \le \operatorname{dist}(x) + 1$. Thus, if x is an ancestor of x' then $xx' \in E(T)$, which is a contradiction since $xx' \in E(H)$. Hence x is not an ancestor of x'. Let y be an ancestor of x' in C (which exists since x' is not maximal). Then $\operatorname{dist}(y) < \operatorname{dist}(x') \le \operatorname{dist}(x) + 1$, implying $\operatorname{dist}(y) \le \operatorname{dist}(x)$. We repeatedly use these facts below.

First, suppose that there is a unique maximal vertex x in C. Let x' be a neighbour of x in C. Since x' is not maximal, some ancestor of x' is in C. As proved above, x is not an ancestor of x' in T, which contradicts the assumption that x is the only maximal vertex in C.

Next, suppose there are exactly two maximal vertices x and y in C. Let P be an x–y path in C that is not the edge xy (if it exists). Let x' be the neighbour of x in P, and let y' be the neighbour of y in P. Thus $x' \ne y$ and $y' \ne x$. Hence neither x' nor y' are maximal. As proved above, y is an ancestor of x' and $\operatorname{dist}(y) \le \operatorname{dist}(x)$, and x is an ancestor of y' and $\operatorname{dist}(x) \le \operatorname{dist}(y)$. Thus $\operatorname{dist}(x) = \operatorname{dist}(y)$. Hence $\operatorname{dist}(x') \le \operatorname{dist}(y)+1$ and $\operatorname{dist}(y') \le \operatorname{dist}(x)+1$, which implies that $x'y$ and $y'x$ are both edges of T, and $x' \ne y'$. Now, the cycle C plus these two edges gives a K_4-model with $|C| \le g\left(\frac{\varepsilon}{2}\right) \cdot \log|G| \le h(\varepsilon) \cdot \log|G|$ vertices.

Finally, suppose that C contains three maximal vertices x, y, z. For $w \in \{x, y, z\}$, let P_w be the unique v–w path in T. Then $C \cup P_x \cup P_y \cup P_z$ contains a K_4-model with at most $|C| + |P_x - x| + |P_y - y| + |P_z - z| \le |C| + 3k \le h(\varepsilon) \cdot \log|G|$ vertices. □

4. Small K_t-models

The following theorem establishes our main result (Theorem 1.1).

Theorem 4.1. *There is a function h such that for every integer $t \ge 2$ and real $\varepsilon > 0$, every graph G with average degree at least $2^t + \varepsilon$ contains a K_t-model with at most $h(t, \varepsilon) \cdot \log|G|$ vertices.*

Proof. We prove the following slightly stronger statement: Every graph G with average degree at least $2^t + \varepsilon$ contains a K_t-model with at most $h(t, \varepsilon) \cdot \log|G|$ vertices such that each branch set of the model contains at least two vertices.

The proof is by induction on t. For $t = 2$, let $h(t, \varepsilon) := 2$. Here we need only assume average degree at least $2+\varepsilon$. Some component of G is neither a tree nor a cycle, as otherwise G would have average degree at most 2. It is easily seen that this component contains a path on 4 vertices, yielding a K_2-model in which each branch set contains two vertices. This model has $4 \le h(t, \varepsilon) \cdot \log|G|$ vertices, as desired. (Observe that $|G| \ge 4$, since G contains a vertex with degree at least 3.)

Now assume $t \ge 3$ and the claim holds for smaller values of t. Using Lemma 3.1, let G' be a subgraph of G with average degree at least $2^t + \frac{\varepsilon}{2}$ and diameter at most $p\left(2^t+\varepsilon, 2^t+\frac{\varepsilon}{2}\right) \cdot \log|G|$. Let $h(t, \varepsilon) := 2 + (t-1)\, p\left(2^t+\varepsilon, 2^t+\frac{\varepsilon}{2}\right) + h\left(t-1, \frac{\varepsilon}{4}\right)$.

Choose an arbitrary edge uv of G'. Define the *depth* of a vertex $w \in V(G')$ to be the minimum distance in G' between w and a vertex in $\{u, v\}$. Note that the depths of the endpoints of each edge differ by at most 1. The *depth* of an edge $xy \in E(G')$ is the minimum of the depth of x and the depth of y.

Considering edges of G' with even depth on one hand, and with odd depth on the other, we obtain two edge-disjoint spanning subgraphs of G'. Since G' has average degree at least $2^t + \frac{\varepsilon}{2}$, one of these two subgraphs has average degree at least $2^{t-1} + \frac{\varepsilon}{4}$. Let H be a component of this subgraph with average degree at least $2^{t-1} + \frac{\varepsilon}{4}$. Observe that every edge of H has the same depth k in G.

If $k = 0$, then $E(H)$ is precisely the set of edges incident to u or v (or both). Thus, every vertex in $V(H) \setminus \{u, v\}$ has degree at most 2 in H. Hence H has average degree less than $4 < 2^{t-1} + \frac{\varepsilon}{4}$, a contradiction. Therefore $k \geq 1$.

Now, by induction, H contains a K_{t-1}-model with at most $h\left(t-1, \frac{\varepsilon}{4}\right) \cdot \log|G'|$ vertices such that each of the $t-1$ branch sets $B_1, \ldots, B_{t-1}$ has at least two vertices. Thus, each B_i contains an edge of H. Hence, there is a vertex v_i in B_i having depth k in G'. Therefore, there is a path P_i of length k in G' between v_i and some vertex in $\{u, v\}$. Let P_{uv} be the trivial path consisting of the edge uv. Let

$$B_t := P_{uv} \cup \bigcup_{1 \leq i \leq t-1} (P_i - v_i).$$

The subgraph B_t is connected, contains at least two vertices (namely, u and v), and is vertex disjoint from B_i for all $i \in \{1, \ldots, t-1\}$. Moreover, there is an edge between B_t and each B_i, and

$$\begin{aligned}
\sum_{1 \leq i \leq t} |B_i| &\leq |B_t| + h\left(t-1, \frac{\varepsilon}{4}\right) \cdot \log|G'| \\
&\leq 2 + \sum_{1 \leq i \leq t-1} |P_i - v_i| + h\left(t-1, \frac{\varepsilon}{4}\right) \cdot \log|G| \\
&\leq 2 + (t-1)k + h\left(t-1, \frac{\varepsilon}{4}\right) \cdot \log|G| \\
&\leq 2 + (t-1)\, p\left(2^t + \varepsilon, 2^t + \frac{\varepsilon}{2}\right) \cdot \log|G| + h\left(t-1, \frac{\varepsilon}{4}\right) \cdot \log|G| \\
&\leq h(t, \varepsilon) \cdot \log|G|.
\end{aligned}$$

Hence, adding B_t to our K_{t-1}-model gives the desired K_t-model of G. □

Observe that one obstacle to reducing the lower bound on the average degree in Theorem 4.1 is the case $t = 3$, which we address in the following result.

Lemma 4.2. *There is a function h such that for every real $\varepsilon > 0$, every graph G with average degree at least $4 + \varepsilon$ contains a K_3-model with at most $h(\varepsilon) \cdot \log|G|$ vertices, such that each branch set contains at least two vertices.*

Proof. The proof is by induction on $|G| + \|G\|$. We may assume that no proper subgraph of G has average degree at least $4 + \varepsilon$, since otherwise we are done by induction. This implies that G is connected. Note that $|G| \geq 6$ since G has average degree >4.

First, suppose that G contains a K_4 subgraph with vertex set X.

Case 1. All edges between X and $V(G) \setminus X$ in G are incident to a common vertex $v \in X$: Let $Y := X \setminus \{v\}$. Then

$$2\|G - Y\| = 2\|G\| - 12 \geq (4 + \varepsilon)|G| - 12 \geq (4 + \varepsilon)|G - Y|,$$

implying that $G - Y$ also has average degree at least $4 + \varepsilon$, a contradiction.

Case 2. There are two independent edges uu' and vv' between X and $V(G) \setminus X$ in G, where $u, v \in X$: Then $\{u, u'\}, \{v, v'\}, X \setminus \{u, v\}$ is the desired K_3-model.

Case 3. Some vertex $w \in V(G) \setminus X$ is adjacent to two vertices $u, v \in X$: No vertex in X has a neighbour in $V(G) \setminus (X \cup \{w\})$, as otherwise Case 2 would apply. Since G is connected and $|G| \geq 6$, it follows that

w has a neighbour w' outside X. Let x, y be the two vertices in $X \setminus \{u, v\}$. Then $\{w, w'\}, \{u, x\}, \{v, y\}$ is the desired K_3-model.

This concludes the case in which G contains a K_4 subgraph. Now, assume that G is K_4-free. By Theorem 3.3, G contains a K_4-model $B_1, \dots, B_4$ with at most $h(\varepsilon) \cdot \log |G|$ vertices. Without loss of generality, $|B_1| \geq |B_2| \geq |B_3| \geq |B_4|$ and $|B_1| \geq 2$.

Case 1. $|B_2| \geq 2$: Then $B_1, B_2, B_3 \cup B_4$ is the desired K_3-model. Now assume that $B_i = \{x_i\}$ for all $i \in \{2, 3, 4\}$.

Case 2. Some x_i is adjacent to some vertex w not in $B_1 \cup B_2 \cup B_3 \cup B_4$: If $i = 2$ then $\{x_2, w\}, B_1, B_3 \cup B_4$ is the desired K_3-model. Similarly for $i \in \{3, 4\}$.

Case 3. $|B_1| \geq 3$. Then there are two independent edges in G between B_1 and $\{x_2, x_3, x_4\}$, say ux_2 and vx_3 with $u, v \in B_1$ (otherwise, there would be a K_4 subgraph). There is a vertex $w \in B_1 \setminus \{u, v\}$ adjacent to at least one of u, v, say u. Let C be the vertex set of the component of $G[B_1] - \{u, w\}$ containing v. Then $\{u, w\}, C \cup \{x_3\}, \{x_2, x_4\}$ is the desired K_3-model.

Case 4. $B_1 = \{u, v\}$. As in the previous cases, there are two independent edges in G between $\{u, v\}$ and $\{x_2, x_3, x_4\}$, say ux_2 and vx_3. At least one of u, v, say u, is adjacent to some vertex w outside $\{u, v, x_2, x_3, x_4\}$, because G is connected with at least 6 vertices, and none of x_2, x_3, x_4 has a neighbour outside $\{u, v, x_2, x_3, x_4\}$. Then $\{u, w\}, \{v, x_3\}, \{x_2, x_4\}$ is the desired K_3-model. $\square$

Note that average degree greater than 4 is required in Lemma 4.2 because of the disjoint union of K_5's. Lemma 4.2 enables the following improvement to Theorem 4.1.

Theorem 4.3. *There is a function h such that for every integer $t \geq 2$ and real $\varepsilon > 0$, every graph G with average degree at least $2^{t-1} + \varepsilon$ contains a K_t-model with at most $h(t, \varepsilon) \cdot \log |G|$ vertices.*

Proof. As before, we prove the following stronger statement: Every graph G with average degree at least $2^{t-1} + \varepsilon$ contains a K_t-model with at most $h(t, \varepsilon) \cdot \log |G|$ vertices such that each branch set of the model contains at least two vertices.

The proof is by induction on t. The $t = 2$ case is handled in the proof of Theorem 4.1. Lemma 4.2 implies the $t = 3$ case. Now assume $t \geq 4$ and the claim holds for smaller values of t. The proof proceeds as in the proof of Theorem 4.1. We obtain a subgraph G' of G with average degree at least $2^{t-1} + \frac{\varepsilon}{2}$ and diameter at most $p\left(2^{t-1} + \varepsilon, 2^{t-1} + \frac{\varepsilon}{2}\right) \cdot \log |G|$. Choose an edge uv of G' and define the depth of edges with respect to uv. We obtain a connected subgraph H with average degree at least $2^{t-2} + \frac{\varepsilon}{4}$, such that every edge of H has the same depth k. If $k = 0$, then $E(H)$ is precisely the set of edges incident to u or v (or both), implying H has average degree less than $4 < 2^{t-2} + \frac{\varepsilon}{4}$. Now assume $k \geq 1$. The remainder of the proof is the same as that of Theorem 4.1. $\square$

Thomassen [26] first observed that high girth (and minimum degree 3) forces a large complete graph as a minor; see [14] for the best known bounds. We now show that high girth (and minimum degree 3) forces a *small* model of a large complete graph.

Theorem 4.4. *Let k be a positive integer. Let G be a graph with girth at least $8k+3$ and minimum degree $r \geq 3$. Let t be an integer such that $r(r-1)^k \geq 2^{t-1} + 1$. Then G contains a K_t-model with at most $h'(k, r) \cdot \log |G|$ vertices, for some function h'.*

Proof. Mader [19] proved that G contains a minor H of minimum degree at least $r(r-1)^k$, such that each branch set has radius at most $2k$; see [3, Lemma 7.2.3]. Let $V(H) = \{b_1, \dots, b_{|H|}\}$, and let $B_1, \dots, B_{|H|}$ be the corresponding branch sets in G. Let r_i be a centre of B_i. For each vertex v in B_i, let $P_{i,v}$ be a path between r_i and v in B_i of length at most $2k$.

By Theorem 4.3, H contains a K_t-model with at most $h(t) \cdot \log |H|$ vertices. Let $C_1, \dots, C_t$ be the corresponding branch sets. Say C_i has n_i vertices. Thus $\sum_{i=1}^{t} n_i \leq h(t) \cdot \log |H|$. We now construct a K_t-model $X_1, \dots, X_t$ in G.

For $i \in \{1, \dots, t\}$, let T_i be a spanning tree of C_i. Each edge $b_j b_\ell$ of T_i corresponds to an edge vw of G, for some v in B_j and w in B_ℓ. Add to X_i the $r_i r_j$-path $P_{j,v} \cup \{vw\} \cup P_{\ell,w}$. This path has at most $4k+2$ vertices. Thus X_i is a connected subgraph of G with at most $(4k+2)(n_i-1)$ vertices (since T_i has $n_i - 1$ edges).

For distinct $i, i' \in \{1, \dots, t\}$ there is an edge between C_i and $C_{i'}$ in H. This edge corresponds to an edge vw of G, where v is in some branch set B_j in C_i, and w is in some branch set $B_{j'}$ in $C_{i'}$. Add the path $P_{j,v}$ to X_i, and add the path $P_{j',w}$ to $X_{i'}$. Thus v in X_i is adjacent to w in X_j.

Hence $X_1, \dots, X_t$ is a K_t-model in G with at most $\sum_{i=1}^{t}(4k+2)(n_i-1) \leq (4k+2) \cdot h(t) \cdot \log|H|$ vertices from the first step of the construction, and at most $\binom{t}{2}(4k+2)$ vertices from the second step. Since t is bounded by a function of r and k, there are at most $h'(k,r) \cdot \log|G|$ vertices in total, for some function h'. □

Corollary 4.5. *Let k be a positive integer. Let G be a graph with girth at least $8k+3$ and minimum degree at least 3. Then G contains a K_k-model with at most $h(k) \cdot \log|G|$ vertices, for some function h.*

5. Planar graphs

In this section we prove that sufficiently dense planar graphs have K_3-models and K_4-models of bounded size. We start with the K_3 case.

Theorem 5.1. *Let $\varepsilon \in (0, 4)$. Every planar graph G with average degree at least $2+\varepsilon$ has girth at most $1 + \lceil \frac{4}{\varepsilon} \rceil$.*

Proof. Let H be a connected component of G with average degree at least $2+\varepsilon$. Thus H is not a tree. Say H has n vertices and m edges. Fix an embedding of H in the plane with r faces. Let ℓ be the minimum length of a facial walk. Thus $\ell \geq 3$ and $2m \geq r\ell = (2+m-n)\ell$, implying

$$n-2 \geq m\left(1-\frac{2}{\ell}\right) \geq \frac{1}{2}(2+\varepsilon)n\left(1-\frac{2}{\ell}\right) > \frac{1}{2}(2+\varepsilon)(n-2)\left(1-\frac{2}{\ell}\right).$$

It follows that $\ell < 2 + \frac{4}{\varepsilon}$. Since ℓ is an integer, $\ell \leq 1 + \lceil \frac{4}{\varepsilon} \rceil$. Since H is not a tree, every facial walk contains a cycle. Thus H and G have girth at most $1 + \lceil \frac{4}{\varepsilon} \rceil$. □

To prove our results for K_4-models in embedded graphs, the notion of visibility will be useful (and of independent interest). Distinct vertices v and w in an embedded graph are *visible* if v and w appear on a common face; we say *v sees w*.

Lemma 5.2. *Let v be a vertex of a plane graph G, such that $\deg(v) \geq 3$, v is not a cut-vertex, and v is in no cut-pair. Then v and the vertices seen by v induce a subgraph containing a K_4-minor.*

Proof. We may assume that G is connected. Since v is not a cut-vertex, $G - v$ is connected. Let f be the face of $G - v$ that contains v in its interior. Let F be the facial walk around f. Suppose that F is not a simple cycle. Then F has a repeated vertex w. Say $(a, w, b, \dots, c, w, d)$ is a subwalk of F. Then there is a Jordan curve C from v to w, arriving at w between the edges wa and wb, then leaving w from between the edges wc and wd, and back to v. Thus C contains b in its interior and a in its exterior. Hence v, w is a cut-pair. This contradiction proves that F is a simple cycle. Hence v and the vertices seen by v induce a subdivided wheel with $\deg(v)$ spokes. Since $\deg(v) \geq 3$ this subgraph contains a subdivision of K_4. □

Recall that $F(G, v)$ is the multiset of faces incident to a vertex v in an embedded graph G, where the multiplicity of a face f in $F(G, v)$ equals the multiplicity of v in the facial walk around f.

Lemma 5.3. *Each vertex v in an embedded graph G sees at most*

$$\sum_{f \in F(G,v)} (|f| - 2)$$

other vertices.

Proof. The vertex v only sees the vertices in the faces in $F(G, v)$. Each $f \in F(G, v)$ contributes at most $|f| - 1$ vertices distinct from v. Moreover, each neighbour of v is counted at least twice. Thus v sees at most $\sum_{f \in F(G,v)}(|f| - 1) - \deg(v)$ other vertices, which equals $\sum_{f \in F(G,v)}(|f| - 2)$. □

The 4-regular planar graph C_{2n}^2 has an embedding in the plane, in which each vertex sees $n + 1$ other vertices; see Fig. 1. On the other hand, we now show that every plane graph with minimum degree 5 has a vertex that sees a bounded number of vertices.

Lemma 5.4. *Every plane graph G with minimum degree 5 has a vertex that sees at most 7 other vertices.*

Proof. For each vertex v of G, associate a charge of

$$2 - \deg(v) + \sum_{f \in F(G,v)} \frac{2}{|f|}.$$

By Euler's formula, the total charge is $2|G| - 2\|G\| + 2|F(G)| = 4$. Thus some vertex v has positive charge. That is,

$$2 \sum_{f \in F(G,v)} \frac{1}{|f|} > \deg(v) - 2.$$

Now $\frac{1}{|f|} \leq \frac{1}{3}$. Thus $\frac{2}{3}\deg(v) > \deg(v) - 2$, implying $\deg(v) < 6$ and $\deg(v) = 5$. If some facial walk containing v has length at least 6, then

$$3 = 2\left(\frac{4}{3} + \frac{1}{6}\right) \geq 2 \sum_{f \in F(G,v)} \frac{1}{|f|} > 3,$$

which is a contradiction. Hence each facial walk containing v has length at most 5. If two facial walks containing v have length at least 4, then

$$3 = 2\left(\frac{3}{3} + \frac{2}{4}\right) \geq 2 \sum_{f \in F(G,v)} \frac{1}{|f|} > 3,$$

which is a contradiction. Thus no two facial walks containing v each have length at least 4. Hence all the facial walks containing v are triangles, except for one, which has length at most 5. Thus v sees at most 7 vertices. □

The bound in Lemma 5.4 is tight since there is a 5-regular planar graph with triangular and pentagonal faces, where each vertex is incident to exactly one pentagonal face (implying that each vertex sees exactly 7 vertices). The corresponding polyhedron is called the *snub dodecahedron*; see Fig. 2.

Lemmas 5.2 and 5.4 imply:

Theorem 5.5. *Every 3-connected planar graph with minimum degree 5 contains a K_4-model with at most 8 vertices.*

Theorem 5.5 is best possible since it is easily seen that every K_4-model in the snub dodecahedron contains at least 8 vertices. Also note that no result like Theorem 5.5 holds for planar graphs with minimum degree 4 since every K_4-model in the 4-regular planar graph C_{2n}^2 has at least n vertices.

We now generalise Lemma 5.4 for graphs with average degree greater than 4.

Lemma 5.6. *Let $\varepsilon \in (0, 2)$. Every plane graph G with minimum degree at least 3 and average degree at least $4 + \varepsilon$ has a vertex v that sees at most $1 + \lceil \frac{8}{\varepsilon} \rceil$ other vertices.*

Proof. For each vertex v of G, associate a charge of

$$(8 + 2\varepsilon) - (8 + 3\varepsilon)\deg(v) + (24 + 6\varepsilon) \sum_{f \in F(G,v)} \frac{1}{|f|}.$$

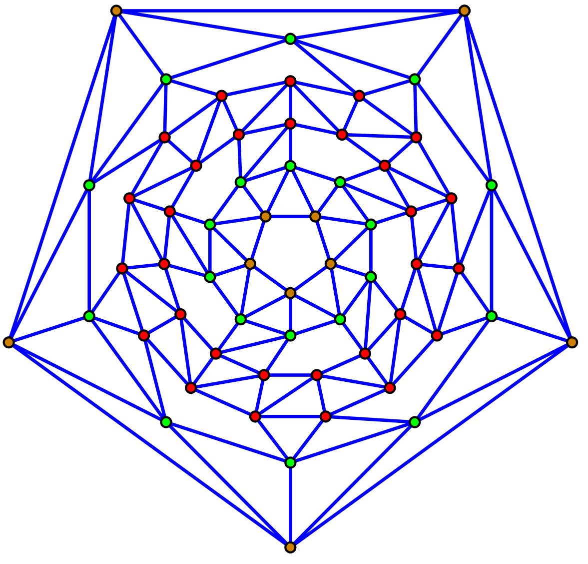

Fig. 2. The snub dodecahedron.

By Euler's formula, the total charge is

$$\begin{aligned}&(8+2\varepsilon)|G| - (16+6\varepsilon)\,\|G\| + (24+6\varepsilon)\,|F(G)|\\ &\quad = (8+2\varepsilon)|G| - (16+6\varepsilon)\,\|G\| + (24+6\varepsilon)\,(\|G\| - |G| + 2)\\ &\quad = 4(2\|G\| - (4+\varepsilon)|G|) + 2\,(24+6\varepsilon)\\ &\quad \geq 2\,(24+6\varepsilon)\,.\end{aligned}$$

Thus some vertex v has positive charge. That is,

$$(24+6\varepsilon)\sum_{f\in F(G,v)} \frac{1}{|f|} > (8+3\varepsilon)\deg(v) - (8+2\varepsilon).$$

That is,

$$\sum_{f\in F(G,v)} \frac{1}{|f|} > \left(\frac{1}{3}+\frac{1}{\alpha}\right)\deg(v) - \frac{1}{3},$$

where $\alpha := 6 + \frac{24}{\varepsilon}$. We have proved that $\deg(v)$ and the lengths of the facial walks incident to v satisfy Lemma A.1. Thus

$$\sum_{f\in F(G,v)} (|f|-2) \leq \left\lceil \frac{\alpha}{3} \right\rceil - 1 = 1 + \left\lceil \frac{8}{\varepsilon} \right\rceil.$$

The result follows from Lemma 5.3. □

Lemmas 5.2 and 5.6 imply:

Theorem 5.7. *Let $\varepsilon \in (0,2)$. Every 3-connected planar graph G with average degree at least $4+\varepsilon$ contains a K_4-model with at most $2 + \lceil \frac{8}{\varepsilon} \rceil$ vertices.*

We now prove that the 3-connectivity assumption in Theorem 5.7 can be dropped, at the expense of a slightly weaker bound on the size of the K_4-model.

Theorem 5.8. *Let $\varepsilon \in (0,2)$. Every planar graph G with average degree at least $4+\varepsilon$ contains a K_4-model with at most $\lceil \frac{8}{\varepsilon} \rceil + \lceil \frac{2}{\varepsilon} \rceil$ vertices. Moreover, this bound is within a constant factor of being optimal.*

Proof. If G has at most $2+\lceil\frac{2}{\varepsilon}\rceil$ vertices, then we are done since $m > 2n$ implies G contains a K_4-model, which necessarily has at most $2+\lceil\frac{2}{\varepsilon}\rceil < \lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil$ vertices.

We now proceed by induction on n with the following hypothesis: Let G be a planar graph with $n \geq 2+\lceil\frac{2}{\varepsilon}\rceil$ vertices and m edges, such that

$$2m > (4+\varepsilon)(n-2). \tag{3}$$

Then G contains a K_4-model with at most $\lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil$ vertices.

This will imply the theorem since $2m \geq (4+\varepsilon)n > (4+\varepsilon)(n-2)$.

Suppose that $n \leq \lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil$. Since $n \geq 2+\frac{2}{\varepsilon}$,

$$2m > (4+\varepsilon)(n-2) = 4n-8+\varepsilon(n-2) \geq 4n-6.$$

Thus $m > 2n-3$, implying G contains a K_4-model, which has at most $n \leq \lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil$ vertices. Now assume that $n \geq \lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil + 1$.

Suppose that $\deg(v) \leq 2$ for some vertex v. Thus $G-v$ satisfies (3) since

$$2\|G-v\| = 2(m-\deg(v)) > (4+\varepsilon)(n-2)-4 > (4+\varepsilon)(n-3).$$

Now $n-1 \geq \lceil\frac{8}{\varepsilon}\rceil + \lceil\frac{2}{\varepsilon}\rceil > 2+\lceil\frac{2}{\varepsilon}\rceil$. Thus, by induction, $G-v$ and hence G contains the desired K_4-minor. Now assume that $\deg(v) \geq 3$ for every vertex v.

Suppose that G contains a separation $\{G_1, G_2\}$ of order at most 2. Let $S := V(G_1 \cap G_2)$. Say each G_i has n_i vertices and m_i edges. Thus $n_1+n_2 \leq n+2$ and $m_1+m_2 \geq m$. Eq. (3) is satisfied for G_1 or G_2, as otherwise

$$(4+\varepsilon)(n-2) < 2m \leq 2m_1+2m_2 \leq (4+\varepsilon)(n_1+n_2-4) \leq (4+\varepsilon)(n-2).$$

Without loss of generality, G_1 satisfies (3). Thus we are done by induction if $n_1 \geq 2+\lceil\frac{2}{\varepsilon}\rceil$. Now assume that $n_1 \leq 1+\lceil\frac{2}{\varepsilon}\rceil$. Also assume that $m_1 \leq 2n_1-3$, as otherwise G_1 contains a K_4-model, which has at most $n_1 \leq 1+\lceil\frac{2}{\varepsilon}\rceil$ vertices.

Suppose that $S = \{v\}$ for some cut-vertex v. Since every vertex in G has degree at least 3, every vertex in G_1, except v, has degree at least 3 in G_1. Since $n_1 \geq 2$, G_1 contains a K_4-model, which has at most $n_1 \leq 1+\lceil\frac{2}{\varepsilon}\rceil$ vertices. Now assume that G is 2-connected.

Suppose that $S = \{v, w\}$ for some adjacent cut-pair v, w. Thus $n_1+n_2 = n+2$ and $m = m_1+m_2-1$ and

$$\begin{aligned} 2m_2 = 2m+2-2m_1 &> (4+\varepsilon)(n-2)+2-2(2n_1-3) \\ &= (4+\varepsilon)(n_1+n_2-4)-4n_1+8 \\ &= (4+\varepsilon)(n_2-4)+\varepsilon n_1+8 \\ &\geq (4+\varepsilon)(n_2-4)+2(4+\varepsilon) \\ &= (4+\varepsilon)(n_2-2). \end{aligned}$$

That is, G_2 satisfies (3). Also,

$$n_2 = n-n_1+2 \geq \left(\left\lceil\frac{8}{\varepsilon}\right\rceil + \left\lceil\frac{2}{\varepsilon}\right\rceil\right)+1-\left(1+\left\lceil\frac{2}{\varepsilon}\right\rceil\right)+2 = 2+\left\lceil\frac{8}{\varepsilon}\right\rceil > 2+\left\lceil\frac{2}{\varepsilon}\right\rceil.$$

Hence, by induction G_2 and thus G contains the desired K_4-model. Now assume that every cut-pair of vertices are not adjacent.

Suppose that $S = \{v, w\}$ for some non-adjacent cut-pair v, w and $m_1 \leq 2n_1-4$: Thus $n_1+n_2 = n+2$ and $m_1+m_2 = m$ and

$$\begin{aligned} 2m_2 = 2m-2m_1 &> (4+\varepsilon)(n-2)-2(2n_1-4) \\ &= (4+\varepsilon)(n_1+n_2-4)-4n_1+8 \\ &= (4+\varepsilon)(n_2-4)+\varepsilon n_1+8 \\ &\geq (4+\varepsilon)(n_2-4)+2\varepsilon+8 \\ &= (4+\varepsilon)(n_2-2). \end{aligned}$$

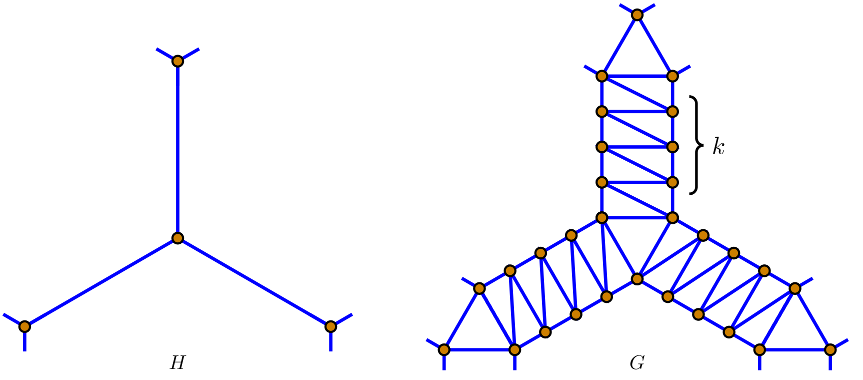

Fig. 3. Construction of G.

That is, G_2 satisfies (3). As proved above, $n_2 > 2 + \lceil \frac{2}{\varepsilon} \rceil$. Hence, by induction G_2 and thus G contains the desired K_4-model. Now assume that for every cut-pair v, w we have $vw \notin E(G)$, and if $\{G_1, G_2\}$ is the corresponding separation with G_1 satisfying (3), then $m_1 = 2n_1 - 3$ and $n_1 \leq 1 + \lceil \frac{2}{\varepsilon} \rceil$.

Fix an embedding of G. By Lemma 5.6, there is a vertex v in G that sees at most $1 + \lceil \frac{8}{\varepsilon} \rceil$ other vertices. If v is in no cut-pair then by Lemma 5.2 and since G is 2-connected, v plus the vertices seen by v induce a subgraph that contains a K_4-model, which has at most $2 + \lceil \frac{8}{\varepsilon} \rceil \leq \lceil \frac{8}{\varepsilon} \rceil + \lceil \frac{2}{\varepsilon} \rceil$ vertices. Now assume that v, w is a cut-pair. Thus $vw \notin E(G)$, and if $\{G_1, G_2\}$ is the corresponding separation, then $m_1 = 2n_1 - 3$ and $n_1 \leq 1 + \lceil \frac{2}{\varepsilon} \rceil$. Since v, w is a cut-pair, there is a vw-path P contained in G_2, such that P is contained in a single face of G. Every vertex in P is seen by v, and v sees at least 2 vertices in $G_1 - w$. Thus P has at most $\lceil \frac{8}{\varepsilon} \rceil - 2$ internal vertices. Let H be the minor of G obtained by contracting P into the edge vw, and deleting all the other vertices in G_2. Thus H has n_1 vertices and $2n_1 - 2$ edges. Hence H contains a K_4-minor. The corresponding K_4-model in G is contained in $G_1 \cup P$, and thus has at most $\left(1 + \lceil \frac{2}{\varepsilon} \rceil\right) + \left(\lceil \frac{8}{\varepsilon} \rceil - 2\right) < \lceil \frac{2}{\varepsilon} \rceil + \lceil \frac{8}{\varepsilon} \rceil$ vertices.

We now prove the lower bound. Assume that $\varepsilon \in (0, 1]$ and $k := \frac{1}{\varepsilon} - 1$ is a non-negative integer. Let H be a cubic plane graph in which the length of every facial walk is at least 5 (for example, the dual of a minimum degree 5 plane triangulation). Say H has p vertices. Let G be the plane graph obtained by replacing each vertex of H by a triangle, and replacing each edge of H by $2k$ vertices, as shown in Fig. 3. Thus G has $3p$ vertices with degree 5 and $3kp$ vertices with degree 4. Thus $|G| = 3p + 3pk = \frac{3p}{\varepsilon}$ and $2\|G\| = 3p \cdot 5 + 3pk \cdot 4 = 4|G| + 3p = (4 + \varepsilon)|G|$. Thus G has average degree $4 + \varepsilon$. Every K_4-model in G includes a cycle that surrounds a 'big' face with more than $5k$ vertices. Thus every K_4-model has more than $5k = \frac{5}{\varepsilon} - 5$ vertices. Similar constructions are possible for $\varepsilon > 1$ starting with a 4- or 5-regular planar graph. □

6. Higher genus surfaces

We now extend our results from Section 5 for graphs embedded on other surfaces.

Lemma 6.1. *Let $\varepsilon > 0$. Let G be a graph with average degree at least $2 + \varepsilon$. Suppose that G is embedded in a surface with Euler genus at most g. Then some facial walk has length at most $\left(\frac{4}{\varepsilon} + 2\right)(g + 1)$. Moreover, this bound is tight up to lower order terms.*

Proof. Say G has n vertices, m edges, and r faces. Let ℓ be the minimum length of a facial walk. Thus $2m \geq r\ell$. By Euler's formula, $n - m + r = 2 - g$. Hence

$$(2 + \varepsilon)n \leq 2m$$
$$(2 + \varepsilon)(2 - g) = (2 + \varepsilon)(n - m + r)$$
$$\frac{\varepsilon}{2}(r\ell) \leq \frac{\varepsilon}{2}(2m).$$

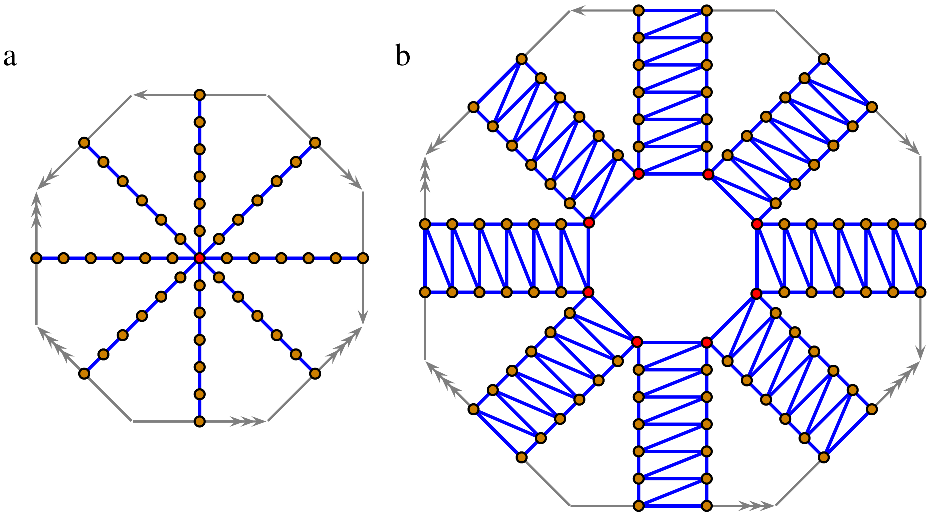

Fig. 4. Graphs embedded in $\mathbb{S}_2$: (a) average degree $2+\varepsilon$ and one face, and (b) average degree $4+\varepsilon$ and every vertex on one face.

Summing gives $\frac{\varepsilon}{2}(r\ell) \le (2+\varepsilon)(g+r-2)$. Since $r \ge 1$,

$$\ell \le \frac{2}{\varepsilon r}(2+\varepsilon)(g+r-2) = \left(\frac{4}{\varepsilon}+2\right)\left(\frac{g}{r}+\frac{r-2}{r}\right) < \left(\frac{4}{\varepsilon}+2\right)(g+1).$$

Hence some facial walk has length at most $(\frac{4}{\varepsilon}+2)(g+1)$.

Now we prove the lower bound. Assume that $g = 2h \ge 2$ is a positive even integer, and that $0 < \varepsilon \le 1 - \frac{3}{2g+1}$. Let $k := \left\lfloor \frac{2}{\varepsilon} - \frac{2}{\varepsilon g} - \frac{1}{g} \right\rfloor$. Thus $k \ge 2$. Let G be the graph consisting of g cycles of length $k+1$ with exactly one vertex in common. Thus

$$\begin{aligned} 2\|G\| &= 2g(k+1) = 2gk + 2 + \varepsilon + \varepsilon g\left(\frac{2}{\varepsilon} - \frac{2}{\varepsilon g} - \frac{1}{g}\right) \ge 2gk + 2 + \varepsilon + \varepsilon gk \\ &= (2+\varepsilon)(gk+1) \\ &= (2+\varepsilon)|G|. \end{aligned}$$

Hence G has average degree at least $2+\varepsilon$. As illustrated in Fig. 4(a), G has an embedding in $\mathbb{S}_h$ (which has Euler genus $2h = g$) with exactly one face. Thus every facial walk in G has length $2\|G\| = 2g(k+1) > 2g\left(\frac{2}{\varepsilon} - \frac{2}{\varepsilon g} - \frac{1}{g}\right) \ge \frac{4(g-1)}{\varepsilon} - 2$. □

Theorem 6.2. *There is a function h, such that for every real $\varepsilon > 0$, every graph G with average degree at least $2+\varepsilon$ and Euler genus g has girth at most $h(\varepsilon)\cdot\log(g+2)$. Moreover, for fixed ε, this bound is within a constant factor of being optimal.*

Proof. Say G has n vertices and m edges. We may assume that every proper subgraph of G has average degree strictly less than $2+\varepsilon$. This implies that G has minimum degree at least 2. Fix an embedding of G with Euler genus g. Let ℓ be the minimum length of a facial walk. By Euler's formula, there are $m-n+2-g$ faces. Thus $2m \ge (m-n+2-g)\ell$, implying $\ell(n+g-2) \ge m(\ell-2) \ge \frac{1}{2}(2+\varepsilon)(\ell-2)n$. Thus $\ell(n+g-2) \ge \frac{1}{2}(2+\varepsilon)(\ell-2)n$, implying $\ell(g-2) \ge \left(\frac{\varepsilon}{2}(\ell-2)-2\right)n$. First suppose that $\ell < 6 + \frac{12}{\varepsilon}$. Since G has no degree-1 vertices, every facial walk contains a cycle. Thus G has girth at most $6+\frac{12}{\varepsilon}$, which is at most $h(\varepsilon)\cdot\log(g+2)$ for some function h. Now assume that $\ell \ge 6+\frac{12}{\varepsilon}$, which implies that $\ell(g-2) \geq \left(\frac{\varepsilon}{2}(\ell-2)-2\right) n \geq \frac{\varepsilon}{3}\ell n$. Thus $n \leq \frac{3}{\varepsilon}(g-2)$. By Lemma 3.2, the girth of G is at most $g(\varepsilon) \cdot \log n \leq g(\varepsilon) \cdot \log\left(\frac{3}{\varepsilon}(g-2)\right)$, which is at most $h(\varepsilon) \cdot \log(g+2)$ for some function h.

Now we prove the lower bound. Let d be the integer such that $d-3 < \varepsilon \leq d-2$. Thus $d \geq 3$. For all $n > 3d$ such that nd is even, Chandran [2] constructed a graph G with n vertices, average degree $d \geq 2+\varepsilon$, and girth at least $(\log_d n) - 1$. Now G has Euler genus $g \leq \frac{dn}{2} - n + 1 \leq dn - 2$. Thus G has girth at least $\left(\log_d \frac{g+2}{d}\right) - 1$. Since $d < 3+\varepsilon$, the girth of G is at least $h(\varepsilon) \cdot \log(g+2)$ for some function h. □

We now extend Lemma 5.6 for sufficiently large embedded graphs.

Lemma 6.3. *Let $\varepsilon \in (0,2)$. Let G be a graph with minimum degree 3 and average degree at least $4+\varepsilon$. Assume that G is embedded in a surface with Euler genus g, such that $|G| \geq \left(\frac{24}{\varepsilon}+6\right) g$. Then G has a vertex v that sees at most $2 + \left\lceil \frac{12}{\varepsilon} \right\rceil$ other vertices.*

Proof. For each vertex v of G, associate a charge of

$$(8+2\varepsilon) - (8+3\varepsilon)\deg(v) + (24+6\varepsilon)\frac{g}{|G|} + (24+6\varepsilon)\sum_{f \in F(G,v)} \frac{1}{|f|}.$$

Thus the total charge is

$$\begin{aligned}
&(8+2\varepsilon)|G| - (16+6\varepsilon)\,\|G\| + (24+6\varepsilon)\,g + (24+6\varepsilon)\,|F(G)| \\
&\quad = (8+2\varepsilon)|G| - (16+6\varepsilon)\,\|G\| + (24+6\varepsilon)\,g + (24+6\varepsilon)\,(\|G\| - |G| - g + 2) \\
&\quad = 4(2\|G\| - (4+\varepsilon)|G|) + 2\,(24+6\varepsilon) \\
&\quad \geq 2\,(24+6\varepsilon)\,.
\end{aligned}$$

Thus some vertex v has positive charge. That is,

$$(8+2\varepsilon) - (8+3\varepsilon)\deg(v) + (24+6\varepsilon)\frac{g}{|G|} + (24+6\varepsilon)\sum_{f \in F(G,v)} \frac{1}{|f|} > 0.$$

Since $\frac{(24+6\varepsilon)g}{|G|} \leq \varepsilon$,

$$(24+6\varepsilon)\sum_{f \in F(G,v)} \frac{1}{|f|} > (8+3\varepsilon)(\deg(v)-1).$$

That is,

$$\sum_{f \in F(G,v)} \frac{1}{|f|} > \left(\frac{1}{3} + \frac{1}{\alpha}\right)(\deg(v)-1),$$

where $\alpha := 6 + \frac{24}{\varepsilon}$. We have proved that $\deg(v)$ and the lengths of the facial walks incident to v satisfy Lemma A.2. Thus

$$\sum_{f \in F(G,v)} (|f|-2) \leq \left\lceil \frac{\alpha}{2} \right\rceil - 1 = 2 + \left\lceil \frac{12}{\varepsilon} \right\rceil.$$

The result follows from Lemma 5.3. □

We now prove that the assumption that $n \in \Omega\left(\frac{g}{\varepsilon}\right)$ in Lemma 6.3 is needed. Assume we are given $\varepsilon \in (0,1]$ such that $k := \frac{1}{\varepsilon} - 1$ is an integer. Hence $k \geq 0$. Consider the graph G shown in Fig. 4(b) with $2g$ vertices of degree 5 and $2gk$ vertices of degree 4. Thus $|G| = 2g(k+1)$ and $2\|G\| = 10g + 8gk = 2g(5+4k) = \frac{|G|}{k+1}(4k+5) = \left(4 + \frac{1}{k+1}\right)|G| = (4+\varepsilon)|G|$. Thus G has average degree $4+\varepsilon$. Observe that every vertex lies on a single face. Thus each vertex sees $|G| - 1 = \frac{2g}{\varepsilon} - 1$ other vertices.

A *k-noose* in an embedded graph G is a noncontractible simple closed curve in the surface that intersects G in exactly k vertices. The *facewidth* of G is the minimum integer k such that G contains a k-noose.

Theorem 6.4. *Let $\varepsilon > 0$. Let G be a 3-connected graph with average degree at least $4+\varepsilon$, such that G has an embedding in a surface with Euler genus g and with facewidth at least 3. Then G contains a K_4-model with at most $q(\varepsilon) \cdot \log(g+2)$ vertices, for some function q. Moreover, for fixed ε, this bound is within a constant factor of being optimal.*

Proof. If $|G| \leq \left(\frac{24}{\varepsilon}+6\right) g$ then the result follows from Theorem 3.3. Otherwise, by Lemma 6.3 some vertex v sees at most $2+\left\lceil\frac{12}{\varepsilon}\right\rceil$ other vertices. The graph $G-v$ is 2-connected and has facewidth at least 2. Thus every face of $G-v$ is a simple cycle [20, Proposition 5.5.11]. In particular, the face of $G-v$ that contains v in its interior is bounded by a simple cycle C. The vertices in C are precisely the vertices that v sees in G. Thus $G[C \cup \{v\}]$ is a subdivided wheel with $\deg(v) \geq 3$ spokes. Hence G contains a K_4-model with at most $2+\left\lceil\frac{12}{\varepsilon}\right\rceil$ vertices, which is at most $q(\varepsilon)\cdot\log(g+2)$ for an appropriate function q.

Now we prove the lower bound. Let d be the integer such that $d-5 < \varepsilon \leq d-4$. Thus $d \geq 5$. For every integer $n > 3d$ such that nd is even, Chandran [2] constructed a graph G with n vertices, average degree $d \geq 4+\varepsilon$, and girth greater than $(\log_d n)-1$. Thus G has Euler genus $g \leq \frac{dn}{2} \leq dn-2$. Since every K_4-model contains a cycle, every K_4-model in G has at least $(\log_d n)-1$ vertices. Since $n \geq \frac{g+2}{d}$ and $d < 5+\varepsilon$, every K_4-model in G has at least $q(\varepsilon)\cdot\log(g+2)$ vertices, for some function q. □

For a class of graphs, an edge is 'light' if both its endpoints have bounded degree. For example, Wernicke [28] proved that every planar graph with minimum degree 5 has an edge vw such that $\deg(v)+\deg(w) \leq 11$; see [1,12,5,6] for extensions. For a class of embedded graphs, we say an edge is 'blind' if both its endpoints see a bounded number of vertices. In a triangulation, a vertex only sees its neighbours, in which case the notions of 'light' and 'blind' are equivalent. But for non-triangulations, a 'blind edge' theorem is qualitatively stronger than a 'light edge' theorem. Hence the following result is a qualitative generalisation of the above theorem of Wernicke [28] (and of Lemma 5.4), and is thus of independent interest. No such result is possible for minimum degree 4 since every edge in C_{2n}^2 sees at least n vertices.

Proposition 6.5. *Let G be a graph with minimum degree 5 embedded in a surface with Euler genus g, such that $|G| \geq 240g$. Then G has an edge vw such that v and w each see at most 12 vertices. Moreover, for plane graphs (that is, $g=0$), v and w each see at most 11 vertices.*

Proof. Consider each vertex x. Let ℓ_x be the maximum length of a facial walk containing x. Let t_x be the number of triangular faces incident to x, unless every face incident to x is triangular, in which case let $t_x := \deg(x)-1$. Say x is *good* if x sees at most 12 vertices, otherwise x is *bad*. Let

$$c_x := 240 - 120\deg(x) + 240\frac{g}{|G|} + 240\sum_{f\in F(G,x)}\frac{1}{|f|}$$

be the charge at x. (c_x is 240 times the *combinatorial curvature* at x.) By Euler's formula, the total charge is

$$240(|G| - \|G\| + g + |F(G)|) = 480.$$

Observe that (since $\ell_x \geq 3$ and $t_x \leq \deg(x)-1$ and $\deg(x) \geq 5$)

$$\begin{aligned} c_x &\leq 240 - 120\deg(x) + 240\frac{g}{|G|} + 240\left(\frac{1}{\ell_x} + \frac{t_x}{3} + \frac{\deg(x)-t_x-1}{4}\right) \\ &\leq 181 - 60\deg(x) + \frac{240}{\ell_x} + 20t_x && (4) \\ &\leq 241 - 40\deg(x) \leq 41. && (5) \end{aligned}$$

For each good vertex x, equally distribute the charge on x to its neighbours. (Bad vertices keep their charge.) Let c'_x be the new charge on each vertex x. Since the total charge is positive, $c'_v > 0$ for some vertex v. If v is good, then all the charge at v was received from its neighbours during the charge distribution phase, implying some neighbour w of v is good, and we are done. Now assume that v is bad. Let D_v be the set of good neighbours of v. By (4) and (5), and since $\deg(w) \geq 5$,

$$0 < c'_v = c_v + \sum_{w \in D_v} \frac{c_w}{\deg(w)} \leq 181 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + \frac{41}{5}|D_v|. \tag{6}$$

We may assume that no two good neighbours of v are on a common triangular face.

Claim 6.6. *$|D_v| \leq \deg(v) - \frac{t_v}{2}$. Moreover, if $|D_v| = \deg(v) - \frac{t_v}{2}$ then some face incident to v is non-triangular, and for every bad neighbour w of v, the edge vw is incident to two triangular faces.*

Proof. First assume that every face incident to v is triangular. Thus no two consecutive neighbours of v are good. Hence $|D_v| \leq \frac{\deg(v)}{2} < \frac{\deg(v)+1}{2} = \deg(v) - \frac{t_v}{2}$, as claimed. This also proves that if $|D_v| = \deg(v) - \frac{t_v}{2}$ then some face incident to v is non-triangular.

We prove the case in which some face incident to v is non-triangular by a simple charging scheme. If w is a good neighbour of v, then charge vw by 1. Charge each triangular face incident to v by $\frac{1}{2}$. Thus the total charge is $|D_v| + \frac{t_v}{2}$. If uvw is a triangular face incident to v, then at least one of u and w, say w, is bad; send the charge of $\frac{1}{2}$ at uvw to vw. Each good edge incident to v gets a charge of 1, and each bad edge incident to v gets a charge of at most $\frac{1}{2}$ from each of its two incident faces. Thus each edge incident to v gets a charge of at most 1. Thus the total charge, $|D_v| + \frac{t_v}{2}$, is at most $\deg(v)$, as claimed.

Finally, assume that $|D_v| = \deg(v) - \frac{t_v}{2}$. Then for every bad neighbour w of v, the edge vw gets a charge of exactly 1, implying vw is incident to two triangular faces. □

Claim 6.6 and (6) imply

$$\begin{aligned} 0 &< 181 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + \frac{41}{5}\deg(v) - \frac{41t_v}{10} \\ &= 181 - \frac{259}{5}\deg(v) + \frac{240}{\ell_v} + \frac{159}{10}t_v. \end{aligned}$$

Since $t_v \leq \deg(v) - 1$ and $\deg(v) \geq 5$,

$$0 < \frac{1651}{10} - \frac{359}{10}\deg(v) + \frac{240}{\ell_v} \leq -\frac{144}{10} + \frac{240}{\ell_v}.$$

implying $\ell_v \in \{3, 4, \ldots, 16\}$. Since $\ell_v \geq 3$,

$$0 < \frac{2451}{10} - \frac{359}{10}\deg(v),$$

implying $\deg(v) \in \{5, 6\}$ and $t_v \in \{0, 1, \ldots, \deg(v) - 1\}$.

We have proved that finitely many values satisfy (6). We now strengthen this inequality in the case that $|D_v| = \deg(v) - \frac{t_v}{2}$.

Let f be a face of length ℓ_v incident to v. Let x and y be two distinct neighbours of v on f. Suppose on the contrary that x is bad. By Claim 6.6, vx is incident to two triangular faces, one of which is vxy. Thus $\ell_v = 3$, and every face incident to v is a triangle, which contradicts the Claim. Hence x is good. Similarly y is good.

Thus $\ell_x \geq \ell_v$. By (4),

$$c_x \leq 181 - 60\deg(x) + \frac{240}{\ell_v} + 20t_x \leq 161 - 40\deg(x) + \frac{240}{\ell_v} \leq \frac{240}{\ell_v} - 39.$$

Similarly, $c_y \le \frac{240}{\ell_v} - 39$. Hence (assuming $|D_v| = \deg(v) - \frac{t_v}{2}$),

$$\begin{aligned} 0 < c'_v &\le 181 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + \frac{c_x}{\deg(x)} + \frac{c_y}{\deg(y)} + \sum_{w \in D_v \setminus \{x,y\}} \frac{c_w}{\deg(w)} \\ &\le 181 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + \frac{\frac{240}{\ell_v} - 39}{\deg(x)} + \frac{\frac{240}{\ell_v} - 39}{\deg(y)} + \sum_{w \in D_v \setminus \{x,y\}} \frac{41}{\deg(w)} \\ &\le 181 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + 2\left(\frac{48}{\ell_v} - \frac{39}{5}\right) + \frac{41}{5}(|D_v| - 2). \end{aligned} \tag{7}$$

Checking all values of $\deg(v)$, t_v and ℓ_v that satisfy (6) and (7) proves that

$$t_v + (\deg(v) - t_v)(\ell_v - 2) \le 12$$

(which is tight for $\deg(v) = 5$ and $t_v = 4$ and $\ell_v = 10$ and $|D_v| = 2$). Thus

$$\sum_{f \in F(G,v)} (|f| - 2) \le t_v(3 - 2) + (\deg(v) - t_v)(\ell_v - 2) \le 12.$$

By Lemma 5.3, v sees at most 12 vertices. Therefore v is good, which is a contradiction.

In the case of planar graphs, we define a vertex to be *good* if it sees at most 11 other vertices. Since $g = 0$, (4) and (5) can be improved to

$$c_x \le 180 - 60\deg(x) + \frac{240}{\ell_x} + 20t_x \le 240 - 40\deg(x) \le 40. \tag{8}$$

Subsequently, (6) is improved to

$$0 < c'_v = 180 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + 8|D_v|, \tag{9}$$

and (7) is improved to

$$0 < c'_v \le 180 - 60\deg(v) + \frac{240}{\ell_v} + 20t_v + 2\left(\frac{48}{\ell_v} - 8\right) + 8(|D_v| - 2). \tag{10}$$

Checking all values of $\deg(v)$, t_v and ℓ_v that satisfy (9) and (10) proves that $t_v + (\deg(v) - t_v)(\ell_v - 2) \le 11$. As in the main proof, it follows that v is good. □

We now prove that the assumption that $|G| \in \Omega(g)$ in Proposition 6.5 is necessary. Let G be the graph obtained from C_{2n}^2 by adding a perfect matching, as shown embedded in $\mathbb{S}_n$ in Fig. 5 (where there is one handle for each pair of crossing edges). This graph is 5-regular, but each vertex is on a facial walk of length n. Thus no vertex sees a bounded number of vertices.

7. Open problems

The first open problem that arises from this work is to determine the best possible function f in Theorem 1.1. In particular, does average degree at least some polynomial in t force a small K_t-model? Even stronger, is there a function h, such that every graph G with average degree at least $f(t) + \varepsilon$ contains a K_t-model with $h(t, \varepsilon) \cdot \log |G|$ vertices, where $f(t)$ is the minimum number such that every graph with average degree at least $f(t)$ contains a K_t-minor? We have answered this question in the affirmative for $t \le 4$. The case $t = 5$ is open. It follows from Wagner's characterisation of graphs with no K_5-minor that average degree at least 6 forces a K_5-minor [27]. Theorem 4.3 proves that average degree at least $16 + \varepsilon$ forces a K_5-model with at most $h(\varepsilon) \cdot \log n$ vertices. We conjecture the following improvement:

Conjecture 7.1. *There is a function h such that for all $\varepsilon > 0$, every graph G with average degree at least $6 + \varepsilon$ contains a K_5-model with at most $h(\varepsilon) \cdot \log |G|$ vertices.*

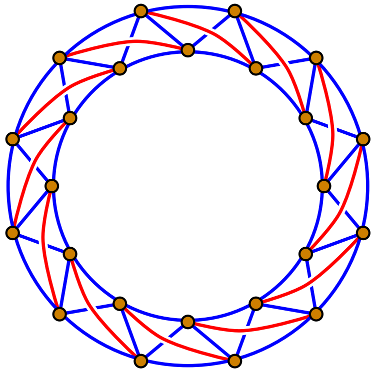

Fig. 5. C_{24}^2 plus a perfect matching, embedded on $\mathbb{S}_{12}$.

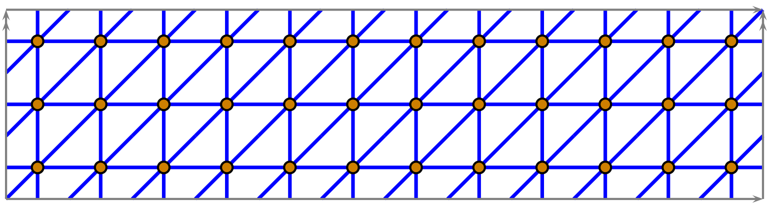

Fig. 6. 6-regular 12×3 triangulated toroidal grid.

This degree bound would be best possible: Let G_n be the 6-regular $n \times 3$ triangulated toroidal grid, as illustrated in Fig. 6. Every K_5-model in G_n intersects every column (otherwise K_5 is planar). Thus every K_5-model in G_n has at least n vertices.

Note that, while in this paper we have only studied small K_t-models, the same questions apply for small H-models, for arbitrary graphs H. This question was studied for $H = K_4 - e$ in [4]. See [25,22,21,10,15] for results about forcing H-minors.

Acknowledgements

The authors are grateful to Michele Conforti for suggesting to study the relationship between average degree and small models, to Paul Seymour for suggesting the example following Lemma 4.2, and to Alexandr Kostochka for pointing out Ref. [11].

The first and second authors were supported in part by the Actions de Recherche Concertées (ARC) fund of the Communauté française de Belgique. The last author was supported by a QEII Research Fellowship and Discovery Project from the Australian Research Council.

Appendix. Some technicalities

Lemma A.1. *Let $\alpha > 0$. Let $d, f_1, \ldots, f_d$ be integers, each at least 3, such that*

$$\sum_{i=1}^{d} \frac{1}{f_i} > \left(\frac{1}{3} + \frac{1}{\alpha}\right) d - \frac{1}{3}.$$

Then

$$\sum_{i=1}^{d} (f_i - 2) \leq \left\lceil \frac{\alpha}{3} \right\rceil - 1.$$

Proof. We may assume that $f_1, \ldots, f_d$ firstly maximise $\sum_i (f_i - 2)$, and secondly maximise $\sum_i \frac{1}{f_i}$. We claim that $f_i = 3$ for all $i \in \{1, \ldots, d\}$ except perhaps one. Suppose on the contrary that $f_j \geq f_k \geq 4$ for distinct $j, k \in \{1, \ldots, d\}$. Let $f'_i := f_i$ for $i \in \{1, \ldots, d\} \setminus \{j, k\}$, $f'_j := f_j + 1$, and $f'_k := f_k - 1$. Then

$$\sum_{i=1}^{d} f'_i = \sum_{i=1}^{d} f_i \quad \text{but} \quad \sum_{i=1}^{d} \frac{1}{f'_i} > \sum_{i=1}^{d} \frac{1}{f_i},$$

implying $f_1, \ldots, f_d$ do not maximise $\sum_j \frac{1}{f_j}$. Thus the claim holds and we may assume $f_i = 3$ for $i \in \{1, \ldots, d-1\}$. Hence

$$\frac{d-1}{3} + \frac{1}{f_d} > \left(\frac{1}{3} + \frac{1}{\alpha}\right) d - \frac{1}{3}.$$

Thus $\frac{1}{f_d} > \frac{d}{\alpha}$, implying $f_d \leq \lceil \frac{\alpha}{d} \rceil - 1$. Since $\frac{\alpha}{d} > f_d \geq 3$ and since $d \geq 3$,

$$\frac{\alpha}{3} = \frac{\alpha}{d}\left(\frac{d}{3} - 1\right) + \frac{\alpha}{d} \geq 3\left(\frac{d}{3} - 1\right) + \frac{\alpha}{d} = d - 3 + \frac{\alpha}{d}.$$

Hence

$$\left\lceil \frac{\alpha}{3} \right\rceil \geq \left\lceil d - 3 + \frac{\alpha}{d} \right\rceil = d - 3 + \left\lceil \frac{\alpha}{d} \right\rceil.$$

Therefore

$$\sum_{i=1}^{d} (f_i - 2) \leq (d-1)(3-2) + \left\lceil \frac{\alpha}{d} \right\rceil - 3 = d - 3 + \left\lceil \frac{\alpha}{d} \right\rceil - 1 \leq \left\lceil \frac{\alpha}{3} \right\rceil - 1.$$

This completes the proof. □

Lemma A.2. *Let $\alpha > 0$. Let $d, f_1, \ldots, f_d$ be integers, each at least 3, such that*

$$\sum_{i=1}^{d} \frac{1}{f_i} > \left(\frac{1}{3} + \frac{1}{\alpha}\right)(d-1).$$

Then

$$\sum_{i=1}^{d} (f_i - 2) \leq \left\lceil \frac{\alpha}{2} \right\rceil - 1.$$

Proof. As in the proof of Lemma A.1, we may assume that $f_j = 3$ for all $j \in \{3, \ldots, d-1\}$. Hence

$$\frac{d-1}{3} + \frac{1}{f_d} > \left(\frac{1}{3} + \frac{1}{\alpha}\right)(d-1).$$

Thus $\frac{1}{f_d} > \frac{d-1}{\alpha}$, implying $f_d \leq \lceil \frac{\alpha}{d-1} \rceil - 1$. Since $\frac{\alpha}{d-1} > f_d \geq 3$ and since $d \geq 3$,

$$\frac{\alpha}{2} \geq \frac{\alpha d}{3(d-1)} = \left(\frac{\alpha}{d-1}\right)\left(\frac{d}{3} - 1\right) + \frac{\alpha}{d-1} \geq 3\left(\frac{d}{3} - 1\right) + \frac{\alpha}{d-1} = d - 3 + \frac{\alpha}{d-1}.$$

Hence

$$\left\lceil \frac{\alpha}{2} \right\rceil \geq \left\lceil d - 3 + \frac{\alpha}{d-1} \right\rceil = d - 3 + \left\lceil \frac{\alpha}{d-1} \right\rceil.$$

Therefore

$$\sum_{i=1}^{d}(f_i - 2) \le (d-1)(3-2) + \left\lceil \frac{\alpha}{d-1} \right\rceil - 3 = d - 3 + \left\lceil \frac{\alpha}{d-1} \right\rceil - 1 \le \left\lceil \frac{\alpha}{2} \right\rceil - 1.$$

This completes the proof. □

References

[1] O.V. Borodin, On the total coloring of planar graphs, J. Reine Angew. Math. 394 (1989) 180–185. http://dx.doi.org/10.1515/crll.1989.394.180.
[2] L.S. Chandran, A high girth graph construction, SIAM J. Discrete Math. 16 (3) (2003) 366–370. http://dx.doi.org/10.1137/S0895480101387893.
[3] R. Diestel, Graph theory, Third ed. in: Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005, http://diestel-graph-theory.com/.
[4] S. Fiorini, G. Joret, U. Pietropaoli, Hitting diamonds and growing cacti, in: Proc. of 14th Conference on Integer Programming and Combinatorial Optimization, IPCO XIV, in: Lecture Notes in Computer Science, vol. 6080, Springer, 2010, pp. 191–204. http://dx.doi.org/10.1007/978-3-642-13036-6_15.
[5] S. Jendrol', T. Madaras, On light subgraphs in plane graphs of minimum degree five, Discuss. Math. Graph Theory 16 (2) (1996) 207–217.
[6] S. Jendrol', H.-J. Voss, Light subgraphs of order at most 3 in large maps of minimum degree 5 on compact 2-manifolds, European J. Combin. 26 (3–4) (2005) 457–471. http://dx.doi.org/10.1016/j.ejc.2004.01.013.
[7] T. Jiang, Compact topological minors in graphs, J. Graph Theory 67 (2) (2011) 139–152. http://dx.doi.org/10.1002/jgt.20522.
[8] A.V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Analiz. 38 (1982) 37–58.
[9] A.V. Kostochka, Lower bound of the Hadwiger number of graphs by their average degree, Combinatorica 4 (4) (1984) 307–316. http://dx.doi.org/10.1007/BF02579141.
[10] A. Kostochka, N. Prince, On $K_{s,t}$-minors in graphs with given average degree, Discrete Math. 308 (19) (2008) 4435–4445. http://dx.doi.org/10.1016/j.disc.2007.08.041.
[11] A. Kostochka, L. Pyber, Small topological complete subgraphs of dense graphs, Combinatorica 8 (1) (1988) 83–86. http://dx.doi.org/10.1007/BF02122555.
[12] A. Kotzig, Contribution to the theory of Eulerian polyhedra, Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955) 101–113.
[13] M. Krivelevich, B. Sudakov, Minors in expanding graphs, Geom. Funct. Anal. 19 (1) (2009) 294–331. http://dx.doi.org/10.1007/s00039-009-0713-z.
[14] D. Kühn, D. Osthus, Minors in graphs of large girth, Random Structures Algorithms 22 (2) (2003) 213–225. http://dx.doi.org/10.1002/rsa.10076.
[15] D. Kühn, D. Osthus, Forcing unbalanced complete bipartite minors, European J. Combin. 26 (1) (2005) 75–81. http://dx.doi.org/10.1016/j.ejc.2004.02.002.
[16] D. Kühn, D. Osthus, Improved bounds for topological cliques in graphs of large girth, SIAM J. Discrete Math. 20 (1) (2006) 62–78. http://dx.doi.org/10.1137/040617765.
[17] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268. http://dx.doi.org/10.1007/BF01364272.
[18] W. Mader, Homomorphiesätze für Graphen, Math. Ann. 178 (1968) 154–168. http://dx.doi.org/10.1007/BF01350657.
[19] W. Mader, Topological subgraphs in graphs of large girth, Combinatorica 18 (3) (1998) 405–412. http://dx.doi.org/10.1007/PL00009829.
[20] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, U.S.A., 2001.
[21] J.S. Myers, The extremal function for unbalanced bipartite minors, Discrete Math. 271 (1–3) (2003) 209–222. http://dx.doi.org/10.1016/S0012-365X(03)00051-7.
[22] J.S. Myers, A. Thomason, The extremal function for noncomplete minors, Combinatorica 25 (6) (2005) 725–753. http://dx.doi.org/10.1007/s00493-005-0044-0.
[23] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (2) (1984) 261–265. http://dx.doi.org/10.1017/S0305004100061521.
[24] A. Thomason, The extremal function for complete minors, J. Combin. Theory Ser. B 81 (2) (2001) 318–338. http://dx.doi.org/10.1006/jctb.2000.2013.
[25] A. Thomason, Extremal functions for graph minors, in: More Sets, Graphs and Numbers, in: Bolyai Soc. Math. Stud., vol. 15, Springer, Berlin, 2006, pp. 359–380.
[26] C. Thomassen, Girth in graphs, J. Combin. Theory Ser. B 35 (2) (1983) 129–141. http://dx.doi.org/10.1016/0095-8956(83)90067-9.
[27] K. Wagner, Über eine Eigenschaft der ebene Komplexe, Math. Ann. 114 (1937) 570–590. http://dx.doi.org/10.1007/BF01594196.
[28] P. Wernicke, Über den kartographischen Vierfarbensatz, Math. Ann. 58 (1904) 413–426. http://dx.doi.org/10.1007/BF01444968.