Drawing a Graph in a Hypercube

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Abstract

A *d-dimensional hypercube drawing* of a graph represents the vertices by distinct points in $\{0,1\}^d$, such that the line-segments representing the edges do not cross. We study lower and upper bounds on the minimum number of dimensions in hypercube drawing of a given graph. This parameter turns out to be related to Sidon sets and antimagic injections.

1 Introduction

Two-dimensional graph drawing [5, 15], and to a lesser extent, three-dimensional graph drawing [6, 17, 27] have been widely studied in recent years. Much less is known about graph drawing in higher dimensions. For research in this direction, see references [3, 8, 9, 26, 27]. This paper studies drawings of graphs in which the vertices are positioned at the points of a hypercube.

We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Consider an injection $\lambda : V(G) \to \{0,1\}^d$. For each edge $vw \in E(G)$, let $\lambda(vw)$ be the open line-segment with endpoints $\lambda(v)$ and $\lambda(w)$. Two distinct edges $vw, xy \in E(G)$ *cross* if $\lambda(vw) \cap \lambda(xy) \neq \emptyset$. We say λ is a *d-dimensional hypercube drawing* of G if no two edges of G cross. A d-dimensional hypercube drawing is said to have *volume* 2^d. That

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is, the volume is the total number of points in the hypercube, and is a measure of the efficiency of the drawing. Let $\mathrm{vol}(G)$ be the minimum volume of a hypercube drawing of a graph G. This paper studies lower and upper bounds on $\mathrm{vol}(G)$.

The remainder of the paper is organised as follows. In Section 2 we review material on Sidon sets and so-called antimagic injections of graphs. In Section 3 we explore the relationship between hypercube drawings and antimagic injections. This enables lower and upper bounds on $\mathrm{vol}(K_n)$ to be proved. In Section 4, we present a simple algorithm for computing an antimagic injection that gives upper bounds on the volume of hypercube drawings in terms of the degeneracy of the graph. In Section 5 we prove a relationship between antimagic injections and queue layouts of graphs that enables an $\mathcal{NP}$-completeness result to be concluded. In Section 6 we relate antimagic injections of graphs to the bandwidth and pathwidth parameters. Finally, in Section 7 we give an asymptotic bound on the volume of hypercube drawings. The proof is based on the Lovász Local Lemma.

2 Sidon Sets and Antimagic Injections

A set $S \subseteq \mathbb{Z}^+$ is called *Sidon* if $a+b=c+d$ implies $\{a,b\}=\{c,d\}$ for all $a,b,c,d \in S$. See the recent survey by O'Bryant [21] for results and numerous references on Sidon sets. A graph in which self-loops are allowed (but no parallel edges) is called a *pseudograph*. For a pseudograph G, an injection $f : V(G) \to \mathbb{Z}^+$ is *antimagic* if $f(v)+f(w) \neq f(x)+f(y)$ for all distinct edges $vw, xy \in E(G)$; see [1, 12, 28]. Let $[k] := \{1,2,\dots,k\}$. Let $\mathrm{mag}(G)$ be the minimum k such that the pseudograph G has an antimagic injection $f : V(G) \to [k]$.

Let K_n^+ be the complete pseudograph; that is, every pair of vertices are adjacent and there is one loop at every vertex. Clearly an antimagic injection of K_n^+ is nothing more than a Sidon set of cardinality n. It follows from results by Singer [23] and Erdős and Turán [11] (see Bollobás and Pikhurko [1]) that

$$\mathrm{mag}(K_n) \;=\; (1+o(1))n^2 \;\text{ and }\; \mathrm{mag}(K_n^+) \;=\; (1+o(1))n^2 \;. \tag{1}$$

Note the following simple lower bound.

Lemma 1. *Every pseudograph G satisfies* $\mathrm{mag}(G) \geq \max\{|V(G)|, \frac{1}{2}(|E(G)|+3)\}$.

Proof. That $\mathrm{mag}(G) \geq |V(G)|$ follows from the definition. Let $\lambda : V(G) \to [k]$ be an antimagic injection of G. For every edge $vw \in E(G)$, $\lambda(v)+\lambda(w)$ is a distinct integer in $\{3,4,\dots,2k-1\}$. Thus $|E(G)| \leq 2k-3$ and $k \geq \frac{1}{2}(|E(G)|+3)$. □

3 Hypercube Drawings

Consider the maximum number of edges in a hypercube drawing. The following observation is a special case of a result by Bose et al. [2] regarding the volume of grid drawings, where the bounding box is unrestricted.

Lemma 2 ([2]). *The maximum number of edges in a d-dimensional hypercube drawing is $3^d - 2^d$.*

Trivially, $\mathrm{vol}(G) \geq |V(G)|$. For dense graphs, we have the following improved lower bound.

Lemma 3. *Every n-vertex m-edge graph G satisfies*

$$\mathrm{vol}(G) \;\geq\; (n+m)^{1/\log_2 3} \;=\; (n+m)^{0.631\dots} \;.$$

Proof. Suppose that G has a d-dimensional hypercube drawing. By Lemma 2 and since $n \leq 2^d$, we have $n + m \leq 3^d$. That is, $d \geq \log_2(n+m)/\log_2 3$, and the volume $2^d \geq (n+m)^{1/\log_2 3}$. □

Now we characterise when two edges cross.

Lemma 4. *Consider an injection $\lambda : V(G) \to \{0,1\}^d$ for some graph G. Two distinct edges $vw, xy \in E(G)$ cross if and only if $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$.*

Proof. Suppose that $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. Then $\frac{1}{2}(\lambda(v) + \lambda(w)) = \frac{1}{2}(\lambda(x) + \lambda(y))$. That is, the midpoint of $\lambda(vw)$ equals the midpoint of $\lambda(xy)$. Hence vw and xy cross. (Note that this idea is used to prove the upper bound in Lemma 2, since the number of midpoints is at most $3^d - 2^d$.) Conversely, suppose that vw and xy cross. Since all vertex coordinates are 0 or 1, the point of intersection between $\lambda(vw)$ and $\lambda(xy)$ is the midpoint of both edges. That is, $\frac{1}{2}(\lambda(v)+\lambda(w)) = \frac{1}{2}(\lambda(x)+\lambda(y))$, and $\lambda(v)+\lambda(w) = \lambda(x)+\lambda(y)$. □

Loosely speaking, Lemma 4 implies that a hypercube drawing of G can be thought of as an antimagic injection of G into a set of binary vectors (where vector addition is *not* modulo 2). Moreover, from an antimagic injection we can obtain a hypercube drawing, and vice versa.

Lemma 5. *Every graph G satisfies* $\mathrm{vol}(G) \leq 2^{\lceil \log_2 \mathrm{mag}(G) \rceil} < 2\,\mathrm{mag}(G)$.

Proof. Let $k := \mathrm{mag}(G)$, and let $f : V(G) \to [k]$ be an antimagic injection of G. For each vertex $v \in V(G)$, let $\lambda(v)$ be the $\lceil \log_2 k \rceil$-bit binary representation of $f(v)$. Suppose that edges vw and xy cross. By Lemma 4, $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. For each $i \in [\lceil \log_2 k \rceil]$, the sum of the i-th coordinates of v and w equals the sum of the i-th coordinates of x and y. Thus $f(v) + f(w) = f(x) + f(y)$, which is the desired contradiction. Therefore no two edges cross, and λ is a $\lceil \log_2 k \rceil$-dimensional hypercube drawing of G. □

Lemma 6. *Every graph G satisfies* $\mathrm{mag}(G) \leq \mathrm{vol}(G)^{\log_2 3} = \mathrm{vol}(G)^{1.585\dots}$.

Proof. Let $\lambda : V(G) \to \{0,1\}^d$ be a hypercube drawing of G, where $d = \log_2 \mathrm{vol}(G)$. For each vertex $v \in V(G)$, define an integer $f(v)$ so that $\lambda(v)$ is the base-3 representation of $f(v)$. Now $\lambda(v) + \lambda(w) \in \{0,1,2\}^d$. Thus $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$ if and only if $f(v) + f(w) = f(x) + f(y)$. Since edges do not cross in λ and by Lemma 4, f is an antimagic injection of G into $[3^d] = [3^{\log_2 \mathrm{vol}(G)}] = [\mathrm{vol}(G)^{\log_2 3}]$. □

Consider the minimum volume of a hypercube drawing of the complete graph K_n.

Lemma 7. *Let $V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of binary d-dimensional vectors. Then V is the vertex set of a hypercube drawing of K_n if and only if $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$.*

Proof. Suppose that V is the vertex set of a hypercube drawing of K_n. Since no two edges cross, by Lemma 4, $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$ with $i \neq j$ and $k \neq \ell$. If $i = j$ and $k = \ell$, then $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ because distinct vertices are mapped to distinct points. If $i = j$ and $k \neq \ell$, then $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$, as otherwise the midpoint of the edge $v_k v_\ell$ would coincide with the vertex v_i, which is clearly impossible. Hence $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$. The converse result follows immediately from Lemma 4. $\square$

Sets of binary vectors satisfying Lemma 7 were first studied by Lindström [18, 19], and more recently by Cohen et al. [4]. Their results can be interpreted as follows, where the lower bound is by Cohen et al. [4], and the upper bound follows from (1) and Lemma 5.

Theorem 1. *Every complete graph K_n satisfies $\mathrm{vol}(K_n) < (2 + o(1))n^2$, and $\mathrm{vol}(K_n) > n^{1.7384\ldots}$ for large enough n.* $\square$

4 Degeneracy

Wood [28] proved that every n-vertex m-edge graph G with maximum degree Δ satisfies $\mathrm{mag}(G) < (\Delta(m - \Delta) + n)$. Thus Lemma 5 implies that

$$\mathrm{vol}(G) < 2(\Delta(m - \Delta) + n) \ . \tag{2}$$

This result by Wood [28] is proved using a greedy algorithm. We can obtain a more precise result as follows. The *degeneracy* of a graph G is the maximum, taken over all induced subgraphs H of G, of the minimum degree of H.

Lemma 8. *Every n-vertex m-edge graph G with degeneracy d satisfies $\mathrm{mag}(G) \leq n + dm$, and thus $\mathrm{vol}(G) < 2n + 2dm$.* $\square$

Proof. We proceed by induction on n' with the hypothesis that "every induced subgraph H of G on n' vertices has $\mathrm{mag}(H) \leq n' + dm$." If $n' = 1$ the result is trivial. Let H be an induced subgraph of G on $n' \geq 2$ vertices. Then H has a vertex v of degree at most d in H. By induction, $H \setminus v$ has an antimagic injection $\lambda : V(H \setminus v) \to [n' - 1 + dm]$. Now

$$\begin{aligned}
&\left|\{\lambda(x) : x \in V(H \setminus v)\} \ \cup \ \{\lambda(x) + \lambda(y) - \lambda(w) : xy \in E(H \setminus v), vw \in E(H)\}\right| \\
&\qquad \leq |V(H \setminus v)| \ + \ \deg_H(v) \cdot |E(H \setminus v)| \\
&\qquad \leq n' - 1 + dm \ .
\end{aligned}$$

Thus there exists an $i \in [n' + dm]$ such that $\lambda(x) \neq i$ for all $x \in V(H \setminus v)$, and $\lambda(x) + \lambda(y) - \lambda(w) \neq i$ for all edges $xy \in E(H \setminus v)$ and $vw \in E(H)$. Let $\lambda(v) := i$. Thus

$\lambda(v) \neq \lambda(x)$ for all $x \in V(H)$, and $\lambda(v)+\lambda(w) \neq \lambda(x)+\lambda(y)$ for all edges $xy \in E(H)$ and $vw \in E(G)$. Thus λ is an antimagic injection of H into $[n'+dm]$, and $\text{mag}(H) \leq n'+dm$. By induction, $\text{mag}(G) \leq n+dm$. $\square$

Planar graphs G are 5-degenerate, and thus satisfy $\text{mag}(G) < 16n$ and $\text{vol}(G) < 32n$ by Lemmas 5 and 8. More generally, Kostochka [16] and Thomason [24, 25] independently proved that a graph G with no K_k minor is $\mathcal{O}(k\sqrt{\log k})$-degenerate, and thus satisfy $\text{mag}(G) \in \mathcal{O}(k^2(\log k)n)$ and $\text{vol}(G) \in \mathcal{O}(k^2(\log k)n)$ by Lemmas 5 and 8. As we now show, a large clique minor does not necessarily force up $\text{mag}(G)$ or $\text{vol}(G)$. Let K'_n be the graph obtained from K_n by subdividing every edge once. Say K'_n has $n' := n + \binom{n}{2}$ vertices. Clearly K'_n is 2-degenerate. If follows from Lemma 8 that $\text{mag}(K'_n) \leq 5n'+o(n')$ and $\text{vol}(K'_n) \leq 10n' + o(n')$, yet K'_n contains a $(\sqrt{2n'} + o(n'))$-clique minor.

5 Queue Layouts and Complexity

Let G be a graph. A bijection $\sigma : V(G) \rightarrow [|V(G)|]$ is called a *vertex ordering* of G. Consider edges $vw, xy \in E(G)$ with no common endpoint. Without loss of generality $\sigma(v) < \sigma(w)$, $\sigma(x) < \sigma(y)$ and $\sigma(v) < \sigma(x)$. We say vw and xy are *nested* in σ if $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$. A *queue* in σ is a set of edges $Q \subseteq E(G)$ such that no two edges in Q are nested in σ. A *k-queue layout* of G consists of a vertex ordering σ of G, and a partition of $E(G)$ into k queues in σ. Heath et al. [13, 14] introduced queue layouts; see [7] for references and a summary of known results.

Lemma 9. *If a graph G has a 1-queue layout, then* $\text{mag}(G) = |V(G)|$.

Proof. Let $\sigma : V(G) \rightarrow [|V(G)|]$ be the vertex ordering in a 1-queue layout of G. If for distinct edges $vw, xy \in E(G)$, we have $\sigma(v) + \sigma(w) = \sigma(x) + \sigma(y)$, then vw and xy are nested. Since no two edges are nested in a 1-queue layout, σ is an antimagic injection of G, and $\text{mag}(G) \leq |V(G)|$. $\square$

Heath and Rosenberg [14] proved that it is $\mathcal{NP}$-complete to determine whether a given graph has a 1-queue layout. Thus, Lemma 9 implies:

Corollary 1. *Testing whether* $\text{mag}(G) = |V(G)|$ *is $\mathcal{NP}$-complete.* $\square$

It is has been widely conjectured that it is $\mathcal{NP}$-complete to recognise graphs that admit certain types of magic and antimagic injections. Corollary 1 is the first result in this direction that we are aware of.

Open Problem 1. Every k-queue graph G on n vertices is $4k$-degenerate [7, 22]. By Lemma 8, $\text{mag}(G) \in \mathcal{O}(k^2n)$ and $\text{vol}(G) \in \mathcal{O}(k^2n)$. Can these bounds be improved to $\mathcal{O}(kn)$?

6 Bandwidth and Pathwidth

Let P_n^k be the k-th power of a path. Thus, P_n^k is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{v_i v_j : 1 \leq |i-j| \leq k\}$. Now P_n^k has $kn - \frac{1}{2}k(k+1)$ edges. By Lemma 1, $\text{mag}(P_n^k) \geq \frac{1}{2}(kn - \frac{1}{2}k(k+1) + 3)$. The following upper bound is a generalisation of the construction of a Sidon set by Erdős and Turán [11].

Lemma 10. *For every prime p,* $\text{mag}(P_n^p) \leq p(2n-1)$.

Proof. If $p = 2$ then $\text{mag}(P_n^2)$ has a 1-queue layout, and $\text{mag}(P_n^2) = n$ by Lemma 9. Now assume that $p > 2$. Let $\lambda(v_i) := 1 + 2pi + (i^2 \bmod p)$ for every vertex v_i, $0 \leq i \leq n-1$. Clearly λ is an injection into $[p(2n-1)]$. Suppose on the contrary, that there are distinct edges $v_i v_\ell$ and $v_j v_k$ with $\lambda(v_i) + \lambda(v_\ell) = \lambda(v_j) + \lambda(v_k)$. Without loss of generality, $i < j < k < \ell \leq i + p$. Then

$$2pi + (i^2 \bmod p) + 2p\ell + (\ell^2 \bmod p) = 2pj + (j^2 \bmod p) + 2pk + (k^2 \bmod p) \ .$$

That is,

$$2p(i + \ell - j - k) = (j^2 \bmod p) + (k^2 \bmod p) - (i^2 \bmod p) - (\ell^2 \bmod p) \ .$$

Now $|(j^2 \bmod p) + (k^2 \bmod p) - (i^2 \bmod p) - (\ell^2 \bmod p)| \leq 2(p-1)$. Thus $i + \ell - j - k = 0$, and

$$(i^2 \bmod p) + (\ell^2 \bmod p) = (j^2 \bmod p) + (k^2 \bmod p) \ .$$

Thus

$$i^2 + \ell^2 \equiv j^2 + k^2 \pmod{p} \ . \tag{3}$$

Let $a := j - i$ and $b := k - i$. Then $0 < a < b < p$. Since $i + \ell = j + k$, we have $\ell = i + a + b$. Rewriting (3),

$$i^2 + (i + a + b)^2 \equiv (i + a)^2 + (i + b)^2 \pmod{p} \ .$$

Hence $2ab \equiv 0 \pmod{p}$. Since p is prime and $p > 2$, $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$, which is a contradiction since $0 < a < b < p$. Hence $\lambda(v_i) + \lambda(v_\ell) \neq \lambda(v_j) + \lambda(v_k)$, and λ is antimagic. □

The *bandwidth* of an n-vertex graph G is the minimum k such that G is a subgraph of P_n^k. By Bertrand's postulate there is a prime $p \leq 2k$. Thus Lemmas 5 and 10 imply:

Corollary 2. *Every n-vertex graph G with bandwidth k has* $\text{mag}(G) \leq 2k(2n-1)$ *and* $\text{vol}(G) < 4k(2n-1)$. □

We have the following technical lemma.

Lemma 11. *Let G be a graph. Let $f_V : V(G) \to [t] \times [r]$ be an injection. Define a function $f_E : E(G) \to \binom{[t]}{2} \times [2r]$ as follows. For every edge $vw \in E(G)$ with $f_V(v) = (a, i)$ and $f_V(w) = (b, j)$, let $f_E(vw) := (\{a, b\}, i + j)$. If f_E is also an injection, then* $\text{mag}(G) \leq (2 + o(1))t^2 r$.

Proof. Singer [23] proved that there is a Sidon set $\{s_1, s_2, \dots, s_t\} \in [(1+o(1))t^2]$. For every vertex $v \in V(G)$ with $f(v) = (a,i)$, let $\lambda(v) := 2r(s_a - 1) + i$. Since f is an injection, λ is an injection into $[(2+o(1))t^2 r]$. We claim that λ is antimagic. Suppose on the contrary that there are distinct edges $vw, xy \in E(G)$ with $\lambda(v)+\lambda(w) = \lambda(x)+\lambda(y)$. Say $f(v) = (a,i)$, $f(w) = (b,j)$, $f(x) = (c,k)$, and $f(y) = (d,\ell)$. Then

$$2r(s_a - 1) + i \;+\; 2r(s_b - 1) + j \;=\; 2r(s_c - 1) + k \;+\; 2r(s_d - 1) + \ell \;. \qquad (4)$$

That is, $2r(s_a + s_b - s_c - s_d) = k + \ell - i - j$. Now $|k + \ell - i - j| < 2r$. Thus $s_a + s_b = s_c + s_d$. Since $\{s_1, s_2, \dots, s_t\}$ is Sidon, $\{a,b\} = \{c,d\}$. By (4), $i + j = k + \ell$. Hence, $f_E(vw) = f_E(xy)$, which is a contradiction since f_E is an injection by assumption. Thus $\lambda(v)+\lambda(w) \neq \lambda(x)+\lambda(y)$, and λ is antimagic. Hence $\operatorname{mag}(G) \leq (2+o(1))t^2 r$. $\square$

Let $\mathcal{S}$ be a set of closed intervals in $\mathbb{R}$. Associated with $\mathcal{S}$, is the *interval graph* with vertex set $\mathcal{S}$ such that two vertices are adjacent if and only if the corresponding intervals have a non-empty intersection. The *pathwidth* of a graph G is the minimum k such that G is a spanning subgraph of an interval graph with no clique on $k+2$ vertices.

Theorem 2. *Every n-vertex graph G with pathwidth k satisfies* $\operatorname{mag}(G) \leq (8+o(1))kn$ *and* $\operatorname{vol}(G) \leq (16+o(1))kn$. *For all k and $n \geq k+1$, there exist n-vertex graphs G with pathwidth k and* $\operatorname{mag}(G) \geq \frac{1}{2}kn - \mathcal{O}(k^2)$.

Proof. Dujmović et al. [6] proved that there is an injection f satisfying Lemma 11 with $t = 2k+2$ and $r = \lceil n/k \rceil$. In fact, they proved the stronger result that for all edges $vw, xy \in E(G)$ with $f(v) = (a,i)$, $f(w) = (b,j)$, $f(x) = (a,k)$, $f(y) = (b,\ell)$, if $i < k$ then $j \leq \ell$ (which implies that $i + j < k + \ell$). By Lemma 11, $\operatorname{mag}(G) \leq (2+o(1))(2k+2)^2 r = (8+o(1))kn$. By Lemma 5, $\operatorname{vol}(G) \leq (16+o(1))kn$. For the lower bound, let $G = P_n^k$ for example. Then G has pathwidth k and $kn - \frac{1}{2}k(k+1)$ edges. By Lemma 1, $\operatorname{mag}(G) \geq \frac{1}{2}kn - \mathcal{O}(k^2)$. $\square$

Open Problem 2. Lemma 8 implies that graphs G of treewidth k satisfy $\operatorname{mag}(G) \in \mathcal{O}(k^2 n)$ and $\operatorname{vol}(G) \in \mathcal{O}(k^2 n)$. Can these bounds be improved to $\mathcal{O}(kn)$? Note that Wood [28] proved that every tree G satisfies $\operatorname{mag}(G) = |V(G)|$, which implies that $\operatorname{vol}(G) < 2|V(G)|$ by Lemma 5.

7 An Asymptotic Upper Bound

Our upper bounds on $\operatorname{vol}(G)$ have thus far been obtained as corollaries of upper bounds on $\operatorname{mag}(G)$. The next theorem, which improves upon (2), only applies to hypercube drawings. In fact, the method used only gives a $\mathcal{O}(n + \Delta m)$ bound on $\operatorname{mag}(G)$.

Theorem 3. *Every n-vertex m-edge graph G with maximum degree Δ satisfies*

$$\operatorname{vol}(G) \;\leq\; \mathcal{O}(n + (\Delta m)^{1/\log_2 8/3}) \;=\; \mathcal{O}(n + (\Delta m)^{0.707\dots}) \;.$$

Theorem 3 is proved using the Local Lemma by Erdős and Lovász [10] (see [20]).

Lemma 12 ([10]). *Let $\mathcal{E} = \{A_1, A_2, \ldots, A_n\}$ be a set of 'bad' events in some probability space, such that each event A_i is mutually independent of $\mathcal{E} \setminus (\{A_i\} \cup \mathcal{D}_i)$ for some $\mathcal{D}_i \subseteq \mathcal{E}$. Suppose that there is a set $\{x_i \in [0,1) : 1 \leq i \leq n\}$, such that for all i,*

$$\mathbf{P}(A_i) \ \leq \ x_i \cdot \prod_{A_j \in \mathcal{D}_i} (1 - x_j) \ . \tag{5}$$

Then

$$\mathbf{P}\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) \ \geq \ \prod_{i=1}^{n} (1 - x_i) \ > \ 0 \ .$$

That is, with positive probability, no event in $\mathcal{E}$ occurs.

Proof of Theorem 3. Let d be a positive integer, to be specified later. For each vertex $v \in V(G)$, let $\lambda(v)$ be a point in $\{0,1\}^d$ chosen randomly and independently. (One can think of this process as d fair coin tosses for each vertex.) We now set up an application of Lemma 12. For all pairs of distinct vertices $v, w \in V(G)$, let $A_{v,w}$ be the event that $\lambda(v) = \lambda(w)$. For all disjoint edges $vw, xy \in E(G)$, let $B_{vw,xy}$ be the event that vw and xy cross.

We will apply Lemma 12 to prove that with positive probability, no event occurs. Hence there exists λ such that no event occurs. No A-event means that λ is an injection. No B-event means that no edges cross. Thus λ is a d-dimensional hypercube drawing.

Observe that $\mathbf{P}(A_{v,w}) = (\frac{1}{2})^d$. It is easily seen that $\mathbf{P}(B_{vw,xy}) \leq (\frac{1}{2})^d$. Below we prove that $\mathbf{P}(B_{vw,xy}) = (\frac{3}{8})^d$. The idea here is that it is unlikely that some edges are involved in a crossing. For example, the actual edges of the hypercube cannot be in a crossing.

Let $M := \{(x_1, x_2, \ldots, x_d) : x_i \in \{0,1,2\}, i \in [d]\}$. Consider an edge $vw \in E(G)$. Clearly $\lambda(v) + \lambda(w) \in M$. The i-coordinate of $\lambda(v) + \lambda(w)$ equals 1 if and only if the i-coordinates of $\lambda(v)$ and $\lambda(w)$ are distinct, which occurs with probability $\frac{1}{2}$. The i-coordinate of $\lambda(v) + \lambda(w)$ equals 0 if and only if the i-coordinates of $\lambda(v)$ and $\lambda(w)$ both equal 0, which occurs with probability $\frac{1}{4}$. The i-coordinate of $\lambda(v) + \lambda(w)$ equals 2 if and only if the i-coordinates of $\lambda(v)$ and $\lambda(w)$ both equal 1, which occurs with probability $\frac{1}{4}$.

Let M_k be the subset of M consisting of those points with exactly k coordinates equal to 1. Thus, for every edge $vw \in E(G)$ and point $p \in M_k$,

$$\mathbf{P}(\lambda(v) + \lambda(w) = p) \ = \ (\tfrac{1}{2})^k (\tfrac{1}{4})^{d-k} \ = \ 2^{k-2d} \ .$$

Hence for all disjoint edges $vw, xy \in E(G)$ and points $p \in M_k$,

$$\mathbf{P}(\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y) = p) \ = \ 2^{2k-4d} \ .$$

Now $|M_k| = \binom{d}{k} 2^{d-k}$. Thus,

$$\mathbf{P}(\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y) \in M_k) \ = \ \binom{d}{k} 2^{d-k} \cdot 2^{2k-4d} \ = \ \binom{d}{k} 2^{k-3d} \ .$$

Thus by Lemma 4,

$$\mathbf{P}(B_{vw,xy}) \;=\; \mathbf{P}(\lambda(v)+\lambda(w)=\lambda(x)+\lambda(y)) \;=\; \sum_{k=0}^{d}\binom{d}{k}2^{k-3d} \;=\; \left(\frac{3}{8}\right)^{d} \;.$$

The base of the natural logarithm $\boldsymbol{e}$ satisfies the following well-known inequality for all $y>0$:

$$\tfrac{1}{\boldsymbol{e}} \;<\; \left(1-\tfrac{1}{y+1}\right)^{y} \;. \tag{6}$$

Now define

$$d \;:=\; \left\lceil \max\left\{\log_2 \boldsymbol{e}(4n+1),\, \log_{8/3}\boldsymbol{e}^2(4\Delta m+1)\right\}\right\rceil \;. \tag{7}$$

For each A-event, let $x_A := 1/(4n+1)$. For each B-event, let $x_B := 1/(4\Delta m+1)$. Thus $0<x_A<1$ and $0<x_B<1$, as required.

Each vertex is involved in at most n A-events, and at most Δm B-events. Each A-event involves two vertices, and is thus dependent on at most $2n$ other A-events, and at most $2\Delta m$ B-events. Each B-event involves four vertices, and is thus dependent on at most $4n$ A-events, and on at most $4\Delta m$ other B-events. We first verify (5) for each event $A_{v,w}$. By (6),

$$\begin{aligned} x_A\,(1-x_A)^{2n}\,(1-x_B)^{2\Delta m} \;&=\; \frac{1}{4n+1}\left(1-\frac{1}{4n+1}\right)^{2n}\left(1-\frac{1}{4\Delta m+1}\right)^{2\Delta m} \\ &\geq\; \frac{1}{\boldsymbol{e}(4n+1)} \;. \end{aligned}$$

By the definition of d in (7), $\frac{1}{\boldsymbol{e}(4n+1)} \geq \frac{1}{2^d}$, and thus

$$x_A\,(1-x_A)^{2n}\,(1-x_B)^{2\Delta m} \;\geq\; \left(\frac{1}{2}\right)^{d} \;=\; \mathbf{P}(A_{v,w}) \;.$$

Now we verify (5) for each event $B_{vw,xy}$. By (6),

$$\begin{aligned} x_B\,(1-x_A)^{4n}\,(1-x_B)^{4\Delta m} \;&=\; \frac{1}{4\Delta m+1}\left(1-\frac{1}{4n+1}\right)^{4n}\left(1-\frac{1}{4\Delta m+1}\right)^{4\Delta m} \\ &\geq\; \frac{1}{\boldsymbol{e}^2(4\Delta m+1)} \;. \end{aligned}$$

Note that (7) implies that $\left(\frac{8}{3}\right)^d \geq \boldsymbol{e}^2(4\Delta m+1)$. Thus,

$$x_B\,(1-x_A)^{4n}\,(1-x_B)^{4\Delta m} \;\geq\; \left(\frac{3}{8}\right)^{d} \;=\; \mathbf{P}(B_{vw,xy}) \;.$$

By Lemma 12, there is a d-dimensional hypercube drawing of G. The volume 2^d is $\mathcal{O}(n+(\Delta m)^{1/\log_2 8/3})$. This completes the proof of Theorem 3. $\square$

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