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# On tree-partition-width

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## ABSTRACT

A *tree-partition* of a graph  $G$  is a proper partition of its vertex set into ‘bags’, such that identifying the vertices in each bag produces a forest. The *width* of a tree-partition is the maximum number of vertices in a bag. The *tree-partition-width* of  $G$  is the minimum width of a tree-partition of  $G$ . An anonymous referee of the paper [Guoli Ding, Bogdan Oporowski, Some results on tree decomposition of graphs, *J. Graph Theory* 20 (4) (1995) 481–499] proved that every graph with tree-width  $k \geq 3$  and maximum degree  $\Delta \geq 1$  has tree-partition-width at most  $24k\Delta$ . We prove that this bound is within a constant factor of optimal. In particular, for all  $k \geq 3$  and for all sufficiently large  $\Delta$ , we construct a graph with tree-width  $k$ , maximum degree  $\Delta$ , and tree-partition-width at least  $(\frac{1}{8} - \epsilon)k\Delta$ . Moreover, we slightly improve the upper bound to  $\frac{5}{2}(k+1)(\frac{7}{2}\Delta - 1)$  without the restriction that  $k \geq 3$ .

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## 1. Introduction

A graph<sup>1</sup>  $H$  is a *partition* of a graph  $G$  if:

- each vertex of  $H$  is a set of vertices of  $G$  (called a *bag*),
- every vertex of  $G$  is in exactly one bag of  $H$ , and
- distinct bags  $A$  and  $B$  are adjacent in  $H$  if and only if there is an edge of  $G$  with one endpoint in  $A$  and the other endpoint in  $B$ .

The *width* of a partition is the maximum number of vertices in a bag. Informally speaking, the graph  $H$  is obtained from a proper partition of  $V(G)$  by identifying the vertices in each part, deleting loops, and replacing parallel edges by a single edge.  $H$  is sometimes called the *touching pattern* or *quotient graph* of the partition of  $V(G)$ .

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<sup>1</sup> All graphs considered are undirected, simple, and finite. Let  $V(G)$  and  $E(G)$  respectively be the vertex set and edge set of a graph  $G$ . Let  $\Delta(G)$  be the maximum degree of  $G$ .

If a forest  $T$  is a partition of a graph  $G$ , then  $T$  is a *tree-partition* of  $G$ . The *tree-partition-width*<sup>2</sup> of  $G$ , denoted by  $\text{tpw}(G)$ , is the minimum width of a tree-partition of  $G$ . Tree-partitions were independently introduced by Seese [2] and Halin [3], and have since been widely investigated [4,1, 5–8]. Applications of tree-partitions include graph drawing [9–13], graph colouring [14], partitioning graphs into subgraphs with only small components [15], monadic second-order logic [16], and network emulations [17–20]. Planar-partitions and other more general structures have also been studied [21,22,13].

What bounds can be proved on the tree-partition-width of a graph? Let  $\text{tw}(G)$  denote the tree-width<sup>3</sup> of a graph  $G$ . [2] proved the lower bound,

$$2 \text{tpw}(G) \geq \text{tw}(G) + 1.$$

In general, tree-partition-width is not bounded from above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width [1]. However, tree-partition-width is bounded for graphs of bounded tree-width *and* bounded degree [5,6]. The best known upper bound is due to an anonymous referee of the paper by Ding and Oporowski [5], who proved that

$$\text{tpw}(G) \leq 24 \text{tw}(G) \Delta(G)$$

whenever  $\text{tw}(G) \geq 3$  and  $\Delta(G) \geq 1$ . Using a similar proof, we make the following improvement to this bound without the restriction that  $\text{tw}(G) \geq 3$ .

**Theorem 1.** *Every graph  $G$  with tree-width  $\text{tw}(G) \geq 1$  and maximum degree  $\Delta(G) \geq 1$  has tree-partition-width*

$$\text{tpw}(G) < \frac{5}{2} (\text{tw}(G) + 1) \left( \frac{7}{2} \Delta(G) - 1 \right).$$

**Theorem 1** is proved in Section 2. Note that **Theorem 1** can be improved in the case of chordal graphs. In particular, a simple extension of a result by Dujmović et al. [11] implies that

$$\text{tpw}(G) \leq \text{tw}(G) (\Delta(G) - 1)$$

for every chordal graph  $G$  with  $\Delta(G) \geq 2$ ; see [8] for a simple proof. Nevertheless, the following theorem proves that  $\mathcal{O}(\text{tw}(G) \Delta(G))$  is the best possible upper bound, even for chordal graphs.

**Theorem 2.** *For every  $\epsilon > 0$  and integer  $k \geq 3$ , for every sufficiently large integer  $\Delta \geq \Delta(k, \epsilon)$ , for infinitely many values of  $N$ , there is a chordal graph  $G$  with  $N$  vertices, tree-width  $\text{tw}(G) \leq k$ , maximum degree  $\Delta(G) \leq \Delta$ , and tree-partition-width*

$$\text{tpw}(G) \geq \left( \frac{1}{8} - \epsilon \right) \text{tw}(G) \Delta(G).$$

**Theorem 2** is proved in Section 3. Note that **Theorem 2** is for  $k \geq 3$ . For  $k = 1$ , every tree is a tree-partition of itself with width 1. For  $k = 2$ , we prove that the upper bound  $\mathcal{O}(\Delta(G))$  is again best possible; see Section 4.

## 2. Upper bound

In this section we prove **Theorem 1**. The proof relies on the following separator lemma by Robertson and Seymour [25].

<sup>2</sup> Tree-partition-width has also been called *strong tree-width* [1,2].

<sup>3</sup> A graph is *chordal* if every induced cycle is a triangle. The *tree-width* of a graph  $G$  can be defined to be the minimum integer  $k$  such that  $G$  is a subgraph of a chordal graph with no clique on  $k + 2$  vertices. This parameter is particularly important in algorithmic and structural graph theory; see [23,24] for surveys.

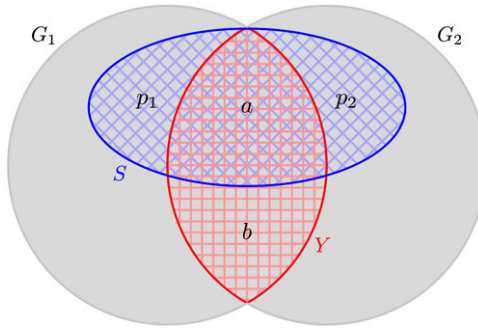


Fig. 1. Illustration of Case 4.

**Lemma 1** ([25]). For every graph  $G$  with tree-width at most  $k$ , for every set  $S \subseteq V(G)$ , there are edge-disjoint subgraphs  $G_1$  and  $G_2$  of  $G$  such that  $G_1 \cup G_2 = G$ ,  $|V(G_1) \cap V(G_2)| \leq k + 1$ , and  $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$  for each  $i \in \{1, 2\}$ .

Theorem 1 is a corollary of the following stronger result.

**Lemma 2.** Let  $\alpha := 1 + 1/\sqrt{2}$  and  $\gamma := 1 + \sqrt{2}$ . Let  $G$  be a graph with tree-width at most  $k \geq 1$  and maximum degree at most  $\Delta \geq 1$ . Then  $G$  has tree-partition-width

$$\text{tpw}(G) \leq \gamma(k + 1)(3\gamma\Delta - 1).$$

Moreover, for each set  $S \subseteq V(G)$  such that

$$(\gamma + 1)(k + 1) \leq |S| \leq 3(\gamma + 1)(k + 1)\Delta,$$

there is a tree-partition of  $G$  with width at most

$$\gamma(k + 1)(3\gamma\Delta - 1),$$

such that  $S$  is contained in a single bag containing at most  $\alpha|S| - \gamma(k + 1)$  vertices.

**Proof.** We proceed by induction on  $|V(G)|$ .

Case 1.  $|V(G)| < (\gamma + 1)(k + 1)$ : Then no set  $S$  is specified, and the tree-partition in which all the vertices are in a single bag satisfies the lemma. Now assume that  $|V(G)| \geq (\gamma + 1)(k + 1)$ , and without loss of generality,  $S$  is specified.

Case 2.  $|V(G) - S| < (\gamma + 1)(k + 1)$ : Then the tree-partition in which  $S$  is one bag and  $V(G) - S$  is another bag satisfies the lemma. Now assume that  $|V(G) - S| \geq (\gamma + 1)(k + 1)$ .

Case 3.  $|S| \leq 3(\gamma + 1)(k + 1)$ : Let  $N$  be the set of vertices in  $G$  that are adjacent to some vertex in  $S$  but are not in  $S$ . Then  $|N| \leq \Delta|S| \leq 3(\gamma + 1)(k + 1)\Delta$ . If  $|N| < (\gamma + 1)(k + 1)$  then add arbitrary vertices from  $V(G) - (S \cup N)$  to  $N$  until  $|N| \geq (\gamma + 1)(k + 1)$ . This is possible since  $|V(G) - S| \geq (\gamma + 1)(k + 1)$ .

By induction, there is a tree-partition of  $G - S$  with width at most  $\gamma(k + 1)(3\gamma\Delta - 1)$ , such that  $N$  is contained in a single bag. Create a new bag only containing  $S$ . Since all the neighbours of  $S$  are in a single bag, we obtain a tree-partition of  $G$ . ( $S$  corresponds to a leaf in the touching pattern.) Since  $|S| \geq (\gamma + 1)(k + 1)$ , it follows that  $|S| \leq \alpha|S| - \gamma(k + 1)$  as desired. Now  $|S| \leq 3(\gamma + 1)(k + 1) < \gamma(k + 1)(3\gamma\Delta - 1)$ . Since the other bags do not change we have the desired tree-partition of  $G$ .

Case 4.  $|S| \geq 3(\gamma + 1)(k + 1)$ : By Lemma 1, there are edge-disjoint subgraphs  $G_1$  and  $G_2$  of  $G$  such that  $G_1 \cup G_2 = G$ ,  $|V(G_1) \cap V(G_2)| \leq k + 1$ , and  $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$  for each  $i \in \{1, 2\}$ . Let  $Y := V(G_1) \cap V(G_2)$ . Let  $a := |S \cap Y|$  and  $b := |Y - S|$ . Thus  $a + b \leq k + 1$ . Let  $p_i := |(S \cap V(G_i)) - Y|$ . Then  $p_1 \leq 2p_2$  and  $p_2 \leq 2p_1$ . Let  $S_i := (S \cap V(G_i)) \cup Y$ . Note that  $|S_i| = p_i + a + b$  (see Fig. 1).

Now  $p_1 + p_2 + a = |S| \geq 3(\gamma + 1)(k + 1)$ . Thus  $3p_i + a \geq 3(\gamma + 1)(k + 1)$  and  $3p_i + 3a + 3b \geq 3(\gamma + 1)(k + 1)$ . That is,  $|S_i| \geq (\gamma + 1)(k + 1)$  for each  $i \in \{1, 2\}$ .

Now  $p_1 + p_2 + a \leq 3(\gamma + 1)(k + 1)\Delta$ . Thus  $\frac{3}{2}p_i + a \leq 3(\gamma + 1)(k + 1)\Delta$  and  $p_i \leq 2(\gamma + 1)(k + 1)\Delta$ . Thus  $p_i + a + b \leq 2(\gamma + 1)(k + 1)\Delta + (k + 1)$ . Hence  $|S_i| = p_i + a + b < 3(\gamma + 1)(k + 1)\Delta$ .

Thus we can apply induction to the set  $S_i$  in the graph  $G_i$  for each  $i \in \{1, 2\}$ . We obtain a tree-partition of  $G_i$  with width at most  $\gamma(k + 1)(3\gamma\Delta - 1)$ , such that  $S_i$  is contained in a single bag  $T_i$  containing at most  $\alpha|S_i| - \gamma(k + 1)$  vertices.

Construct a partition of  $G$  by uniting  $T_1$  and  $T_2$ . Each vertex of  $G$  is in exactly one bag since  $V(G_1) \cap V(G_2) = Y \subseteq S_i \subseteq T_i$ . Since  $G_1$  and  $G_2$  are edge-disjoint, the touching pattern of this partition of  $G$  is obtained by identifying one vertex of the touching pattern of the tree-partition of  $G_1$  with one vertex of the touching pattern of the tree-partition of  $G_2$ . Since the touching patterns of the tree-partitions of  $G_1$  and  $G_2$  are forests, the touching pattern of the partition of  $G$  is a forest, and we have a tree-partition of  $G$ .

Moreover,  $S$  is contained in a single bag  $T_1 \cup T_2$  and

$$\begin{aligned} |T_1 \cup T_2| &= |T_1| + |T_2| - |Y| \\ &\leq \alpha|S_1| - \gamma(k + 1) + \alpha|S_2| - \gamma(k + 1) - (a + b) \\ &= \alpha(p_1 + a + b) - \gamma(k + 1) + \alpha(p_2 + a + b) - \gamma(k + 1) - (a + b) \\ &= \alpha(p_1 + p_2 + a) - 2\gamma(k + 1) + (\alpha - 1)a + (2\alpha - 1)b \\ &\leq \alpha|S| - 2\gamma(k + 1) + (2\alpha - 1)(a + b) \\ &\leq \alpha|S| - 2\gamma(k + 1) + (2\alpha - 1)(k + 1) \\ &= \alpha|S| - \gamma(k + 1). \end{aligned}$$

Thus  $|T_1 \cup T_2| \leq \alpha \cdot 3(\gamma + 1)(k + 1)\Delta - \gamma(k + 1) = \gamma(k + 1)(3\gamma\Delta - 1)$ . Since the other bags do not change we have the desired tree-partition of  $G$ .  $\square$

### 3. General lower bound

The remainder of the paper studies lower bounds on the tree-partition-width. The graphs employed are chordal. We first show that tree-partitions of chordal graphs can be assumed to have certain useful properties.

**Lemma 3.** *Every chordal graph  $G$  has a tree-partition  $T$  with width  $\text{tpw}(G)$ , such that for every independent set  $S$  of simplicial<sup>4</sup> vertices of  $G$ , and for every bag  $B$  of  $T$ , either  $B = \{v\}$  for some vertex  $v \in S$ , or the induced subgraph  $G[B - S]$  is connected.*

**Proof.** Let  $T_0$  be a tree-partition of a chordal graph  $G$  with width  $\text{tpw}(G)$ . Let  $T$  be the partition of  $G$  obtained from  $T_0$  by replacing each bag  $B$  of  $T_0$  by bags corresponding to the connected components of  $G[B]$ . Add an edge between bags  $A$  and  $B$  of  $T$  if and only if there is an edge of  $G$  between  $A$  and  $B$ . Then  $T$  has width at most  $\text{tpw}(G)$ .

To prove that  $T$  is a forest, suppose on the contrary that  $T$  contains an induced cycle  $C$ . Since each bag in  $C$  induces a connected subgraph of  $G$ ,  $G$  contains an induced cycle  $D$  with at least one vertex from each bag in  $C$ . Since  $G$  is chordal,  $D$  is a triangle. Thus  $C$  is a triangle, implying that the vertices in  $D$  were in distinct bags in  $T_0$  (since the bags of  $T$  that replaced each bag of  $T_0$  form an independent set). Hence the bags of  $T_0$  that contain  $D$  induce a triangle in  $T_0$ , which is the desired contradiction since  $T_0$  is a forest. Hence  $T$  is a forest.

Let  $S$  be an independent set of simplicial vertices of  $G$ . Consider a bag  $B$  of  $T$ . By construction,  $G[B]$  is connected. First suppose that  $B \subseteq S$ . Since  $S$  is an independent set and  $G[B]$  is connected,  $B = \{v\}$  for some vertex  $v \in S$ .

Now assume that  $B - S \neq \emptyset$ . Suppose on the contrary that  $G[B - S]$  is disconnected. Thus  $B \cap S$  is a cut-set in  $G[B]$ . Let  $v$  and  $w$  be vertices in distinct components of  $G[B - S]$  such that the distance between  $v$  and  $w$  in  $G[B]$  is minimised. (This is well-defined since  $G[B]$  is connected.) Since  $S$  is an

<sup>4</sup> A vertex is *simplicial* if its neighbourhood is a clique.



Consider a bag  $B$  of  $U$ . Let  $\ell(B)$  be the minimum integer such that some vertex in  $B$  is in the  $\ell(B)$ -column, and let  $r(B)$  be the maximum integer such that some vertex in  $B$  is in the  $r(B)$ -column. Since  $H[B]$  is connected, there is a path in  $B$  from the  $\ell(B)$ -column to the  $r(B)$ -column. By the definition of  $H$ , for each  $x \in [\ell(B), r(B)]$ , the  $x$ -column contains a vertex in  $B$ . Let  $I(B)$  be the closed real interval from  $\ell(B) - \frac{1}{2}$  to  $r(B) + \frac{1}{2}$ . Observe that two bags  $B$  and  $C$  of  $U$  are adjacent if and only if  $I(B) \cap I(C) \neq \emptyset$ . Thus  $\{I(B) : B \text{ is a bag of } U\}$  is an interval representation of the tree  $U$ . Every tree that is an interval graph is a caterpillar<sup>5</sup>; see [26] for example. Thus  $U$  is a caterpillar.

Let  $\preceq$  be the relation on the set of non-leaf bags of  $U$  defined by  $A \preceq B$  if and only if  $\ell(A) \leq \ell(B)$  and  $r(A) \leq r(B)$ . We claim that  $\preceq$  is a total order. It is immediate that  $\preceq$  is reflexive and transitive. To prove that  $\preceq$  is antisymmetric, suppose on the contrary that  $A \preceq B$  and  $B \preceq A$  for distinct non-leaf bags  $A$  and  $B$ . Thus  $\ell(A) = \ell(B)$  and  $r(A) = r(B)$ . Since  $U$  has at least three bags, there is a third bag  $C$  that contains a vertex in the  $(\ell(A) - 1)$ -column or in the  $(r(A) + 1)$ -column. Thus  $\{A, B, C\}$  induce a triangle in  $U$ , which is the desired contradiction. Hence  $\preceq$  is antisymmetric. To prove that  $\preceq$  is total, suppose on the contrary that  $A \not\preceq B$  and  $B \not\preceq A$  for distinct non-leaf bags  $A$  and  $B$ . Now  $A \not\preceq B$  implies that  $\ell(A) > \ell(B)$  or  $r(A) > r(B)$ . Without loss of generality,  $\ell(A) > \ell(B)$ . Thus  $B \not\preceq A$  implies that  $r(B) > r(A)$ . Hence the interval  $[\ell(A), r(A)]$  is strictly within the interval  $[\ell(B), r(B)]$  at both ends. For each  $x \in [\ell(A), r(A)]$ , every vertex in the  $x$ -column is in  $A \cup B$ , as otherwise  $U$  would contain a triangle (since each column is a clique in  $H$ ). Moreover, every vertex in the  $(\ell(A) - 1)$ -column or in the  $(r(A) + 1)$ -column is in  $B$ , as otherwise  $U$  would contain a triangle (since the union of consecutive columns is a clique in  $H$ ). Thus every neighbour of every vertex in  $A$  is in  $B$ . That is,  $A$  is a leaf in  $U$ . This contradiction proves that  $\preceq$  is a total order on the set of non-leaf bags of  $U$ .

Suppose that  $U$  has a 4-vertex path  $(A, B, C, D)$  as a subgraph.

Thus  $B$  and  $C$  are non-leaf bags. Without loss of generality,  $B \prec C$ . If every column contains vertices in both  $B$  and  $C$ , then  $B$  and  $C$  and any other bag would induce a triangle in  $U$  (since each column induces a clique in  $H$ ). Thus some column contains a vertex in  $B$  but no vertex in  $C$ , and some column contains a vertex in  $C$  but no vertex in  $B$ . Let  $p$  be the maximum integer such that some vertex in  $B$  is in the  $p$ -column, but no vertex in  $C$  is in the  $p$ -column. Let  $q$  be the minimum integer such that some vertex in  $C$  is in the  $q$ -column, but no vertex in  $B$  is in the  $q$ -column. Now  $p < q$  since  $B \prec C$ .

We claim that the  $(p + 1)$ -column contains a vertex in  $C$ . If not, then the  $(p + 1)$ -column contains no vertex in  $B$  by the definition of  $p$ . Thus  $r(B) = p$  since  $H[B]$  is connected. Since  $B$  is adjacent to  $C$  in  $U$ ,  $\ell(C) \leq r(B) + 1 = p + 1$ . In particular, the  $(p + 1)$ -column contains a vertex in  $C$ . Since  $H[C]$  is connected, for  $x \in [p + 1, q]$ , each  $x$ -column contains a vertex in  $C$ . In fact,  $\ell(C) = p + 1$  since the  $p$ -column contains no vertex in  $C$ . By symmetry, for  $x \in [p, q - 1]$ , each  $x$ -column contains a vertex in  $B$ , and  $r(C) = q - 1$ .

The union of the  $p$ -column and the  $(p + 1)$ -column only contains vertices in  $B \cup C$ , as otherwise  $U$  would contain a triangle (since the union of two consecutive columns is a clique in  $H$ ). By the definition of  $p$ , no vertex in the  $p$ -column is in  $C$ . Thus every vertex in the  $p$ -column is in  $B$ . By symmetry, every vertex in the  $q$ -column is in  $C$ . Now for each  $y \in [k]$ , the vertices  $(p, y), (p + 1, y), \dots, (q, y)$  are all in  $B \cup C$ , the first vertex  $(p, y)$  is in  $B$ , and the last vertex  $(q, y)$  is in  $C$ . Thus  $(x, y) \in B$  and  $(x + 1, y) \in C$  for some  $x \in [p, q - 1]$ . That is, in every row of  $H$  there is a horizontal edge with one endpoint in  $B$  and the other in  $C$ .

Thus there are at least  $k$  horizontal edges with one endpoint in  $B$  and the other in  $C$  (now considered to be bags of  $T$ ). For each such horizontal edge  $vw$ , each vertex of  $G - H$  adjacent to  $v$  and  $w$  is in  $B \cup C$ , as otherwise  $T$  would contain a triangle. There are  $\lceil \frac{1}{2}(\Delta - 3k) \rceil$  such vertices of  $G - H$  for each of the  $k$  horizontal edges between  $B$  and  $C$ . Thus  $|B \cup C| \geq \frac{1}{2}k(\Delta - 3k)$ . Thus one of  $B$  and  $C$  has at least  $\frac{1}{4}k(\Delta - 3k)$  vertices. Hence  $\text{tpw}(G) \geq \frac{1}{4}k(\Delta - 3k)$  as desired.

Now assume that  $U$  has no 4-vertex path as a subgraph.

A tree is a star if and only if it has no 4-vertex path as a subgraph. Hence  $U$  is a star. Let  $R$  be the root bag of  $U$ . If  $R$  contains a vertex in every column then  $|R| \geq n$ , implying  $\text{tpw}(G) \geq n \geq \frac{1}{4}k(\Delta - 3k)$ , as desired. Now assume that for some  $x \in [n]$ , the  $x$ -column of  $H$  contains no vertex in  $R$ . Let  $B$  be a bag

<sup>5</sup> A caterpillar is a tree such that deleting the leaves gives a path.

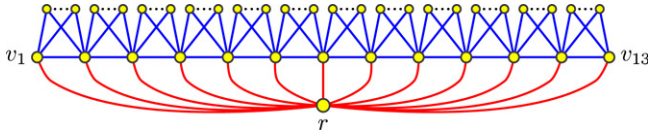


Fig. 3. Illustration for Theorem 3 with  $\Delta = 13$ .

containing some vertex in the  $x$ -column. The  $x$ -column induces a clique in  $H$ , the only bag in  $U$  that is adjacent to  $B$  is  $R$ , and  $R$  contains no vertex in the  $x$ -column. Thus every vertex in the  $x$ -column is in  $B$ . Since  $R$  is the only bag in  $U$  adjacent to  $B$ , there are at least  $k$  horizontal edges with one endpoint in  $B$  and the other endpoint in  $R$ . As in the case when  $U$  contained a 4-vertex path, we conclude that  $\text{tpw}(G) \geq \frac{1}{4}k(\Delta - 3k)$  as desired.  $\square$

**Proof of Theorem 2.** Let  $\ell := \lceil \frac{k}{2} \rceil$ . Thus  $\ell \geq 2$ . By Lemma 4, for each integer  $\Delta \geq \Delta(k, \epsilon) := \max\{3\ell + 1, \frac{3\ell}{8\epsilon}\}$ , there are infinitely many values of  $N$  for which there is a chordal graph  $G$  with  $N$  vertices, tree-width  $\text{tw}(G) = 2\ell - 1 \leq k$ , maximum degree  $\Delta(G) \leq \Delta$ , and tree-partition-width  $\text{tpw}(G) > \frac{1}{4}\ell(\Delta - 3\ell)$ , which is at least  $(\frac{1}{8} - \epsilon)k\Delta$  since  $\Delta \geq \frac{3\ell}{8\epsilon}$ .  $\square$

A *domino tree decomposition*<sup>6</sup> is a tree decomposition in which each vertex appears in at most two bags. The *domino tree-width* of a graph  $G$ , denoted by  $\text{dtw}(G)$ , is the minimum width of a domino tree decomposition of  $G$ . Domino tree-width behaves like tree-partition-width in the sense that  $\text{dtw}(G) \geq \text{tw}(G)$ , and  $\text{dtw}(G)$  is bounded for graphs of bounded tree-width and bounded degree [1]. The best upper bound is

$$\text{dtw}(G) \leq (9 \text{tw}(G) + 7) \Delta(G) (\Delta(G) + 1) - 1,$$

which is due to Bodlaender [4], who also constructed a graph  $G$  with

$$\text{dtw}(G) \geq \frac{1}{12} \text{tw}(G) \Delta(G) - 2.$$

Tree-partition-width and domino tree-width are related in that every graph  $G$  satisfies

$$\text{dtw}(G) \geq \text{tpw}(G) - 1,$$

as observed by Bodlaender and Engelfriet [1]. Thus Theorem 2 provides examples of graphs  $G$  with

$$\text{dtw}(G) \geq (\frac{1}{8} - \epsilon) \text{tw}(G) \Delta(G).$$

This represents a small constant-factor improvement over the above lower bound by Bodlaender [4].

4. Lower bound for tree-width 2

We now prove a lower bound on the tree-partition-width of graphs with tree-width 2.

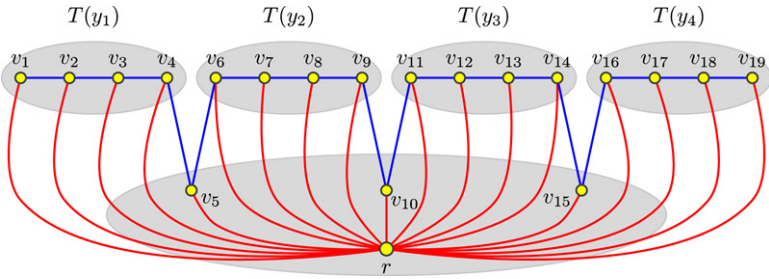
**Theorem 3.** For all odd  $\Delta \geq 11$  there is a chordal graph  $G$  with tree-width 2, maximum degree  $\Delta$ , and tree-partition-width  $\text{tpw}(G) \geq \frac{2}{3}(\Delta - 1)$ .

**Proof.** As illustrated in Fig. 3, let  $G$  be the graph with

$$\begin{aligned} V(G) &:= \{r\} \cup \{v_i : i \in [\Delta]\} \cup \{w_{i,\ell} : i \in [\Delta - 1], \ell \in [\frac{1}{2}(\Delta - 3)]\}, \quad \text{and} \\ E(G) &:= \{rv_i : i \in [\Delta]\} \cup \{v_i v_{i+1} : i \in [\Delta - 1]\} \\ &\quad \cup \{v_i w_{i,\ell}, v_{i+1} w_{i,\ell} : i \in [\Delta - 1], \ell \in [\frac{1}{2}(\Delta - 3)]\}. \end{aligned}$$

Observe that  $G$  has maximum degree  $\Delta$ . Clearly every induced cycle of  $G$  is a triangle. Thus  $G$  is chordal. Observe that  $G$  has no 4-vertex clique. Thus  $G$  has tree-width 2.

<sup>6</sup> See [27] for an introduction to tree decompositions.



**Fig. 4.** Illustration for [Theorem 3](#) with  $\Delta = 19$  and  $d = 4$ .

Let  $T$  be the tree-partition of  $G$  from [Lemma 3](#). Then  $T$  has width  $\text{tpw}(G)$ , and every bag induces a connected subgraph of  $G$ . Let  $R$  be the bag containing  $r$ . Let  $B_1, \dots, B_d$  be the bags, not including  $R$ , that contain some vertex  $v_i$ . Thus  $R$  is adjacent to each  $B_j$  (since  $r$  is adjacent to each  $v_i$ ). Since  $\{w_{i,\ell} : i \in [\Delta - 1], \ell \in [\frac{1}{2}(\Delta - 3)]\}$  is an independent set of simplicial vertices, by [Lemma 3](#), for each  $j \in [d]$ , the vertices  $\{v_1, v_2, \dots, v_\Delta\} \cap B_j$  induce a (connected) subpath of  $G$ .

First suppose that  $d = 0$ . Then the  $\Delta + 1$  vertices  $\{r, v_1, \dots, v_\Delta\}$  are contained in one bag  $R$ . Thus  $\text{tpw}(G) \geq \Delta + 1 \geq \frac{2}{3}(\Delta - 1)$ .

Now suppose that  $d = 1$ . Thus  $\{r, v_1, \dots, v_\Delta\} \subseteq R \cup B_1$ . In addition, at least one edge  $v_i v_{i+1}$  has one endpoint in  $R$  and the other endpoint in  $B_1$ . Thus  $w_{i,\ell} \in R \cup B_1$  for each  $\ell \in [\frac{1}{2}(\Delta - 3)]$ . Hence  $1 + \Delta + \frac{1}{2}(\Delta - 3)$  vertices are contained in two bags. Thus one bag contains at least  $\frac{1}{4}(3\Delta - 1)$  vertices, and  $\text{tpw}(G) \geq \frac{1}{4}(3\Delta - 1) \geq \frac{2}{3}(\Delta - 1)$ .

Finally suppose that  $d \geq 2$ . Since  $\{v_1, v_2, \dots, v_\Delta\} \cap B_j$  induce a subpath in each bag  $B_j$ , we can assume that  $\{v_1, v_2, \dots, v_\Delta\} \cap B_j = \{v_i : i \in [f(j), g(j)]\}$ , where

$$1 \leq f(1) \leq g(1) < f(2) \leq g(2) < \dots < f(d) \leq g(d) \leq \Delta.$$

Distinct  $B_j$  bags are not adjacent (since  $T$  is a tree). Thus  $v_{f(j)-1} \in R$  for each  $j \in [2, d]$ . Similarly,  $v_{g(j)+1} \in R$  for each  $j \in [d - 1]$ . Thus  $w_{f(j)-1,\ell} \in R \cup B_j$  for each  $j \in [2, d]$  and  $\ell \in [\frac{1}{2}(\Delta - 3)]$ . Similarly,  $w_{g(j),\ell} \in R \cup B_j$  for each  $j \in [d - 1]$  and  $\ell \in [\frac{1}{2}(\Delta - 3)]$  (see [Fig. 4](#)).

Hence the bags  $R, B_1, \dots, B_d$  contain at least

$$1 + \Delta + 2(d - 1) \cdot \frac{1}{2}(\Delta - 3)$$

vertices. Therefore one of these bags has at least

$$(1 + \Delta + (d - 1)(\Delta - 3))/(d + 1)$$

vertices, which is at least  $\frac{2}{3}(\Delta - 1)$ . Hence  $\text{tpw}(G) \geq \frac{2}{3}(\Delta - 1)$ .  $\square$

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