Queue Layouts, Tree-Width, and Three-Dimensional Graph Drawing*

David R. Wood

School of Computer Science
Carleton University
Ottawa, Canada
davidw@scs.carleton.ca

Abstract. A *three-dimensional* (*straight-line grid*) *drawing* of a graph represents the vertices by points in $\mathbb{Z}^3$ and the edges by non-crossing line segments. This research is motivated by the following open problem due to Felsner, Liotta, and Wismath [Graph Drawing '01, *Lecture Notes in Comput. Sci.*, 2002]: *does every n-vertex planar graph have a three-dimensional drawing with $O(n)$ volume*? We prove that this question is almost equivalent to an existing one-dimensional graph layout problem. A *queue layout* consists of a linear order σ of the vertices of a graph, and a partition of the edges into queues, such that no two edges in the same queue are nested with respect to σ. The minimum number of queues in a queue layout of a graph is its *queue-number*. Let G be an n-vertex member of a proper minor-closed family of graphs (such as a planar graph). We prove that G has a $O(1) \times O(1) \times O(n)$ drawing if and only if G has $O(1)$ queue-number. Thus the above question is almost equivalent to an open problem of Heath, Leighton, and Rosenberg [*SIAM J. Discrete Math.*, 1992], who ask whether every planar graph has $O(1)$ queue-number? We also present partial solutions to an open problem of Ganley and Heath [*Discrete Appl. Math.*, 2001], who ask whether graphs of bounded tree-width have bounded queue-number? We prove that graphs with bounded path-width, or both bounded tree-width and bounded maximum degree, have bounded queue-number. As a corollary we obtain three-dimensional drawings with optimal $O(n)$ volume, for series-parallel graphs, and graphs with both bounded tree-width and bounded maximum degree.

1 Introduction

A celebrated result independently due to de Fraysseix, Pach, and Pollack [6] and Schnyder [27] states that every n-vertex planar graph has a (two-dimensional) straight-line grid drawing with $O(n^2)$ area. Motivated by applications in information visualisation, VLSI circuit design and software engineering, there is a growing body of research in three-dimensional graph drawing (see [12] for example). One might expect that in three dimensions, planar graphs would admit straight-line grid drawings with $o(n^2)$ volume. However, this question has remained an

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elusive open problem. The main contribution of this paper is to prove that this question of three-dimensional graph drawing is almost equivalent to an existing one-dimensional graph layout problem regarding queue layouts. Furthermore, we establish new relationships between queue-number, tree-width and path-width; and obtain $O(n)$ volume three-dimensional drawings of series-parallel graphs, and graphs with both bounded tree-width and bounded degree.

1.1 Definitions and Notation

Throughout this paper, all graphs G are undirected, simple, connected, and finite with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and maximum degree of G are respectively denoted by $n = |V(G)|$ and $\Delta(G)$. For all disjoint subsets $A, B \subseteq V(G)$, the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$ is denoted by $G[A, B]$.

A *tree-decomposition* of a graph G consists of a tree T and a collection $\{T_x : x \in V(T)\}$ of subsets T_x (called *bags*) of $V(G)$ indexed by the nodes of T such that:

- $\displaystyle\bigcup_{x \in V(T)} T_x = V(G)$,
- $\forall$ edges $vw \in E(G)$, $\exists$ node $x \in V(T)$ such that $\{v, w\} \subseteq T_x$, and
- $\forall$ nodes $x, y, z \in V(T)$, if y is on the xz-path in T, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree-decomposition is one less than the maximum size of a bag. A *path-decomposition* is a tree-decomposition where the tree T is a path. The *path-width* (respectively, *tree-width*) of a graph G, denoted by $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$), is the minimum width of a path- (tree-) decomposition of G.

1.2 Three-Dimensional Straight-Line Grid Drawing

A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *three-dimensional drawing*, represents the vertices by distinct points in $\mathbb{Z}^3$, and represents each edge as a line-segment between its end-vertices, such that edges only intersect at common end-vertices. In contrast to the case in the plane, it is well known that every graph has a three-dimensional drawing. We therefore are interested in optimising certain measures of the aesthetic quality of a drawing. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. We study three-dimensional drawings with small volume.

Cohen, Eades, Lin, and Ruskey [5] proved that every graph has a three-dimensional drawing with $O(n^3)$ volume, and this bound is asymptotically tight for the complete graph K_n. Calamoneri and Sterbini [4] proved that every 4-colourable graph has a three-dimensional drawing with $O(n^2)$ volume. Generalising this result, Pach, Thiele, and Tóth [23] proved that every k-colourable graph, for fixed $k \geq 2$, has a three-dimensional drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. The first linear volume bound was established by Felsner, Wismath, and Liotta [14], who proved that every outerplanar graph has a drawing with $O(n)$ volume. Poranen [25] proved that series-parallel digraphs have upward three-dimensional drawings with $O(n^3)$ volume, and that this bound can be improved to $O(n^2)$ and $O(n)$ in certain special cases. di Giacomo, Liotta, and Wismath [7] proved that series-parallel graphs with maximum degree three have three-dimensional drawings with $O(n)$ volume. Dujmović, Morin, and Wood [12] proved that every graph G has a three-dimensional drawing with $O(n \cdot \mathsf{pw}(G)^2)$ volume. This implies $O(n \log^2 n)$ volume drawings for graphs of bounded tree-width, such as series-parallel graphs.

Since a planar graph G is 4-colourable and has $\mathsf{pw}(G) \in O(\sqrt{n})$, by the above results of Calamoneri and Sterbini [4], Pach *et al.* [23], and Dujmović *et al.* [12], every planar graph has a three-dimensional drawing with $O(n^2)$ volume. This result also follows from the classical algorithms of de Fraysseix *et al.* [6] and Schnyder [27] for producing plane grid drawings. This paper is motivated by the following open problem due to Felsner *et al.* [14].

Open Problem 1 ([14]). Does every planar graph have a three-dimensional drawing with $O(n)$ volume? In fact, any $o(n^2)$ bound would be of interest.

In this paper we prove that Open Problem 1 is almost equivalent to an existing open problem in the theory of queue layouts.

1.3 Queue Layouts

For a graph G, a linear order of $V(G)$ is called a *vertex-ordering* of G. A *queue layout* of G consists of a vertex-ordering σ of G, and a partition of $E(G)$ into *queues*, such that no two edges in the same queue are *nested* with respect to σ. That is, there are no edges vw and xy in a single queue with $v <_\sigma x <_\sigma y <_\sigma w$. The minimum number of queues in a queue layout of G is called the *queue-number* of G, and is denoted by $\mathsf{qn}(G)$. A similar concept is that of a *stack layout* (or *book embedding*), which consists of a vertex-ordering of G, and a partition of $E(G)$ into *stacks* (or *pages*) such that there are no edges vw and xy in a single stack with $v <_\sigma x <_\sigma w <_\sigma y$. The minimum number of stacks in a stack layout of G is called the *stack-number* (or *page-number*) of G, and is denoted by $\mathsf{sn}(G)$.

Motivated by applications in VLSI layout, fault-tolerant processing, parallel processing, matrix computations, and sorting networks, queue layouts have been extensively studied [19, 20, 24, 26, 29]. Heath and Rosenberg [20] characterised graphs admitting 1-queue layouts as the 'arched leveled planar' graphs, and proved that it is NP-complete to recognise such graphs. This result is in contrast to the situation for stack layouts — the graphs admitting 1-stack layouts are precisely the outerplanar graphs, which can be recognised in polynomial time. On the other hand, it is NP-hard to minimise the number of stacks in a stack layout which respects a given vertex-ordering [17]. However the analogous problem for queue layouts can be solved as follows. As illustrated in Fig. 1, a *k-rainbow*

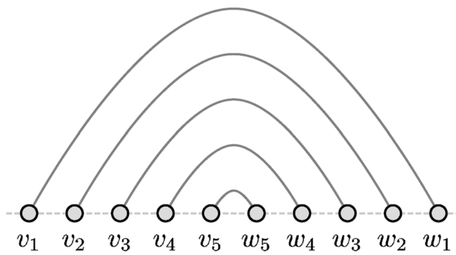

Fig. 1. A rainbow of five edges in a vertex-ordering.

in a vertex-ordering σ consists of a matching $\{v_i w_i : 1 \leq i \leq k\}$ such that $v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1$.

A vertex-ordering containing a k-rainbow needs at least k queues. A straightforward application of Dilworth's Theorem [9] proves the converse. That is, a fixed vertex-ordering admits a k-queue layout where k is the size of the largest rainbow. (Heath and Rosenberg [20] describe an $O(m \log\log n)$ time algorithm to compute the queue assignment.) Thus determining $\mathsf{qn}(G)$ can be viewed as the following vertex layout problem.

Lemma 1 ([20]). *The queue-number* $\mathsf{qn}(G)$ *of a graph* G *is the minimum, taken over all vertex-orderings* σ *of* G*, of the maximum size of a rainbow in* σ*.*

The relationship between tree-width and stack and queue layouts has previously been studied in [16, 26]. Rengarajan and Veni Madhavan [26] prove that a graph of tree-width at most two (that is, a graph with series-parallel biconnected components [2]) has a 2-stack layout and a 3-queue layout. In the special case of an outerplanar graph a 2-queue layout is constructed. More generally, Ganley and Heath [16] prove that the stack-number $\mathsf{sn}(G) \leq \mathsf{tw}(G)+1$, and ask whether a similar relationship holds for the queue-number.

Open Problem 2 ([16]). Does every graph of bounded tree-width have bounded queue-number?

2 Our Results

This paper contributes the following two theorems. The first, proved in Section 3, provides a partial answer to Open Problem 2.

Theorem 1. *The following classes of graphs have bounded queue-number:*
(1) graphs of bounded path-width, and
(2) graphs of bounded tree-width and bounded maximum degree.
In particular, $\mathsf{qn}(G) \leq \mathsf{pw}(G)$ *and* $\mathsf{qn}(G) \leq 36\,\mathsf{tw}(G)\Delta(G)$ *for every graph* G*.*

A similar upper bound to (1) is obtained by Heath and Rosenberg [20], who show that every graph G has $\mathsf{qn}(G) \leq \lceil \frac{1}{2}\mathsf{bw}(G)\rceil$, where $\mathsf{bw}(G)$ is the bandwidth of G. In many cases this result is weaker than (1) since $\mathsf{pw}(G) \leq \mathsf{bw}(G)$ (see

[8]). Note that since $\mathsf{pw}(G) \in O(\mathsf{tw}(G) \cdot \log n)$ [2], the queue-number $\mathsf{qn}(G) \in O(\mathsf{tw}(G) \cdot \log n)$.

Theorem 2 below relates the volume of a three-dimensional drawing of a graph to its queue-number, and is proved in Section 4. While our motivation is for three-dimensional drawings of planar graphs, the theorem applies to any *proper minor-closed* family of graphs; that is, a graph family which is not the class of all graphs, and is closed under edge-contraction, edge-deletion, and deleting isolated vertices.

Theorem 2. *Let $\mathcal{G}$ be a proper minor-closed family of graphs, and let $F(n)$ be a set of functions closed under taking polynomials (for example, $O(1)$ or $O(\text{polylog}\, n)$). For every graph $G \in \mathcal{G}$, G has a $F(n) \times F(n) \times O(n)$ drawing if and only if G has queue number $\mathsf{qn}(G) \in F(n)$.*

Graphs with constant queue-number include de Bruijn graphs, FFT and Beneš network graphs [20]. By the above-mentioned result of Rengarajan and Veni Madhavan [26], and since graphs with tree-width at most some constant form a proper minor-closed family, Theorems 1 and 2 together imply the following. Part (2) is proved without using queue layouts in [12].

Corollary 1. *The following graphs have three-dimensional drawings with $O(n)$ volume:*
(1) de Bruijn graphs, FFT and Beneš network graphs,
(2) graphs of bounded path-width [12]*,*
(3) graphs of tree-width at most two (series-parallel graphs), and
(4) graphs of bounded tree-width and bounded maximum degree.

Corollary 1 improves and/or generalises the above-mentioned results for three-dimensional drawings of outerplanar graphs, series-parallel graphs, and graphs of bounded tree-width in [7, 12, 14, 25]. Note that the algorithm by Felsner *et al.* [14] closely parallels the construction of 2-queue layouts of outerplanar graphs due to Rengarajan and Veni Madhavan [26], both of which are based on breadth-first search, as is one of our proofs to follows.

3 Queue Layouts and Tree-Width

In this section we prove Theorem 1. Consider a vertex-ordering σ of a graph G. The *vertex cut* in σ at a vertex $v \in V(G)$ is defined to be $|\{x \in V(G) : \exists xy \in E(G), x \leq_\sigma v <_\sigma y\}|$. The *vertex separation number* of G is the minimum, taken over all vertex-orderings σ of G, of a maximum vertex cut in σ. A k-rainbow in σ implies σ has a vertex cut of size k. Thus the queue-number of a graph is at most its vertex separation number by Lemma 1. The next result immediately follows, since the vertex separation number of a graph equals its path-width (see [8]).

Lemma 2. *Graphs of bounded path-width have bounded queue-number. In particular, $\mathsf{qn}(G) \leq \mathsf{pw}(G)$ for every graph G.*

To establish our next result we employ a structure called a tree-partition [3, 10, 11, 18, 28]. Let G be a graph, let T be a tree, and let $\{T_x : x \in V(T)\}$ be a partition of $V(G)$ into sets (called *bags*) indexed by the nodes of T. We denote the bag containing a vertex $v \in V(G)$ by $T_{\alpha(v)}$. The pair $(T, \{T_x\})$ is a *tree-partition* of G if for every edge $vw \in E(G)$, either $\alpha(v) = \alpha(w)$ or $\alpha(v)\alpha(w) \in E(T)$. We call vw an *intra-bag* edge if $\alpha(v) = \alpha(w)$ and an *inter-bag* edge otherwise. The *width* of the tree-partition is the maximum size of a bag T_x. The *tree-partition-width* of a graph G, denoted by $\mathsf{tpw}(G)$, is the minimum width of a tree-partition of G. Note that tree-partition-width has also been called *strong tree-width* [3, 28].

Lemma 3. *Graphs of bounded tree-partition-width, which includes graphs of bounded tree-width and bounded maximum degree, have bounded queue-number. In particular,* $\mathsf{qn}(G) \leq \frac{3}{2}\mathsf{tpw}(G) \leq 36\,\mathsf{tw}(G)\Delta(G)$ *for every graph* G.

Proof. Let $(T, \{T_x\})$ be a tree-partition of G with width $\mathsf{tpw}(G)$. Let π be a vertex-ordering of T determined by a lexicographical breadth-first-search of T starting from an arbitrary root node. Then no two edges of T are nested in π. (This is why trees have queue-number one.) Also observe that each node $x \in V(T)$ has at most one incident edge xy with $y <_\pi x$.

Let σ be a vertex-ordering of G such that $v <_\sigma w$ implies $\alpha(v) \leq_\pi \alpha(w)$. Suppose edges e_1 and e_2 of G are nested in σ. If e_1 and e_2 are both intra-bag edges then their end-vertices are all in a common bag. Thus there are at most $\frac{1}{2}\mathsf{tpw}(G)$ intra-bag edges in a rainbow of σ. If e_1 and e_2 are both inter-bag edges then the left end-vertex of e_1 and the left end-vertex of e_2 are in a common bag. Thus there are at most $\mathsf{tpw}(G)$ inter-bag edges in a rainbow of σ. Therefore a rainbow in σ can have at most $\frac{3}{2}\mathsf{tpw}(G)$ edges.

The result follows from Lemma 1, and since Ding and Oporowski [10] proved that $\mathsf{tpw}(G) \leq 24\,\mathsf{tw}(G)\Delta(G)$ for every graph G. □

Lemmata 2 and 3 establish Theorem 1.

4 Queue Layouts and Three-Dimensional Drawings

In this section we prove Theorem 2. Our proof depends on the following structure introduced by Dujmović *et al.* [12]. An *ordered k-layering* of a graph G consists of a partition $V_1, V_2, \dots, V_k$ of $V(G)$ into *layers*, and a total order $<_i$ of each V_i, such that for every edge vw, if $v <_i w$ then there is no vertex x with $v <_i x <_i w$. The *span* of an edge vw is $|i - j|$ where $v \in V_i$ and $w \in V_j$. An *intra-layer* edge is an edge with zero span. An *X-crossing* consists of two edges vw and xy such that for distinct layers i and j, $v <_i x$ and $y <_j w$. Dujmović *et al.* [12] proved the following (see Fig. 2).

Lemma 4 ([12]). *Let $F(n)$ be a set of functions closed under taking polynomials. Then a graph G has a $F(n) \times F(n) \times O(n)$ drawing if and only if G has an ordered k-layering with no X-crossing, for some $k \in F(n)$. Furthermore, if G has an ordered layering with no X-crossing and maximum edge span s then G has a $O(s) \times O(s) \times O(n)$ drawing.*

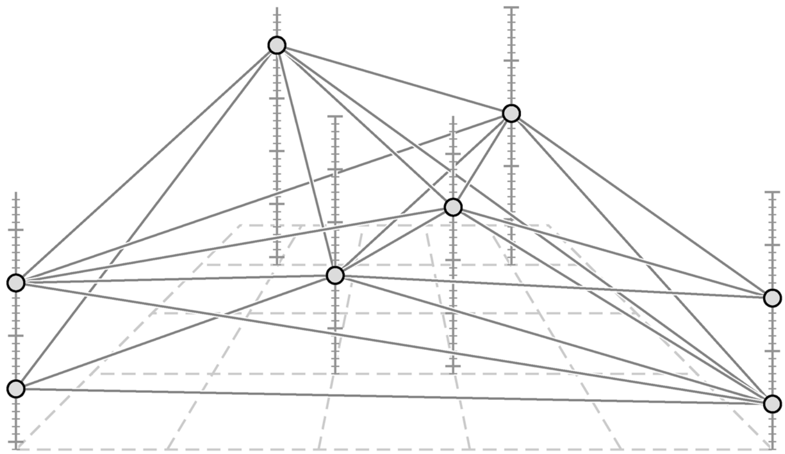

Fig. 2. A three-dimensional drawing produced from an ordered layering with no X-crossing; vertices in each layer are placed on a vertical 'rod'.

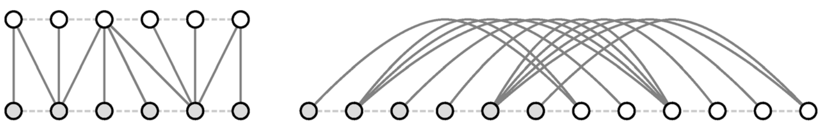

Fig. 3. An ordered 2-layering and a 1-queue layout of a bipartite graph.

Dujmović *et al.* [12] proved that a graph G has an ordered $(\mathsf{pw}(G)+1)$-layering with no X-crossing. That G has a three-dimensional drawing with $O(n \cdot \mathsf{pw}(G)^2)$ volume follows from Lemma 4. A result of Felsner *et al.* [14] also fits into this framework. To construct three-dimensional drawings of outerplanar graphs with $O(n)$ volume, they proved that such a graph has an ordered layering with no X-crossing and maximum edge span at most one. Note that the plane grid graph, which has $\Theta(\sqrt{n})$ path-width and tree-width, has an obvious ordered layering with no X-crossing and maximum edge span one. The 'nested triangles' graph which provides an $\Omega(n^2)$ lower bound on the area of plane grid drawings [6], has an ordered 3-layering with no X-crossing. Thus both of these important examples of planar graphs have three-dimensional drawings with $O(n)$ volume.

Lemma 4 implies that Theorem 2 can be proved if we show that $\mathsf{qn}(G) \in F(n)$ if and only if G has an ordered k-layering with no X-crossing, for some $k \in F(n)$. The next lemma highlights the inherent relationship between ordered layerings and queue layouts. Its proof follows immediately from the definitions (see Fig. 3).

Lemma 5. *A bipartite graph $G = (A, B; E)$ has an ordered 2-layering with no X-crossing and no intra-layer edges if and only if G has a 1-queue layout such that in the corresponding vertex-ordering, the vertices in A appear before the vertices in B.*

We now show that a queue layout can be obtained from an ordered layering with no X-crossing. This result can be viewed as a generalisation of the

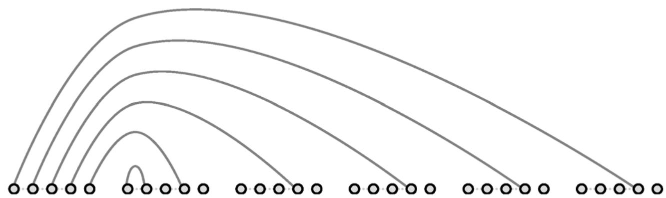

Fig. 4. Maximum rainbow in a vertex-ordering from an ordered layering.

construction of a 2-queue layout of an outerplanar graph by Rengarajan and Veni Madhavan [26] (with $s = 1$).

Lemma 6. *Let G be a graph with an ordered k-layering $\{(V_i, <_i) : 1 \leq i \leq k\}$ with no X-crossing and maximum edge span s. Then $\mathsf{qn}(G) \leq s+1$, and if there are no intra-layer edges then $\mathsf{qn}(G) \leq s$.*

Proof. Let $\sigma = V_1, \dots, V_k$, with each V_i ordered by $<_i$. Let R be the largest rainbow in σ. By Lemma 5, between each pair of layers there is at most one edge in R. A simple inductive argument shows that there is at most s non-intra-layer edges in R; see Fig. 4. No two intra-layer edges are nested in σ. Thus R has at most $s+1$ edges. By Lemma 1, $\mathsf{qn}(G) \leq s+1$. If there are no intra-layer edges then R has at most s edges and $\mathsf{qn}(G) \leq s$. □

We now prove a converse result to Lemma 6. Consider an ordered k-layering with no X-crossing and no intra-layer edges. It is easily seen that the subgraph induced by two layers is a forest of caterpillars. A slightly smaller family of graphs is a forest of stars. A proper vertex-colouring of a graph is called a *star colouring* if each bichromatic subgraph is a forest of stars; that is, every path on four vertices receives at least three distinct colours. The minimum number of colours in a star colouring of a graph G is called the *star chromatic number* of G, and is denoted by $\chi_{\rm st}(G)$. Nešetřil and Ossona de Mendez [22] proved that every planar graph G has $\chi_{\rm st}(G) \leq 30$. Many other graph families have bounded star chromatic number, including graphs with bounded maximum degree [1], and graphs with bounded tree-width [15]. In particular, Fertin *et al.* [15] proved that $\chi_{\rm st}(G) \leq \frac{1}{2}\mathsf{tw}(G)(\mathsf{tw}(G)+3)+1$. More generally, Nešetřil and Ossona de Mendez [22] proved that G has bounded star chromatic number if and only if G is a member of a proper minor-closed family of graphs. In this case, $\chi_{\rm st}(G)$ is at most a quadratic function of the maximum chromatic number of a minor of G.

Lemma 7. *Let G be a graph with star chromatic number $\chi_{\rm st}(G) \leq c$, and queue-number $\mathsf{qn}(G) \leq q$. Then G has an ordered t-layering with no X-crossing where*

$$t \;\leq\; c\big(2(c-1)q+1\big)^{c-1} \quad .$$

Proof. Let $V_1, \dots, V_c$ be the colour classes of a star colouring of G. Pemmaraju [24] proved that a q-queue graph layout can be 'separated' by a vertex c-colouring to produce a $2(c-1)q$-queue layout with the vertices in each colour class consecutive in the vertex-ordering. (The proof is a straightforward application of

Lemma 1.) Applying this result to the given queue layout and star colouring, we obtain a q'-queue layout of G with vertex-ordering $\sigma = V_1, \dots, V_c$, where $q' = 2(c-1)q$.

For every vertex $v \in V_i$, $1 \leq i \leq c$, and $j \in \{1, \dots, c\} \setminus \{i\}$, let $d_j(v)$ be the degree of v in $G[V_i, V_j]$. Define the jth *label* of v, denoted by $\phi_j(v)$, as follows. If $d_j(v) \geq 2$ then let $\phi_j(v) =$ 'r' (v is the root of a star in $G[V_i, V_j]$). If $d_j(v) = 1$ then let $\phi_j(v)$ be the queue containing the edge in $G[V_i, V_j]$ incident to v. If $d_j(v) = 0$ then let $\phi_j(v)$ be some arbitrary queue. Let the *label* of $v \in V_i$ be $\phi(v) = (\phi_1(v), \dots, \phi_{i-1}(v), \phi_{i+1}(v), \dots, \phi_c(v))$. Let S_i be the set of possible labels for a vertex in V_i. Then $|S_i| = (q'+1)^{c-1}$.

Now group the vertices with the same colour and the same label. Let $V_{i,L} = \{v \in V_i : \phi(v) = L\}$ for all labels $L \in S_i$ and $1 \leq i \leq c$, and consider each $V_{i,L}$ to be ordered by σ. Thus $\{V_{i,L} : 1 \leq i \leq c, L \in S_i\}$ is an ordered layering of G. We denote the jth label of $L \in S_i$ by $L[j]$.

Consider a subgraph $G[V_{i,P}, V_{j,Q}]$ for some $1 \leq i < j \leq c$ and labels $P \in S_i$ and $Q \in S_j$. We claim that all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. If $P[j] =$ 'r' and $Q[i] =$ 'r' then $G[V_{i,P}, V_{j,Q}]$ has no edges. If $P[j] =$ 'r' and $Q[i] = q_a$ for some queue q_a, then all edges in $G[V_{i,P}, V_{j,Q}]$ are in q_a. Similarly, if $Q[i] =$ 'r' and $P[j] = q_a$ for some queue q_a, then all edges in $G[V_{i,P}, V_{j,Q}]$ are in q_a. Finally, consider the case in which $P[j] = q_a$ and $Q[i] = q_b$ for some queues q_a and q_b. If $a \neq b$ then there are no edges in $G[V_{i,P}, V_{j,Q}]$, and if $a = b$ then all edges in $G[V_{i,P}, V_{j,Q}]$ are in queue $q_a (= q_b)$. In each case, all edges in $G[V_{i,P}, V_{j,Q}]$ are in a single queue. By Lemma 5, $V_{i,P}$ and $V_{j,Q}$ form an ordered 2-layering of $G[V_{i,P}, V_{j,Q}]$ with no X-crossing. In general, $\{V_{i,L} : 1 \leq i \leq c, L \in S_i\}$ is an ordered layering of G with no X-crossing. The number of layers is $c(q'+1)^{c-1} = c(2(c-1)q+1)^{c-1}$. □

Lemmata 4, 6 and 7 together with the result of Nešetřil and Ossona de Mendez [22] establish Theorem 2.

5 Conclusion

Theorem 2 implies that a planar graph has a three-dimensional drawing with $O(n)$ volume if it has $O(1)$ queue-number. Thus an affirmative answer to the following open problem due to Heath *et al.* [19] would solve Open Problem 1. In fact, the two problems are almost equivalent. It is possible, however, that a planar graph has non-constant queue-number, yet has say a $O(n^{1/3}) \times O(n^{1/3}) \times O(n^{1/3})$ drawing.

Open Problem 3 ([19, 20]). Does every planar graph have $O(1)$ queue-number?

In 1992, Heath and Rosenberg [20] and Heath *et al.* [19] conjectured that every planar graph *does* have $O(1)$ queue-number. More recently, Pemmaraju [24] provided 'evidence' that the planar graph obtained by repeated stellation of K_3 (that is, by adding a degree three vertex to every face) has non-constant queue-number. This graph does have $O(\log n)$ queue-number [24]. Pemmaraju [24] and Heath [private communication, 2002] conjecture that every planar graph has $O(\log n)$ queue-number. By Theorem 2, this would imply that every planar graph has a three-dimensional drawing with $O(n \operatorname{polylog} n)$ volume. Note that if the stellated K_3 graph, which has tree-width three, has non-constant queue-number then Open Problem 2 would also have a negative answer [16].

The best known upper bound on the queue-number of a planar graph is $O(\sqrt{n})$, which follows from Lemma 2 and the fact that the path-width of a planar graph is $O(\sqrt{n})$ (see [2]). This result can also be proved using a variant of the randomised algorithm of Malitz [21] (see [19]), or the derandomised algorithm of Shahrokhi and Shi [29].

As a final word, we estimate the constants in the $O(n)$ volume bound of Corollary 1. Take a graph G with bounded tree-width $\mathsf{tw}(G) \leq k$ and bounded maximum degree $\Delta(G) \leq d$. Then $\chi_{\mathrm{st}}(G) \leq \frac{1}{2}k^2 + o(k^2)$ [15] and $\mathsf{qn}(G) \leq 36kd$ by Lemma 3. By Lemma 7, G has an ordered layering with no X-crossing and approximately $k^2(36k^3d)^{k^2/2}$ layers. By Lemma 4, G has a three-dimensional drawing with approximately $O(k^4(36k^3d)^{k^2} \cdot n)$ volume. As another example, a series-parallel graph G has $\mathsf{tw}(G) \leq 2$ [2], $\mathsf{qn}(G) \leq 3$ [26], and $\chi_{\mathrm{st}}(G) \leq 6$ [15]. By Lemma 7, G has an ordered layering with no X-crossing and at most $6 \cdot 31^5$ layers. By Lemma 4, the constant in the $O(n)$ volume bound of Corollary 1 for series-parallel graphs is at least $36 \cdot 31^{10} \approx 2.9 \times 10^{16}$. It is an interesting open problem to construct linear volume three-dimensional drawings with a smaller constant in the $O(n)$ volume bound.

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Note added in proof: Dujmović and Wood [13] recently solved Open Problem 2. That is, graphs of bounded tree-width have bounded queue-number, and hence have three-dimensional drawings with linear volume.

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