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Contractibility and the Hadwiger Conjecture

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ABSTRACT

Consider the following relaxation of the Hadwiger Conjecture: For each t there exists N_t such that every graph with no K_t-minor admits a vertex partition into $\lceil \alpha t + \beta \rceil$ parts, such that each component of the subgraph induced by each part has at most N_t vertices. The Hadwiger Conjecture corresponds to the case $\alpha = 1$, $\beta = -1$ and $N_t = 1$. Kawarabayashi and Mohar [K. Kawarabayashi, B. Mohar, A relaxed Hadwiger's conjecture for list colorings, J. Combin. Theory Ser. B 97 (4) (2007) 647–651. URL: http://dx.doi.org/10.1016/j.jctb.2006.11.002] proved this relaxation with $\alpha = \frac{31}{2}$ and $\beta = 0$ (and N_t a huge function of t). This paper proves this relaxation with $\alpha = \frac{7}{2}$ and $\beta = -\frac{3}{2}$. The main ingredients in the proof are: (1) a list colouring argument due to Kawarabayashi and Mohar, (2) a recent result of Norine and Thomas that says that every sufficiently large $(t + 1)$-connected graph contains a K_t-minor, and (3) a new sufficient condition for a graph to have a set of edges whose contraction increases the connectivity.

1. Introduction

In 1943, Hadwiger [5] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory; see Ref. [25] for a survey.[1]

Hadwiger Conjecture. Every graph with no K_t-minor is $(t - 1)$-colourable.

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[1] All graphs in this paper are undirected, simple and finite. Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$. For $v \in V(G)$, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$. If $X \subseteq V(G)$ then $G[X]$ denotes the subgraph induced by X. If vw is an edge of G then G/vw is the graph obtained from G by contracting vw; that is, the edge vw is deleted and the vertices v and w are identified. A *minor* of G is a graph that can be obtained from a subgraph of G by contracting edges. A *k-colouring* of G is a function that assigns one of at most k colours to each vertex of G, such that adjacent vertices receive distinct colours. G is *k-colourable* if G admits a k-colouring.

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The Hadwiger Conjecture holds[2] for $t \leq 6$. Kostochka [11,12] and Thomason [22,23] independently proved that for some constant c, every graph G with no K_t-minor has a vertex of degree at most $ct\sqrt{\log t}$ (and this bound is best possible). It follows that G is $ct\sqrt{\log t}$-colourable. This is the best known such upper bound. In particular, the following conjecture is unsolved:

Weak Hadwiger Conjecture. For some constant c, every graph with no K_t-minor is ct-colourable.

This conjecture motivated Kawarabayashi and Mohar [8] to prove the following relaxation; see Ref. [7] for a recent extension to graphs with no odd K_t-minor.

Theorem 1.1 ([8]). *For each $t \in \mathbb{Z}^+$ there exists $N_t \in \mathbb{Z}^+$ such that every graph with no K_t-minor admits a vertex partition into $\lceil \frac{31}{2}t \rceil$ parts, and each connected component of the subgraph induced by each part has at most N_t vertices.*

With $N_t = 1$ the vertex partition in Theorem 1.1 is a colouring. So Theorem 1.1 is a relaxation of the Weak Hadwiger Conjecture. It would be interesting to improve the bound of $\frac{31}{2}t$ in Theorem 1.1. Indeed, Kawarabayashi and Mohar [8] write:

> "The $\frac{31}{2}t$ bound can be improved slightly by fine-tuning parts of the proof in [1]. However, new methods would be needed to go below $10t$."

The main contribution of this paper is to improve $\frac{31}{2}$ in Theorem 1.1 to $\frac{7}{2}$.

Theorem 1.2. *For each $t \in \mathbb{Z}^+$ there exists $N_t \in \mathbb{Z}^+$ such that every graph with no K_t-minor admits a vertex partition into $\lceil \frac{7t-3}{2} \rceil$ parts, and each connected component of the subgraph induced by each part has at most N_t vertices.*

There are three main ingredients to the proof of Theorem 1.2. The first ingredient is a list colouring argument due to Kawarabayashi and Mohar [8], which is described in Section 2. The second ingredient is a sufficient condition for a graph to have a set of edges whose contraction increases the connectivity. This condition generalises previous results given by Mader [16], and is presented in Section 3. The third ingredient, the "new methods" alluded to in the above quote, is the following recent result by Norine and Thomas [18].

Theorem 1.3 ([18]). *For each $t \in \mathbb{Z}^+$ there exists $N_t \in \mathbb{Z}^+$ such that every $(t + 1)$-connected graph with at least N_t vertices has a K_t-minor.*

2. List colouring

A key tool in the proofs of Theorems 1.1 and 1.2 is the notion of list colouring, independently introduced by Vizing [26] and Erdős et al. [3]. A *list assignment* of a graph G is a function L that assigns to each vertex v of G a set $L(v)$ of colours. G is *L-colourable* if there is a colouring of G such that the colour assigned to each vertex v is in $L(v)$. G is *k-choosable* if G is L-colourable for every list assignment L with $|L(v)| \geq k$ for each vertex v of G. If G is k-choosable then G is also k-colourable—just use the same set of k colours for each vertex. See Ref. [29] for a survey on list colouring.

As well as being of independent interest, list colourings enable inductive proofs about ordinary colourings that might be troublesome without using lists. Most notable is the proof by Thomassen [24] that every planar graph is 5-choosable. This proof, unlike most proofs of the 5-colourability of planar graphs, does not use the fact that every planar graph has a vertex of degree at most 5. Given that

[2] If G has no K_1-minor then $V(G) = \emptyset$ and G is 0-colourable. If G has no K_2-minor then $E(G) = \emptyset$ and G is 1-colourable. If G has no K_3-minor then G is a forest, which is 2-colourable. Hadwiger [5] and Dirac [2] independently proved that if G has no K_4-minor (so-called *series-parallel* graphs) then G is 3-colourable. The Hadwiger Conjecture with $t = 5$ implies the Four-Colour Theorem, since planar graphs contain no K_5-minor. In fact, Wagner [27] proved that the Hadwiger Conjecture with $t = 5$ is equivalent to the Four-Colour Theorem, and therefore holds [4,19]. Robertson et al. [20] proved that the Hadwiger Conjecture with $t = 6$ also is a corollary of the Four-Colour Theorem.

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there are graphs with no K_t-minor and minimum degree $\Omega(t\sqrt{\log t})$, this suggests that list colourings might provide an approach for attacking the Hadwiger Conjecture. List colourings also provide a way to handle small separators—first colour one side of the separator, and then colour the second side with the vertices of the separator precoloured. This idea is central in the proofs of Theorems 1.1 and 1.2.

We need the following definitions. Let G be a graph. For $A, B \subseteq V(G)$, the pair $\{A, B\}$ is a *separation* of G if $G = G[A] \cup G[B]$ and $A - B \neq \emptyset$ and $B - A \neq \emptyset$. In particular, there is no edge between $A - B$ and $B - A$. The set $A \cap B$ is called a *separator*, and each of $A - B$ and $B - A$ are called *fragments*. If $|A \cap B| \leq t$ then $\{A, B\}$ is called a t-*separation* and $A \cap B$ is called a t-*separator*. By Menger's Theorem, G is t-connected if and only if G has no $(t-1)$-separation and $|V(G)| \geq t+1$. For $Z \subseteq V(G)$, a separation $\{A, B\}$ of G is Z-*good* if $\{A - Z, B - Z\}$ is also a separation of $G - Z$; otherwise it is Z-*bad*. Observe that $\{A, B\}$ is Z-*bad* if and only if $A - B \subseteq Z$ or $B - A \subseteq Z$.

Theorem 1.2 follows from the next lemma (with $Z = \emptyset$ and $L(v) = \{1, \ldots, \lceil \frac{7t-3}{2} \rceil\}$ for each $v \in V(G)$).

Lemma 2.1. *Let G be a graph containing no K_t-minor. Let $Z \subseteq V(G)$ with $|Z| \leq 2t - 1$. Let L be a list assignment of G such that:*

- $|L(v)| = 1$ *for each vertex $v \in Z$ (said to be "precoloured"),*
- $|L(w)| \geq \frac{7t-3}{2}$ *for each vertex $w \in V(G) - Z$.*

Then there is a function f such that:

(C1) *$f(v) \in L(v)$ for each vertex $v \in V(G)$,*
(C2) *for each colour i, if $V_i := \{v \in V(G) : f(v) = i\}$ then each component of $G[V_i]$ has at most $N_t + 2t - 1$ vertices (where N_t comes from Theorem 1.3), and*
(C3) *$f(v) \neq f(w)$ for all $v \in Z$ and $w \in N_G(v) - Z$.*

Proof. We proceed by induction on $|V(G)|$.

Case I: First suppose that $|V(G)| \leq N_t + 2t - 1$. For each vertex $v \in Z$, let $f(v)$ be the element of $L(v)$. For each vertex $w \in V(G) - Z$, choose $f(w) \in L(w)$ such that $f(w) \neq f(v)$ for every vertex $v \in Z$. This is possible since $|L(w)| \geq \frac{7t-3}{2} > 2t - 1 \geq |Z|$. Thus (C1) and (C3) are satisfied. (C2) is satisfied since $|V(G)| \leq N_t + 2t - 1$. Now assume that $V(G) \geq N_t + 2t - 1$.

Case II: Suppose that some vertex $x \in V(G) - Z$ has degree less than $\frac{7t-3}{2}$ in G. Let f be the function obtained by induction applied to $G - x$ with Z precoloured. Choose $f(x) \in L(x)$ such that $f(x) \neq f(y)$ for each $y \in N_G(x)$. This is possible since $|L(x)| \geq \frac{7t-3}{2} > \deg(x)$. Thus x is in its own monochromatic component. Hence (C1), (C2) and (C3) are maintained. Now assume that every vertex in $V(G) - Z$ has degree at least $\frac{7t-3}{2}$.

Case III: Suppose that G has a Z-good t-separation $\{A, B\}$. Let $P := Z - B$ and $Q := Z \cap A \cap B$ and $R := Z - A$ and $X := (A \cap B) - Z$. Thus P, Q, R, Z are pairwise disjoint. Since $Z = P \cup Q \cup R$, we have $|P| + |Q| + |R| \leq 2t - 1$. Since $A \cap B = Q \cup X$, we have $|Q| + |X| \leq t$ and $|Q| + 2|X| \leq 2t$. Thus $|P| + |R| + 2|Q| + 2|X| = (|P| + |Q| + |R|) + (|Q| + 2|X|) \leq 4t - 1$. Without loss of generality, $|P| \leq |R|$. Thus $2|P| + 2|Q| + 2|X| \leq 4t - 1$, implying $|P| + |Q| + |X| \leq 2t - 1$. That is, $|A \cap (B \cup Z)| \leq 2t - 1$.

Now $B \cup Z \neq V(G)$, as otherwise $A - B \subseteq Z$ and $\{A, B\}$ would be Z-bad. Thus the induction hypothesis is applicable to $G[B \cup Z]$ with Z precoloured. (This is why we need to consider Z-good and Z-bad separations.) Hence there is a function f such that:

(C1′) $f(v) \in L(v)$ for each vertex $v \in B \cup Z$,
(C2′) for each colour i, if $V'_i := \{v \in B \cup Z : f(v) = i\}$ then each component of $G[V'_i]$ has at most $N_t + 2t - 1$ vertices, and
(C3′) $f(v) \neq f(w)$ for all $v \in Z$ and $w \in (B \cap N_G(v)) - Z$.

Let $L'(w) := \{f(w)\}$ for each vertex $w \in A \cap (B \cup Z)$. Let $L'(v) := L(v)$ for each vertex $v \in A - (B \cup Z)$. Now apply induction to $G[A]$ with list assignment L', and $A \cap (B \cup Z)$ precoloured. This is possible since $|A \cap (B \cup Z)| \leq 2t - 1$. Hence there is a function f such that:

(C1″) $f(v) \in L'(v)$ for each vertex $v \in A$,
(C2″) for each colour i, if $V''_i := \{v \in A : f(v) = i\}$ then each component of $G[V''_i]$ has at most $N_t + 2t - 1$ vertices, and
(C3″) $f(v) \neq f(w)$ for all neighbours $v \in A - (B - Z)$ and $w \in A \cap (B \cup Z)$.

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Since $L'(v) \subseteq L(v)$, conditions (C1′) and (C1″) imply (C1). Since there is no edge between $A - B$ and $B - A$ in G, (C3′) and (C3″) imply that every component of $G[V_i]$ is a component of $G[V_i']$ or $G[V_i'']$ or $G[Z]$. Since $N_t + 2t - 1 \geq |Z|$, conditions (C2′) and (C2″) imply (C2). Hence (C1), (C2) and (C3) are satisfied. Now assume that every t-separation of G is Z-bad.

Case IV: Every vertex in $V(G) - Z$ has degree at least $\frac{7t-3}{2} \geq \frac{3}{2}k + |Z| - 2$, where $k := t + 1$. Thus Theorem 3.3 below implies that G has a $(t+1)$-connected minor H with at least $|V(G)| - |Z| \geq N_t$ vertices. By Theorem 1.3, H, and thus G, has a K_t-minor. This contradiction completes the proof. □

3. Contractibility

The main result in this section is Theorem 3.3, which was used in the proof of Lemma 2.1. The proof reduces to questions about contractibility that are of independent interest. Mader [16] proved the following sufficient condition for a given vertex to be incident to an edge whose contraction maintains connectivity.[3] See Refs. [14,17] for surveys of results in this direction.

Theorem 3.1 ([16]). *Let v be a vertex in a k-connected graph G, such that every neighbour of v has degree at least $\frac{3}{2}k - 1$. Then G/vw is k-connected for some edge vw incident to v.*

The following strengthening of Theorem 3.1 describes a scenario when there is an edge whose contraction *increases* connectivity.

Theorem 3.2. *Let v be a vertex in graph G, such that $N_G(v)$ is the only minimal $(k-1)$-separator, and every neighbour of v has degree at least $\frac{3}{2}k - 1$. Then G/vw is k-connected for some edge vw incident to v.*

The first condition in Theorem 3.2 is equivalent to saying that every $(k-1)$-separation of G is $\{v\}$-bad. Thus Theorem 3.2 is a special case of the following theorem (with $Z = \{v\}$).

Theorem 3.3. *Suppose that G is a graph and for some $Z \subset V(G)$,*

- *every $(k-1)$-separation of G is Z-bad, and*
- *every vertex in $\cup\{N_G(v) - Z : v \in Z\}$ has degree at least $\frac{3}{2}k + |Z| - 2$ in G.*

Then G has a set of at most $|Z|$ edges, each with one endpoint in Z, whose contraction gives a k-connected graph.

Proof. We proceed by induction on $|Z|$. If $Z = \emptyset$, or $N_G(v) \subseteq Z$ for each $v \in Z$, then $G - Z$ is k-connected. Now assume that $N_G(v) \not\subseteq Z$ for some $v \in Z$. By assumption, every vertex in $N_G(v) - Z$ has degree at least $\frac{3}{2}k + |Z| - 2$ in G. By Lemma 3.4 below there is an edge vw with $w \in N_G(v) - Z$ such that every $(k-1)$-separation of G/vw is $(Z - \{v\})$-bad. For every vertex $x \in V(G/vw)$, if contracting vw decreases the degree of some vertex x, then x is a common neighbour of v and w, and $\deg_{G/vw}(x) = \deg_G(x) - 1$. Thus $\deg_{G/vw}(x) \geq \frac{3}{2}k + |Z - \{v\}| - 2$. By induction, G/vw has a set S of at most $|Z - \{v\}|$ edges whose contraction gives a k-connected graph. Thus $S \cup \{vw\}$ is a set of at most $|Z|$ edges in G whose contraction gives a k-connected graph. □

Lemma 3.4. *Suppose that G is a graph and for some $Z \subset V(G)$ and for some vertex $v \in Z$ with $N_G(v) - Z \neq \emptyset$,*

- *every $(k-1)$-separation of G is Z-bad, and*
- *every vertex in $N_G(v) - Z$ has degree at least $\frac{3}{2}k + |Z| - 2$ in G.*

Then there is an edge vw with $w \in N_G(v) - Z$, such that

- *every $(k-1)$-separation of G/vw is $(Z - \{v\})$-bad.*

[3] Theorem 3.1 is a special case of Theorem 1 in [16] with $\mathfrak{S} = \{\{v, w\} : w \in N_G(v)\}$. Ref. [16] cites Ref. [15] for the proof of Theorem 1 in [16]. The proof of our Theorem 3.2 was obtained by following a treatment of Mader's work by Kriesell [13].

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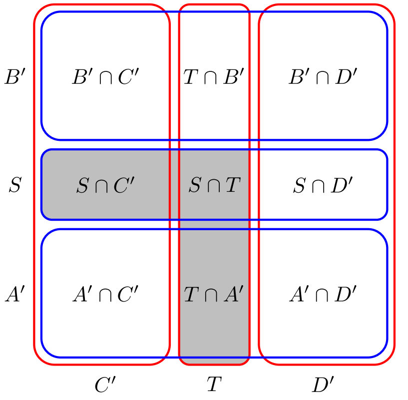

Fig. 1. Separator S and its fragments A' and B'. Separator T and its fragments C' and D'. The induced separator U is shaded.

Proof. Suppose on the contrary that for each $w \in N_G(v) - Z$, the contracted graph G/vw has a $(Z - \{v\})$-good $(k-1)$-separator. This separator must contain the vertex obtained by contracting vw. Thus G has a Z-good k-separator containing v and w. Let $\mathbb{S}$ be the set of Z-good k-separations $\{A, B\}$ of G such that $v \in A \cap B$ and $A \cap B \cap (N_G(v) - Z) \neq \emptyset$. We say $\{A, B\} \in \mathbb{S}$ *belongs* to x for each $x \in A \cap B \cap (N_G(v) - Z)$. As proved above, for each $w \in N_G(v) - Z$, some separation in $\mathbb{S}$ belongs to w.

For each separation $\{A, B\} \in \mathbb{S}$,

$$(A - B) \cap (N_G(v) - Z) \neq \emptyset \quad \text{and} \quad (B - A) \cap (N_G(v) - Z) \neq \emptyset; \tag{1}$$

otherwise $\{A - \{v\}, B\}$ or $\{A, B - \{v\}\}$ would be a Z-good $(k-1)$-separation of G.

Say $\{A, B\} \in \mathbb{S}$ belongs to $x \in N_G(v) - Z$, and $\{C, D\} \in \mathbb{S}$ belongs to $y \in (N_G(v) - Z) - \{x\}$. Let $S := A \cap B$ and $T := C \cap D$ be the corresponding separators in G. Let $A' := A - B$ and $B' := B - A$ and $C' := C - D$ and $D' := D - C$ be the corresponding fragments in G. Let $U := (S \cap C') \cup (S \cap T) \cup (T \cap A')$. Thus U separates $A' \cap C'$ and $B' \cup D'$, as illustrated in Fig. 1.

Suppose that $A' \cap C' \nsubseteq Z$. Since $\{A, B\}$ is Z-good, $B' \nsubseteq Z$. Since $B' \cup D' \nsubseteq Z$,

$$\mathcal{U} := \left\{(A' \cap C') \cup U, B' \cup D' \cup U\right\}$$

is a Z-good separation of G, whose separator is U. Thus $|U| \geq k$. That is, $|S \cap C'| + |S \cap T| + |T \cap A'| \geq k$. Now $|S \cap C'| + |S \cap T| = |S| - |S \cap D'| \leq k - |S \cap D'|$. Hence $k - |S \cap D'| + |T \cap A'| \geq k$, implying $|T \cap A'| \geq |S \cap D'|$. Similarly, $|S \cap C'| \geq |T \cap B'|$. By symmetry,

$$A' \cap C' \nsubseteq Z \Longrightarrow |T \cap A'| \geq |S \cap D'| \quad \text{and} \quad |S \cap C'| \geq |T \cap B'| \tag{2}$$

$$A' \cap D' \nsubseteq Z \Longrightarrow |T \cap A'| \geq |S \cap C'| \quad \text{and} \quad |S \cap D'| \geq |T \cap B'| \tag{3}$$

$$B' \cap C' \nsubseteq Z \Longrightarrow |T \cap B'| \geq |S \cap D'| \quad \text{and} \quad |S \cap C'| \geq |T \cap A'| \tag{4}$$

$$B' \cap D' \nsubseteq Z \Longrightarrow |T \cap B'| \geq |S \cap C'| \quad \text{and} \quad |S \cap D'| \geq |T \cap A'|. \tag{5}$$

Choose a separation $\{A, B\} \in \mathbb{S}$ that minimises $\min\{|A - B|, |B - A|\}$. Let x be a vertex in $N_G(v) - Z$ such that $\{A, B\}$ belongs to x. Define the separator S, and the fragments A' and B' as above. Without loss of generality, $|A'| \leq |B'|$. By (1), there is a vertex $y \in (N_G(v) - Z) \cap (A - B)$. Let $\{C, D\}$ be a separator in $\mathbb{S}$ that belongs to y. Define the separator T, and the fragments C' and D' as above.

Suppose that $A' \cap C' \nsubseteq Z$ and $B' \cap D' \nsubseteq Z$. By (2) and (5), $|T \cap A'| = |S \cap D'|$. Define U and $\mathcal{U}$ as above. Thus $\mathcal{U}$ is a Z-good separation of G, whose separator is U. Now $|U| = |S \cap C'| + |S \cap T| + |S \cap D'| = |S| \leq k$. Thus $\mathcal{U}$ is a Z-good k-separation. Observe that $v \in S \cap T \subseteq U$ and $y \in A' \cap T \subseteq U$. Thus $\mathcal{U} \in \mathbb{S}$ and $\mathcal{U}$ belongs to y. One fragment of $\mathcal{U}$ is $A' \cap C' \subseteq A' - \{y\}$ since $y \in T$. Thus $|A' \cap C'| < |A'|$, which contradicts the choice of $\{A, B\}$.

Thus $A' \cap C' \subseteq Z$ or $B' \cap D' \subseteq Z$. By symmetry, $A' \cap D' \subseteq Z$ or $B' \cap C' \subseteq Z$. It follows that $A' - Z \subseteq T$ or $B' - Z \subseteq T$ or $C' - Z \subseteq S$ or $D' - Z \subseteq S$. The choice of $\{A, B\}$ will not be used in the remainder. So without loss of generality, $A' - Z \subseteq T$.

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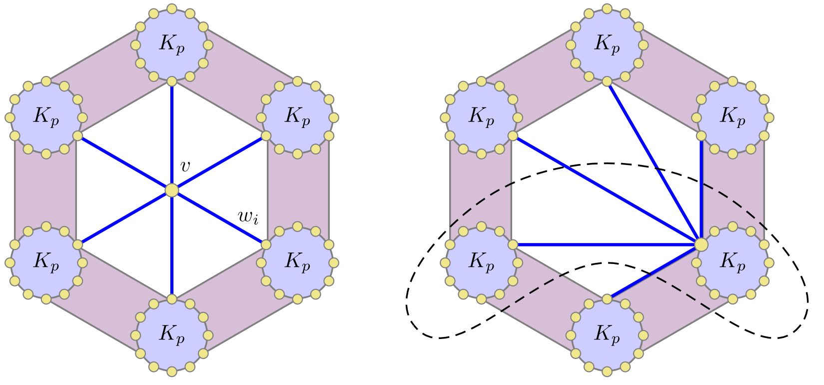

Fig. 2. Contracting vw_i produces a $(k-1)$-separation.

We claim that $A'-Z$ or $B'-Z$ or $C'-Z$ or $D'-Z$ has at most $\frac{1}{2}\max\{|S-T|,|T-S|\}$ vertices. If $B'-Z\subseteq T$, then $(A'-Z)\cup(B'-Z)\subseteq T-S$, implying that $A'-Z$ or $B'-Z$ has at most $\frac{1}{2}|T-S|$ vertices, as claimed. Now assume that $B'-Z\not\subseteq T$. Without loss of generality, $B'\cap C'\not\subseteq Z$. By (4), $|S\cap C'|\geq|T\cap A'|=|A'-Z|$. If $|A'-Z|\leq\frac{1}{2}|S-T|$ then the claim is proved. Otherwise, $|S\cap C'|\geq|A'-Z|>\frac{1}{2}|S-T|$. Thus $|S\cap D'|<\frac{1}{2}|S-T|$ (since $S-T$ is the disjoint union of $S\cap C'$ and $S\cap D'$). If $D'-Z\subseteq S$ then $|D'-Z|\leq|D'\cap S|<\frac{1}{2}|S-T|$. So assume that $D'-Z\not\subseteq S$. Thus $D'\cap B'\not\subseteq Z$. By (5), $|S\cap D'|\geq|T\cap A'|=|A'-Z|>\frac{1}{2}|S-T|$, which is a contradiction.

Hence $|Q-Z|\leq\frac{1}{2}\max\{|S-T|,|T-S|\}$ for some fragment $Q\in\{A',B',C',D'\}$. Now $\max\{|S-T|,|T-S|\}=\max\{|S|,|T|\}-|S\cap T|\leq k-1$ since $v\in S\cap T$. Thus $|Q|\leq\frac{1}{2}(k-1)+|Z|$. By (1), there is a vertex $w\in(N_G(v)-Z)\cap Q$. Then $N_G(w)\subseteq Q\cup S$ or $N_G(w)\subseteq Q\cup T$. Since $v\in S\cap T\cap Z$ and $|S-\{v\}|\leq k-1$ and $|T-\{v\}|\leq k-1$ and $w\in Q$, we have $\deg(w)\leq\frac{1}{2}(k-1)+|Z|+(k-1)-1=\frac{3k-5}{2}+|Z|$. This contradicts the assumption that each vertex in $N_G(v)-Z$ has degree at least $\frac{3}{2}k+|Z|-2$. □

We now show that the degree bound in Theorem 3.2 is best possible. The proof is an adaptation of a construction by Watkins [28] that shows that the degree bound in Theorem 3.1 is best possible. For odd $k\geq 5$ and $n\in[4,k-1]$, let $p:=\frac{1}{2}(k-1)$. Start with the lexicographic product $C_n\cdot K_p$, which consists of n disjoint copies $H_1,\dots,H_n$ of K_p, where every vertex in H_i is adjacent to every vertex in H_{i+1}, and H_j means $H_{j\bmod n}$. Let G be the graph obtained by adding a new vertex v adjacent to one vertex w_i in each H_i, as illustrated in Fig. 2. It is straightforward to verify that there are k internally disjoint paths in G between each pair of distinct vertices in $V(G)-\{v\}$. Thus $N_G(v)$ is the only minimal $(k-1)$-separator in G (since $\deg(v)=n\leq k-1$). For each neighbour w_i of v, observe that $\deg(w_i)=(p-1)+2p+1=\frac{3}{2}(k-1)$, but in G/vw_i the set $V(H_i)\cup V(H_{i+2})$ is a $2p$-separator, implying that G/vw_i is not k-connected. Thus the degree bound of $\frac{3}{2}k-1$ in Theorem 3.2 is best possible.

4. Final remarks

Seymour and Thomas conjectured the following strengthening of Theorem 1.3.

Conjecture 4.1 (*Seymour and Thomas*)**.** *For each $t\in\mathbb{Z}^+$ there exists $N_t\in\mathbb{Z}^+$ such that every t-connected graph G with at least N_t vertices and no K_t minor contains a set S of $t-5$ vertices such that $G-S$ is planar.*

Kawarabayashi et al. [9,10] proved this conjecture for $t\leq 6$. Recently, Norine and Thomas [18] proved it for $t\leq 8$. If true, Conjecture 4.1 can be used instead of Theorem 1.3 to make small improvements to Theorem 1.2.

Given that list colourings are a useful tool in attacking the Hadwiger Conjecture, it is interesting to ask what is the least function f such that every graph with no K_t-minor is $f(t)$-choosable. Since every

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graph with no K_t-minor has a vertex of degree at most $ct\sqrt{\log t}$, it follows that $f(t) \le ct\sqrt{\log t}$, and this is the best known bound. In particular, the following conjecture of Kawarabayashi and Mohar [8] is unsolved.

Weak List Hadwiger Conjecture. For some constant c, every graph with no K_t-minor is ct-choosable.

Kawarabayashi and Mohar [8] write that this conjecture may hold with $c = 1$, and that they believe that it holds with $c = \frac{3}{2}$. We dare to conjecture the following.

List Hadwiger Conjecture. Every graph with no K_t-minor is t-choosable.

This conjecture holds for $t \le 5$ [6,21]. The $t = 6$ case is open.

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