A SIMPLE PROOF OF THE FÁRY-WAGNER THEOREM

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The purpose of this note is to give a simple proof of the following fundamental result independently due to Fáry [1] and Wagner [2]. A *plane graph* is a simple graph embedded in the plane without edge crossings. Combinatorially speaking, there is a circular ordering of the edges incident to each vertex, and a nominated outerface.

Theorem. Every plane graph has a drawing in which every edge is straight.

Proof. A *triangulation* is a plane graph in which every face is bounded by three edges. Edges can be added to a plane graph to obtain a plane triangulation. Thus it suffices to prove the theorem for plane triangulations G. We proceed by induction on $|V(G)|$. The base case with $|V(G)| = 3$ is trivial. Now suppose that $|V(G)| \geq 4$. A *separating triangle* of G is a 3-cycle that contains a vertex in its interior and in its exterior. If G has no separating triangles, then let vw be any edge of G. Otherwise, let vw be an edge incident to a vertex that is in the interior of an innermost separating triangle of G. Now vw is on the boundary of two faces, say vwp and vwq. Since vw is not in a separating triangle, p and q are the only common neighbours of v and w. Let $(vp, vw, vq, vx_1, vx_2, \dots, vx_k)$ and $(wq, wv, wp, wy_1, wy_2, \dots, wy_\ell)$ be the clockwise ordering of the edges incident to v and w respectively[1].

Let G' be the plane triangulation obtained from G by contracting the edge vw into a single vertex s. Replace the pairs of parallel edges $\{vp, wp\}$ and $\{vq, wq\}$ in G by edges sp and sq in G'. The clockwise ordering of the edges of G' incident to s is $(sp, sy_1, sy_2, \dots, sy_\ell, sq, sx_1, sx_2, \dots, sx_k)$. By induction, G' has a drawing in which every edge is straight (and the circular ordering of the edges incident to s are preserved). For all $\epsilon > 0$, let $C_\epsilon(s)$

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[1] In fact, for every vertex v there is an edge incident to v whose endpoints have at most two common neighbours. This is because the neighbourhood of v has no K_4-minor (it is even outerplanar), and every graph with no K_4-minor has a vertex of degree at most two.

denote the circle of radius ϵ centred at s. For each neighbour t of s in G', let $R_\epsilon(t)$ denote the region consisting of the union of all open segments between t and a point in $C_\epsilon(s)$. There is an $\epsilon > 0$ such that all neighbours t of s are in the exterior of $C_\epsilon(s)$ and the only edges of G' that intersect $R_\epsilon(t)$ are incident to s.

There is a line L through s with p on one side of L and q on the other side, as otherwise the edges sp and sq would overlap. Now sp and sq break $C_\epsilon(s)$ into two arcs, one that intersects the edges $\{sx_i : 1 \leq i \leq k\}$, and one that intersects the edges $\{sy_j : 1 \leq j \leq \ell\}$. The set $L \cap C_\epsilon(s)$ consists of two points. Position v and w at these two points, with v on the side of $C_\epsilon(s)$ that intersects the edges $\{sx_i : 1 \leq i \leq k\}$, and with w on the other side. Delete s and its incident edges. Draw the edges of G incident to v or w straight. Thus vw is contained in L. Since p and q are on different sides of L, the edges incident to v or w do not cross. By the choice of ϵ, edges incident to v or w do not cross other edges of G. Thus we obtain the desired drawing of G. □

References

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