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Bounded degree acyclic decompositions of digraphs ☆

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Abstract

An acyclic decomposition of a digraph is a partition of the edges into acyclic subgraphs. Trivially every digraph has an acyclic decomposition into two subgraphs. It is proved that for every integer $s \geqslant 2$ every digraph has an acyclic decomposition into s subgraphs such that in each subgraph the outdegree of each vertex v is at most $\left\lceil \frac{\deg(v)}{s-1} \right\rceil$. For all digraphs this degree bound is optimal.

Keywords: Digraph; Acyclic subgraph; Acyclic decomposition; Arboricity; Diarboricity

1. Introduction

Throughout this paper a *digraph* $G = (V, E)$ is a finite loopless directed graph without parallel edges (but possibly with 2-cycles). $E(v)$ denotes the set of outgoing edges incident to a vertex $v \in V$, and $\deg(v) = |E(v)|$ denotes the *outdegree* of v. For some set of edges $E' \subseteq E$, $G[E']$ denotes the subdigraph $G' = (V, E')$.

A *vertex-ordering* π of a digraph $G = (V, E)$ is a total order $(v_1, v_2, \ldots, v_n)$ of V. For each vertex v_i, let $\mathrm{succ}_\pi(v_i) = \{(v_i, v_j) \in E(v_i) : i < j\}$ and $\mathrm{pred}_\pi(v_i) = \{(v_i, v_j) \in E(v_i) : j < i\}$. Edges $(v_i, v_j) \in \mathrm{succ}_\pi(v_i)$ (respectively, $\mathrm{pred}_\pi(v_i)$) are called *successor* (*predecessor*) *edges* of v_i in π, and v_j is called a *successor* (*predecessor*) of v_i in π.

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It is well known that a digraph G is acyclic if and only if G has a vertex-ordering π such that all edges are successor edges. π is then called a *topological ordering* (or *topological sort*) of G.

An *acyclic decomposition* of a digraph $G = (V, E)$ is a partition $\{E_1, E_2, \ldots, E_s\}$ of E such that each $G[E_i]$ is acyclic. Every digraph G has an acyclic decomposition into two subgraphs. To see this, take an arbitrary vertex-ordering π of G, and let $E_1 = \bigcup_v \text{succ}_\pi(v)$ and $E_2 = \bigcup_v \text{pred}_\pi(v)$. Then π and its reverse are topological orderings of $G[E_1]$ and $G[E_2]$, respectively. Thus $G[E_1]$ and $G[E_2]$ are acyclic.

Our aim is to produce acyclic decompositions with a prescribed number of subgraphs such that in each subgraph each vertex has small outdegree proportional to its original outdegree. Such acyclic decompositions have been implicitly used in algorithms for orthogonal graph drawing [2,7]. For a given acyclic decomposition $\{E_1, E_2, \ldots, E_s\}$ of G, the outdegree of each vertex v in $G[E_i]$ is denoted by $\deg_i(v)$.

Theorem 1. *For every integer* $s \geqslant 2$, *every digraph* $G = (V, E)$ *has an acyclic decomposition* $\{E_1, E_2, \ldots, E_s\}$ *such that* $\deg_i(v) \leqslant \left\lceil \frac{\deg(v)}{s-1} \right\rceil$ *for all vertices* $v \in V$ *and all* $i \in \{1, 2, \ldots, s\}$.

Since every acyclic digraph has some vertex with outdegree zero, in every acyclic decomposition $\{E_1, E_2, \ldots, E_s\}$ of G, there is a subgraph $G[E_i]$ and a vertex v with $\deg_i(v) \geqslant \left\lceil \frac{\deg(v)}{s-1} \right\rceil$. Thus Theorem 1 is optimal for all G and s. Note that Theorem 1 also holds when replacing outdegree by indegree—just reverse each edge and apply the outdegree result.

Theorem 1 is related to the arboricity of a digraph. See [3–6] for various 'diarboricity' definitions. Here the aim is to find acyclic decompositions with a small number of subgraphs, each satisfying some prescribed upper bound on the indegree or outdegree (or both). Define $\text{arb}_d(G)$ (respectively, $\text{arb}^d(G)$) to be the minimum number of subgraphs in an acyclic decomposition of a digraph G into subgraphs with maximum indegree (outdegree) at most d. Let $\Delta^-(G)$ and $\Delta^+(G)$ be the maximum indegree and outdegree of G. Karejan [3] showed that $\text{arb}_1(G) \in \{\Delta^-(G), \Delta^-(G)+1\}$, and thus $\text{arb}^1 G \in \{\Delta^+(G), \Delta^+(G)+1\}$. It follows from Theorem 1 with $s = \left\lceil \frac{\Delta^+(G)}{d} \right\rceil + 1$ that $\text{arb}^d(G) \leqslant \left\lceil \frac{\Delta^+(G)}{d} \right\rceil + 1$. By the above lower bound, this result is optimal for out-regular digraphs. Similarly one shows that $\text{arb}_d(G) \leqslant \left\lceil \frac{\Delta^-(G)}{d} \right\rceil + 1$, which matches the above bound of Karejan for $d = 1$ and is optimal for in-regular digraphs.

2. Proof of Theorem 1

We prove Theorem 1 by induction on s with the following induction hypothesis: *For a given vertex-ordering* $\pi = (v_1, v_2, \ldots, v_n)$ of a digraph $G = (V, E)$, *there is an acyclic decomposition* $\{E_1, E_2, \ldots, E_s\}$ *of* G *such that for all* $i \in \{1, 2, \ldots, s\}$ *and all* *vertices* $v \in V$,

$$\deg_i(v) \leqslant D_{s,\pi}(v) \stackrel{\text{def}}{=} \begin{cases} \left\lceil \frac{\deg(v)}{s} \right\rceil & \text{if } |\text{pred}_\pi(v)| = \deg(v), \\ \left\lceil \frac{\deg(v) - \lfloor |\text{pred}_\pi(v)|/s \rfloor}{s-1} \right\rceil & \text{if } |\text{pred}_\pi(v)| < \deg(v). \end{cases}$$

Proof. Construct a vertex-ordering σ of G as follows. Start with an empty vertex-ordering, and add the vertices $v_1, v_2, \ldots, v_n$ into σ in this order, inserting v_i so that

$$|\text{pred}_\sigma(v_i)| = \begin{cases} \left\lceil \frac{s-1}{s} |\text{pred}_\pi(v_i)| \right\rceil & \text{if } |\text{pred}_\pi(v_i)| < \deg(v_i), \\ \left\lfloor \frac{s-1}{s} |\text{pred}_\pi(v_i)| \right\rfloor & \text{if } |\text{pred}_\pi(v_i)| = \deg(v_i). \end{cases}$$

Note that the predecessors of v_i in π are already in σ when v_i is inserted. Since each vertex v with $|\text{pred}_\pi(v)| = \deg(v)$ has no successors in π, in the final vertex-ordering σ,

$$|\text{pred}_\sigma(v)| = \left\lfloor \frac{s-1}{s} \deg(v) \right\rfloor \text{ and } |\text{succ}_\sigma(v)| = \left\lceil \frac{\deg(v)}{s} \right\rceil. \tag{1}$$

For each vertex v with $|\text{pred}_\pi(v)| < \deg(v)$, regardless of where in σ the successors of v in π are inserted,

$$\begin{aligned} \left\lceil \frac{s-1}{s} |\text{pred}_\pi(v)| \right\rceil \leqslant |\text{pred}_\sigma(v)| &\leqslant \left\lceil \frac{s-1}{s} |\text{pred}_\pi(v)| \right\rceil + |\text{succ}_\pi(v)| \\ &= \left\lceil \frac{s-1}{s} |\text{pred}_\pi(v)| \right\rceil + \deg(v) - |\text{pred}_\pi(v)| \\ &= \deg(v) - \left\lfloor \frac{|\text{pred}_\pi(v)|}{s} \right\rfloor \end{aligned} \tag{2}$$

and

$$\left\lfloor \frac{|\text{pred}_\pi(v)|}{s} \right\rfloor \leqslant |\text{succ}_\sigma(v)| \leqslant \deg(v) - \left\lceil \frac{s-1}{s} |\text{pred}_\pi(v)| \right\rceil. \tag{3}$$

Consider the base case with $s = 2$. For each vertex v, let $E_1(v) = \text{succ}_\sigma(v)$ and $E_2(v) = \text{pred}_\sigma(v)$. Then σ and its reverse are topological orderings of $G[E_1]$ and $G[E_2]$, respectively. Thus $\{E_1, E_2\}$ is an acyclic decomposition of G. For each $i \in \{1, 2\}$, every vertex v with $|\text{pred}_\pi(v)| = \deg(v)$ has

$$\deg_i(v) \leqslant \max\{|\text{pred}_\sigma(v)|, |\text{succ}_\sigma(v)|\} = \left\lceil \tfrac{1}{2} \deg(v) \right\rceil = D_{2,\pi}(v)$$

by Eq. (1), and every vertex v with $|\text{pred}_\pi(v)| < \deg(v)$ has

$$\deg_i(v) \leqslant \max\{|\text{pred}_\sigma(v)|, |\text{succ}_\sigma(v)|\} \leqslant \deg(v) - \left\lfloor \tfrac{1}{2} |\text{pred}_\pi(v)| \right\rfloor = D_{2,\pi}(v)$$

by Eqs. (2) and (3). Thus the induction hypothesis holds for $s = 2$.

Now assume $s \geqslant 3$ and the induction hypothesis holds for $s - 1$. For each vertex v, let $E_s(v)$ be a set of $\min\{D_{s,\pi}(v), |\text{succ}_\sigma(v)|\}$ edges from $\text{succ}_\sigma(v)$. Let $E_s = \bigcup_v E_s(v)$. Then σ is a topological ordering of $G[E_s]$, and thus $G[E_s]$ is acyclic. By the choice of

$E_s(v)$, the claimed degree bound of $\deg_s(v) \leqslant D_{s,\pi}(v)$ holds. Let $G' = G \backslash E_s$, and denote the outdegree in G' of each vertex v by $\deg'(v) = \deg(v) - |E_s(v)|$.

By the induction hypothesis for $s-1$ applied to G' with vertex-ordering σ, G' has an acyclic decomposition $\{E_1, E_2, \ldots, E_{s-1}\}$ such that for all $i \in \{1, 2, \ldots, s-1\}$ and vertices v,

$$\deg_i(v) \leqslant \begin{cases} \left\lceil \dfrac{\deg'(v)}{s-1} \right\rceil & \text{(a)} \quad \text{if } |\mathrm{pred}_\sigma(v)| = \deg'(v), \\ \left\lceil \dfrac{\deg'(v) - \lfloor |\mathrm{pred}_\sigma(v)|/(s-1) \rfloor}{s-2} \right\rceil & \text{(b)} \quad \text{if } |\mathrm{pred}_\sigma(v)| < \deg'(v). \end{cases} \tag{4}$$

We now show that each vertex v has $\deg_i(v) \leqslant D_{s,\pi}(v)$ for all $i \in \{1, 2, \ldots, s-1\}$.

Case 1. $|\mathrm{pred}_\pi(v)| = \deg(v)$: Then $D_{s,\pi}(v) = \left\lceil \frac{\deg(v)}{s} \right\rceil = |\mathrm{succ}_\sigma(v)|$ by Eq. (1). Thus $E_s(v) = \mathrm{succ}_\sigma(v)$ and $\deg'(v) = |\mathrm{pred}_\sigma(v)| = \left\lfloor \frac{s-1}{s} \deg(v) \right\rfloor$ by Eq. (1). By Eq. (4)(a),

$$\deg_i(v) \leqslant \left\lceil \frac{1}{s-1} \left\lfloor \frac{s-1}{s} \deg(v) \right\rfloor \right\rceil \leqslant \left\lceil \frac{\deg(v)}{s} \right\rceil = D_{s,\pi}(v).$$

Case 2. $|\mathrm{pred}_\pi(v)| < \deg(v)$ and $|\mathrm{succ}_\sigma(v)| \leqslant D_{s,\pi}(v)$: Then $E_s(v) = \mathrm{succ}_\sigma(v)$ and $\deg'(v) = |\mathrm{pred}_\sigma(v)| \leqslant \deg(v) - \left\lfloor \frac{|\mathrm{pred}_\pi(v)|}{s} \right\rfloor$ by Eq. (2). By Eq. (4)(a),

$$\deg_i(v) \leqslant \left\lceil \frac{\deg(v) - \lfloor |\mathrm{pred}_\pi(v)|/s \rfloor}{s-1} \right\rceil = D_{s,\pi}(v).$$

Case 3. $|\mathrm{pred}_\pi(v)| < \deg(v)$ and $D_{s,\pi}(v) < |\mathrm{succ}_\sigma(v)|$: Then $|E_s(v)| = D_{s,\pi}(v)$ and thus,

$$\begin{aligned} \deg'(v) &= \deg(v) - D_{s,\pi}(v) \\ &= \deg(v) - \left\lceil \frac{\deg(v) - \lfloor |\mathrm{pred}_\pi(v)|/s \rfloor}{s-1} \right\rceil \\ &\leqslant \frac{(s-2)\deg(v) + \lfloor |\mathrm{pred}_\pi(v)|/s \rfloor}{s-1}. \end{aligned} \tag{5}$$

Since $|E_s(v)| < |\mathrm{succ}_\sigma(v)|$, $\deg'(v) > |\mathrm{pred}_\sigma(v)|$. Thus by Eqs. (4)(b) and (5),

$$\deg_i(v) \leqslant \left\lceil \frac{1}{s-2} \left(\frac{(s-2)\deg(v) + \lfloor |\mathrm{pred}_\pi(v)|/s \rfloor}{s-1} - \left\lfloor \frac{|\mathrm{pred}_\sigma(v)|}{s-1} \right\rfloor \right) \right\rceil.$$

By Eq. (2), $\frac{|\text{pred}_\sigma(v)|}{s-1} \geqslant \frac{|\text{pred}_\pi(v)|}{s}$. Thus,

$$\begin{aligned}\deg_i(v) &\leqslant \left\lceil \frac{\deg(v)}{s-1} + \frac{1}{s-2}\left(\frac{\lfloor |\text{pred}_\pi(v)|/s \rfloor}{s-1} - \left\lfloor \frac{|\text{pred}_\pi(v)|}{s} \right\rfloor\right)\right\rceil \\ &= \left\lceil \frac{\deg(v)}{s-1} + \frac{1}{s-2}\left(-\frac{s-2}{s-1}\left\lfloor \frac{|\text{pred}_\pi(v)|}{s} \right\rfloor\right)\right\rceil \\ &= \left\lceil \frac{\deg(v) - \lfloor |\text{pred}_\pi(v)|/s \rfloor}{s-1} \right\rceil \\ &= D_{s,\pi}(v).\end{aligned}$$

Hence every vertex v has $\deg_i(v) \leqslant D_{s,\pi}(v)$ for all $i \in \{1, 2, \ldots, s-1\}$. Thus $\{E_1, E_2, \ldots, E_s\}$ is an acyclic decomposition with $\deg_i(v) \leqslant D_{s,\pi}(v)$ for all vertices v and all $i \in \{1, 2, \ldots, s\}$. This completes the proof of the induction hypothesis. □

Theorem 1 follows immediately since $D_{s,\pi}(v) \leqslant \left\lceil \frac{\deg(v)}{s-1} \right\rceil$.

Finally, note that using a similar approach to the median placement algorithm of Bied et al. [1], each inductive step in the proof of Theorem 1 can be implemented in $O(|V| + |E|)$ time. Thus the desired acyclic decomposition can be determined in $O(s(|V| + |E|))$ time.

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