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Graphs and Combinatorics

On the Maximum Number of Cliques in a Graph

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Abstract. A *clique* is a set of pairwise adjacent vertices in a graph. We determine the maximum number of cliques in a graph for the following graph classes: (1) graphs with n vertices and m edges; (2) graphs with n vertices, m edges, and maximum degree Δ; (3) d-degenerate graphs with n vertices and m edges; (4) planar graphs with n vertices and m edges; and (5) graphs with n vertices and no K_5-minor or no $K_{3,3}$-minor. For example, the maximum number of cliques in a planar graph with n vertices is $8(n-2)$.

Key words. extremal graph theory, Turán's Theorem, clique, Complete subgraph, Degeneracy, Graph minor, Planar graph, K_5-minor, $K_{3,3}$-minor.

1. Introduction

The typical question of extremal graph theory asks for the maximum number of edges in a graph in a certain family; see the surveys [2, 38–40]. For example, a celebrated theorem of Turán [47] states that the maximum number of edges in a graph with n vertices and no $(k+1)$-clique is $\frac{1}{2}(1-\frac{1}{k})n^2$. Here a *clique* is a (possibly empty) set of pairwise adjacent vertices in a graph. For $k \geq 0$, a k-*clique* is a clique of cardinality k. Since an edge is nothing but a 2-clique, it is natural to consider the maximum number of ℓ-cliques in a graph. The following generalisation of Turán's Theorem, first proved by Zykov [52], has been rediscovered and itself generalised by several authors [8, 10, 15–17, 20, 27, 31, 33, 36].

Theorem 1 ([52]). *For all integers $k \geq \ell \geq 0$, the maximum number of ℓ-cliques in a graph with n vertices and no $(k+1)$-clique is $\binom{k}{\ell}\left(\frac{n}{k}\right)^{\ell}$.*

A simple inductive proof of Theorem 1 is included in Appendix A. In this paper we determine the maximum number of cliques in a graph in each of the following classes:

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- graphs with n vertices and m edges (Section 3),
- graphs with n vertices, m edges, and maximum degree Δ (Section 4),
- d-degenerate graphs with n vertices and m edges (Section 5),
- planar graphs with n vertices and m edges (Section 6), and
- graphs with n vertices and no K_5-minor or no $K_{3,3}$-minor (Section 7).

We now review some related work from the literature. Eckhoff [5, 6] determined the maximum number of cliques in a graph with m edges and no $(k+1)$-clique. Lower bounds on the number of cliques in a graph have also been obtained [4, 13, 14, 22–25]. The number of cliques in a random graph has been studied [3, 29, 37]. Bounds on the number of cliques in a graph have recently been applied in the analysis of an algorithm for finding small separators [32] and in the enumeration of minor-closed families [28].

2. Preliminaries

Every graph G that we consider is undirected, finite, and simple. Let $V(G)$ and $E(G)$ be the vertex and edge sets of G. Let $\Delta(G)$ be the maximum degree of G. We say G is a $(|V(G)|, |E(G)|)$-graph or a $(|V(G)|, |E(G)|, \Delta(G))$-graph.

Let $C(G)$ be the set of cliques in G. Let $c(G) := |C(G)|$. Let $C_k(G)$ be the set of k-cliques in G. Let $c_k(G) := |C_k(G)|$. Our aim is to prove bounds on $c(G)$ and $c_k(G)$.

A clique is not necessarily maximal.[1] In particular, $\emptyset$ is a clique of every graph, $\{v\}$ is a clique for each vertex v, and each edge is a clique. Thus every graph G satisfies

$$c(G) \geq c_0(G) + c_1(G) + c_2(G) = 1 + |V(G)| + |E(G)|. \tag{1}$$

A *triangle* is a 3-clique. Equation (1) implies that

$$c(G) = 1 + |V(G)| + |E(G)| \text{ if and only if } G \text{ is triangle-free.} \tag{2}$$

Triangle-free graphs have the fewest cliques. Obviously the complete graph K_n has the most cliques for a graph on n vertices. In particular, $c(K_n) = 2^n$ since every set of vertices in K_n is a clique.

Say v is a vertex of a graph G. Let G_v be the subgraph of G induced by the neighbours of v. Observe that X is a clique of G containing v if and only if $X = Y \cup \{v\}$ for some clique Y of G_v. Thus the number of cliques of G that contain v is exactly $c(G_v)$. Every clique of G either contains v or is a clique of $G \setminus v$. Thus $C(G) = C(G \setminus v) \cup \{Y \cup \{v\} : Y \in C(G_v)\}$ and

$$c(G) = c(G \setminus v) + c(G_v) \leq c(G \setminus v) + 2^{\deg(v)}. \tag{3}$$

Let G be a graph with induced subgraphs G_1, G_2 and S such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = S$. Then G is obtained by *pasting* G_1 and G_2 on S. Observe that $C(G) = C(G_1) \cup C(G_2)$ and $C(G_1) \cap C(G_2) = C(S)$. Thus

$$c(G) = c(G_1) + c(G_2) - c(S). \tag{4}$$

[1] Moon and Moser [26] proved that the maximum number of *maximal* cliques in a graph with n vertices is approximately $3^{n/3}$; see [9, 11, 12, 18, 19, 34, 35, 42, 50, 51] for related results.

Lemma 1. *Let G be an (n, m)-graph that is obtained by pasting G_1 and G_2 on S. Say G_i has n_i vertices and m_i edges. Say S has n_S vertices and m_S edges. If $c(G_i) \leq xn_i + ym_i + z$ and $c(S) \geq xn_s + ym_S + z$, then $c(G) \leq xn + ym + z$.*

Proof. By Equation (4),

$$\begin{aligned} c(G) &= c(G_1) + c(G_2) - c(S) \\ &\leq (xn_1 + ym_1 + z) + (xn_2 + ym_2 + z) - (xn_s + ym_s + z) \\ &= x(n_1 + n_2 - n_S) + y(m_1 + m_2 - m_S) + z \\ &= xn + ym + z. \end{aligned}$$ □

The following special case of Lemma 1 will be useful.

Corollary 1. *Let G be an (n, m)-graph that is obtained by pasting G_1 and G_2 on a k-clique. Say G_i has n_i vertices and m_i edges. Assume that $c(G_i) \leq xn_i + ym_i + z$ and that $xk + y\binom{k}{2} + z \leq 2^k$. Then $c(G) \leq xn + ym + z$.* □

3. General Graphs

We now determine the maximum number of cliques in an (n, m)-graph.

Theorem 2. *Let n and m be non-negative integers such that $m \leq \binom{n}{2}$. Let d and ℓ be the unique integers such that $m = \binom{d}{2} + \ell$ where $d \geq 1$ and $0 \leq \ell \leq d - 1$. Then the maximum number of cliques in an (n, m)-graph equals $2^d + 2^\ell + n - d - 1$.*

Proof. First we prove the lower bound. Let $V(G) := \{v_1, v_2, \ldots, v_n\}$ and $E(G) := \{v_i v_j : 1 \leq i < j \leq d\} \cup \{v_i v_{d+1} : 1 \leq i \leq \ell\}$, as illustrated in Figure 1. Then G has $\binom{d}{2} + \ell$ edges. Now $\{v_1, v_2, \ldots, v_d\}$ is a clique, which contains 2^d cliques (including $\emptyset$). The neighbourhood of v_{d+1} is an ℓ-clique with 2^ℓ cliques. Thus there are 2^ℓ cliques that contain v_{d+1}. Finally $v_{d+2}, v_{d+3}, \ldots, v_n$ are isolated vertices, which contribute $n - d - 1$ cliques to G. In total, G has $2^d + 2^\ell + n - d - 1$ cliques.

Now we prove the upper bound. That is, every (n, m)-graph G has at most $2^d + 2^\ell + n - d - 1$ cliques. We proceed by induction on $n + m$. For the base case, suppose that $m = 0$. Then $d = 1$, $\ell = 0$, and $c(G) = n+1 = 2^d + 2^\ell + n - d - 1$. Now

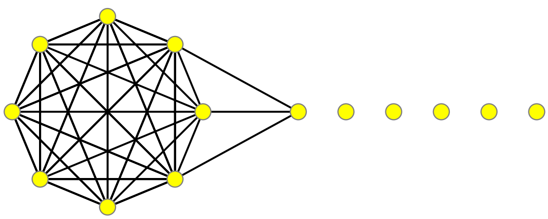

Fig. 1. A (14, 31)-graph with 269 cliques ($d = 8$ and $\ell = 3$)

assume that $m \geq 1$. Let v be a vertex of minimum degree in G. Then $\deg(v) \leq d-1$, as otherwise every vertex has degree at least d, implying $m \geq \frac{dn}{2} \geq \frac{d(d+1)}{2} = \binom{d+1}{2}$, which contradicts the definition of d. By Equation (3), $c(G) \leq c(G \setminus v) + 2^{\deg(v)}$. To apply induction to $G \setminus v$ (which has $n-1$ vertices and $m - \deg(v)$ edges) we distinguish two cases.

First suppose that $\deg(v) \leq \ell$. Thus $m - \deg(v) = \binom{d}{2} + \ell - \deg(v)$. By induction, $c(G) \leq 2^d + 2^{\ell-\deg(v)} + n - 1 - d - 1 + 2^{\deg(v)}$. Hence the result follows if $2^d + 2^{\ell-\deg(v)} + n - 1 - d - 1 + 2^{\deg(v)} \leq 2^d + 2^\ell + n - d - 1$. That is, $2^{\ell-\deg(v)} - 1 \leq (2^{\ell-\deg(v)} - 1)2^{\deg(v)}$, which is true since $0 \leq \deg(v) \leq \ell$.

Otherwise $\ell + 1 \leq \deg(v) \leq d-1$. Thus $m - \deg(v) = \binom{d-1}{2} + d - 1 + \ell - \deg(v)$. By induction, $c(G) \leq 2^{d-1} + 2^{d-1+\ell-\deg(v)} + n - 1 - d + 2^{\deg(v)}$. Hence the result follows if $2^{d-1} + 2^{d-1+\ell-\deg(v)} + n - 1 - d + 2^{\deg(v)} \leq 2^d + 2^\ell + n - d - 1$. That is, $2^\ell(2^{\deg(v)-\ell} - 1) \leq 2^{d-1-\deg(v)+\ell}(2^{\deg(v)-\ell} - 1)$. Since $\deg(v) \geq \ell + 1$, we need $2^\ell \leq 2^{d-1-\deg(v)+\ell}$, which is true since $\deg(v) \leq d-1$. □

4. Bounded Degree Graphs

We now determine the maximum number of cliques in an (n, m, Δ)-graph. West [49] proved a related result.

Theorem 3. *The number of cliques in an (n, m, Δ)-graph G is at most*

$$1 + n + \left(\frac{2^{\Delta+1} - \Delta - 2}{\binom{\Delta+1}{2}}\right) m \leq 1 + \left(\frac{2^{\Delta+1} - 1}{\Delta + 1}\right) n.$$

Proof. G has one 0-clique and n 1-cliques. For $k \geq 2$, each edge is in at most $\binom{\Delta-1}{k-2}$ k-cliques, and each k-clique contains $\binom{k}{2}$ edges. Thus G has at most $m\binom{\Delta-1}{k-2}/\binom{k}{2}$ k-cliques. Thus the number of cliques (not counting 0- and 1-cliques) is at most

$$\begin{aligned}
\sum_{k=2}^{\Delta+1} \frac{m\binom{\Delta-1}{k-2}}{\binom{k}{2}} &= m \sum_{k=2}^{\Delta+1} \frac{2}{k(k-1)} \cdot \frac{(\Delta-1)!}{(k-2)!(\Delta-1-k+2)!} \\
&= \frac{m}{\binom{\Delta+1}{2}} \sum_{k\geq 2}^{\Delta+1} \frac{2(\Delta-1)!\binom{\Delta+1}{2}}{k!(\Delta+1-k)!} \\
&= \frac{m}{\binom{\Delta+1}{2}} \sum_{k=2}^{\Delta+1} \frac{(\Delta+1)!}{k!(\Delta+1-k)!} \\
&= \frac{m}{\binom{\Delta+1}{2}} \left(\left(\sum_{k=0}^{\Delta+1} \binom{\Delta+1}{k} \right) - \frac{(\Delta+1)!}{1!(\Delta+1-1)!} - \frac{(\Delta+1)!}{0!(\Delta+1-0)!} \right) \\
&= \frac{m}{\binom{\Delta+1}{2}} \left(2^{\Delta+1} - \Delta - 2 \right).
\end{aligned}$$

The result follows since $m \leq \frac{\Delta n}{2}$. □

The bound in Theorem 3 is tight for many values of m.

Proposition 1. *For all n and m such that $m \leq \frac{\Delta n}{2}$ and $m \equiv 0 \pmod{\binom{\Delta+1}{2}}$, there is an (n, m, Δ)-graph G with*

$$c(G) = 1 + n + \left(\frac{2^{\Delta+1} - \Delta - 2}{\binom{\Delta+1}{2}} \right) m.$$

Proof. Let $p := m/\binom{\Delta+1}{2}$. Let G consist of p copies of $K_{\Delta+1}$, plus $n - p(\Delta+1)$ isolated vertices. Then G is an (n, m, Δ)-graph. Each copy of $K_{\Delta+1}$ contributes $2^{\Delta+1} - \Delta - 2$ cliques with at least two vertices. Thus G has $1 + n + (2^{\Delta+1} - \Delta - 2)p$ cliques. □

5. Degenerate Graphs

A graph G is d-degenerate if every subgraph of G has a vertex with degree at most d. The following simple result is well known; see [7, 32] for example.

Proposition 2. *Every d-degenerate graph G with $n \geq d$ vertices has at most $2^d(n - d + 1)$ cliques.*

Proof. We proceed by induction on n. If $n = d$ then $c(G) \leq 2^d = 2^d(n - d + 1)$. Now assume that $n \geq d+1$. Let v be a vertex of G with $\deg(v) \leq d$. By Equation (3), $c(G) \leq c(G \setminus v) + 2^{\deg(v)}$. Now $G \setminus v$ is d-degenerate since it is a subgraph of G. Moreover, $G \setminus v$ has at least d vertices. By induction, $c(G \setminus v) \leq 2^d(n - 1 - d + 1)$. Thus $c(G) \leq 2^d(n - 1 - d + 1) + 2^d = 2^d(n - d + 1)$. □

The bound in Proposition 2 is tight.

Proposition 3. *For all $n \geq d$, there is a d-degenerate graph G_n with n vertices and exactly $2^d(n - d + 1)$ cliques (and with a d-clique).*

Proof. Let G_d be the complete graph K_d. Then G_d has the desired properties. For $n \geq d + 1$, let G_n be the graph obtained by adding one new vertex v adjacent to every vertex in some d-clique in G_{n-1}. Then G_n is d-degenerate and contains a d-clique. (G_n is a chordal graph called a d-*tree*; see [1].) By Equation (3), $c(G_n) = c(G_{n-1}) + 2^{\deg(v)} = 2^d(n - 1 - d + 1) + 2^d = 2^d(n - d + 1)$. □

Proposition 2 can be made sensitive to the number of edges as follows.

Theorem 4. *For all $d \geq 1$, every d-degenerate graph G with n vertices and $m \geq \binom{d}{2}$ edges has at most*

$$n + \frac{(2^d - 1)m}{d} - \frac{(d-3)2^d + d + 1}{2}$$

cliques.

Proof. We proceed by induction on $n+m$. For the base case, suppose that $m = \binom{d}{2}+\ell$ where $d \geq 1$ and $0 \leq \ell \leq d-1$. Thus $c(G) \leq 2^d + 2^\ell + n - d - 1$ by Theorem 2, and the result follows if

$$2^d + 2^\ell + n - d - 1 \leq n + \frac{(2^d-1)m}{d} - \frac{(d-3)2^d + d + 1}{2}.$$

That is, $d(2^\ell - 1) \leq \ell(2^d - 1)$, which we prove in Lemma 2 below.

Now assume that $m \geq \binom{d+1}{2}$. Now G has a vertex v with $\deg(v) \leq d$. By Equation (3), $c(G) \leq c(G\backslash v) + 2^{\deg(v)}$. The graph $G\backslash v$ has $m - \deg(v) \geq \binom{d}{2}$ edges, and is d-degenerate since it is a subgraph of G. By induction,

$$c(G \setminus v) \leq n - 1 + \frac{(2^d-1)(m-\deg(v))}{d} - \frac{(d-3)2^d + d + 1}{2}.$$

Thus the result follows if

$$-1 + \frac{(2^d-1)(m-\deg(v))}{d} + 2^{\deg(v)} \leq \frac{(2^d-1)m}{d}.$$

That is, $d(2^{\deg(v)} - 1) \leq (2^d - 1)\deg(v)$, which holds by Lemma 2 below. □

Lemma 2. $d(2^\ell - 1) \leq \ell(2^d - 1)$ *for all integers* $d \geq \ell \geq 0$.

Proof. The case $\ell = 0$ is trivial. Now assume that $\ell \geq 1$. We proceed by induction on d. The base case $d = \ell$ is trivial. Assume that $d \geq \ell + 1 \geq 2$ and by induction,

$$(d-1)(2^\ell - 1) \leq \ell(2^{d-1} - 1). \tag{5}$$

Since $d \geq 2$,

$$\frac{d}{d-1} \leq 2 < 2 + \frac{1}{2^{d-1}-1} = \frac{2^d - 1}{2^{d-1}-1}. \tag{6}$$

Equations (5) and (6) imply that

$$(d-1)(2^\ell - 1) \cdot \frac{d}{d-1} < \ell(2^{d-1} - 1) \cdot \frac{2^d-1}{2^{d-1}-1}.$$

That is, $d(2^\ell - 1) < \ell(2^d - 1)$, as desired. □

Note that a d-degenerate n-vertex graph has at most $dn - \binom{d+1}{2}$ edges, and Theorem 4 with $m = dn - \binom{d+1}{2}$ is equivalent to Proposition 2.

The bound in Theorem 4 is tight for many values of m.

Proposition 4. *Let* $d \geq 1$. *For all* n *and* m *such that* $\binom{d}{2} \leq m \leq dn - \binom{d+1}{2}$ *and*

$$m \bmod d = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \frac{d}{2} & \text{if } d \text{ is even,} \end{cases}$$

there is a d-degenerate (n, m)-graph G with

$$c(G) = n + \frac{(2^d - 1)m}{d} - \frac{(d-3)2^d + d + 1}{2}.$$

Proof. Let $n' := \frac{m}{d} + \frac{1}{2}(d+1)$. Then n' is an integer and $d \leq n' \leq n$. Let G consist of a d-degenerate n'-vertex graph with $2^d(n' - d + 1)$ cliques (from Proposition 3), plus $n - n'$ isolated vertices. Then G has m edges and $c(G) = 2^d(n' - d + 1) + n - n' = n + (2^d - 1)\frac{m}{d} - \frac{1}{2}((d-3)2^d + d + 1)$. □

A graph is 1-degenerate if and only if it is a forest. Thus Theorem 4 with $d = 1$ implies that every forest has at most $n + m - 1$ cliques, which also follows from Equation (2). In particular, $c(T) = 2n$ for every n-vertex tree T.

Theorem 4 with $d = 2$ implies that every 2-degenerate graph has at most $n + \frac{1}{2}(3m + 1)$ cliques. Outerplanar graphs are 2-degenerate. The construction in Propositions 3 and 4 can produce outerplanar graphs. (Add each new vertex adjacent to two consecutive vertices on the outerface.) Thus this bound is tight for outerplanar graphs.

6. Planar Graphs

Papadimitriou and Yannakakis [30] and Storch [44] proved that every n-vertex planar graph has $\mathcal{O}(n)$ cliques; see [7] for a more general result. The proof is based on the corollary of Euler's Formula that planar graphs are 5-degenerate. By Theorem 4, if G is a planar (n, m)-graph with $m \geq 10$, then $c(G) < n + \frac{31}{5}m < \frac{98}{5}n$. We now prove that the bound for 3-degenerate graphs in Theorem 4 also holds for planar graphs.

Theorem 5. *Every planar (n, m)-graph G with $m \geq 3$ has at most $n + \frac{7}{3}m - 2$ cliques.*

Proof. We proceed by induction on $n + m$. The result is easily verified if $m = 3$.

Suppose that G has a separating triangle T. Thus G is obtained by pasting two induced subgraphs G_1 and G_2 on T. Say G_i has n_i vertices and m_i edges. Then $m_i \geq 3$ since $T \subset G_i$. By induction, $c(G_i) \leq n_i + \frac{7}{3}m_i - 2$. By Corollary 1 with $k = 3$, $x = 1$, $y = \frac{7}{3}$ and $z = -2$, we have $c(G) \leq n + \frac{7}{3}m - 2$ (since $1 \cdot 3 + \frac{7}{3}\binom{3}{2} - 2 = 2^3$). Now assume that G has no separating triangle.

Let v be a vertex of G. We have $c(G) = c(G \backslash v) + c(G_v)$ by Equation (3). The graph $G \backslash v$ has $m - \deg(v)$ edges. Suppose that $m - \deg(v) \leq 2$. (Then we cannot apply induction to $G \backslash v$.) Then G has no 4-clique and at most two triangles. If G has at most one triangle, then $c(G) \leq 1 + n + m + 1 \leq n + \frac{7}{3}m - 2$ since $m \geq 3$. Otherwise G has two triangles, and $c(G) \leq 1 + n + m + 2 < n + \frac{7}{3}m - 2$ since $m \geq 5$.

Now assume that $m - \deg(v) \geq 3$. By (3), applying induction to $G \backslash v$,

$$c(G) = c(G \backslash v) + c(G_v) \leq (n-1) + \tfrac{7}{3}(m - \deg(v)) - 2 + c(G_v).$$

Fix a plane embedding of G. If uw is an edge of G_v, then the edges vu and vw are consecutive in the circular ordering of edges incident to v defined by the embedding (as otherwise G would contain a separating triangle). Thus $\Delta(G_v) \leq 2$ and $c(G_v) \leq 1 + \frac{7}{3}\deg(v)$ by Theorem 3. Hence

$$c(G) \leq (n-1) + (\tfrac{7}{3}(m - \deg(v)) - 2) + (1 + \tfrac{7}{3}\deg(v)) = n + \tfrac{7}{3}m - 2.$$

□

If $n \geq 3$ in Theorem 5 then $m \leq 3(n-2)$ by Euler's Formula. Thus we have the following corollary.

Corollary 2. *Every planar graph with $n \geq 3$ vertices has at most $8(n-2)$ cliques.* □

We now prove bounds on the number of 3- and 4-cliques in a planar graph.

Proposition 5. *For every planar graph G with $n \geq 3$ vertices, $c_3(G) \leq 3n - 8$ and $c_4(G) \leq n - 3$.*

Proof. We proceed by induction on n. The result is trivial if $n \leq 4$. Now assume that $n \geq 5$. First suppose that G has no separating triangle. Then $c_4(G) = 0$, and every triangle of G is a face. By Euler's Formula, $c_3(G) \leq 2n - 4 < 3n - 8$ faces. Now suppose that G has a separating triangle T. Thus G is obtained by pasting two induced subgraphs G_1 and G_2 on T. Say G_i has n_i vertices. Then $n_i \geq 3$ since $T \subset G_i$. By induction, $c_3(G_i) \leq 3n_i - 8$ and $c_4(G_i) \leq n_i - 3$. Every clique of G is a clique of G_1 or G_2. Thus $c_4(G) = c_4(G_1) + c_4(G_2) \leq n_1 - 3 + n_2 - 3 = n - 3$. Moreover, T is a triangle in both G_1 and G_2. Thus $c_3(G) \leq (3n_1 - 8) + (3n_2 - 8) - 1 = 3(n_1 + n_2) - 17 = 3(n+3) - 17 = 3n - 8$. □

Note that Proposition 5 and Euler's Formula (which implies $c_2(G) \leq 3n - 6$) reprove Corollary 2, since $1 + n + 3(n-2) + (3n-8) + (n-3) = 8(n-2)$.

We now show that all our bounds for planar graphs are tight.

Proposition 6. *For all $n \geq 3$ there is a maximal planar n-vertex graph G_n with $c_2(G_n) = 3(n-2)$, $c_3(G_n) = 3n - 8$, $c_4(G_n) = n - 3$, and $c(G_n) = 8(n-2)$.*

Proof. Let $G_3 := K_3$. Then $c_2(G_3) = 3$, $c_3(G_3) = 1$, $c_4(G_3) = 0$, and $c(G_3) = 8$. Say G_{n-1} is a maximal planar $(n-1)$-vertex graph with $c_2(G_{n-1}) = 3(n-3)$, $c_3(G_{n-1}) = 3n - 11$, $c_4(G_{n-1}) = n - 4$, and $c(G_n) = 8(n-3)$. Let G_n be the maximal planar n-vertex graph obtained by adding one new vertex v adjacent to each vertex of some face of G_{n-1}, as illustrated in Figure 2. Then $c_2(G_n) = c_2(G_{n-1}) + 3 = 3(n-2)$, $c_3(G_n) = c_3(G_{n-1}) + 3 = 3n - 8$, $c_4(G_n) = c_4(G_{n-1}) + 1 = n - 3$, and $c(G_n) = c(G_{n-1}) + c(G_n(v)) = 8(n-3) + 8 = 8(n-2)$. (Note that G_n is also an example of a 3-degenerate graph with the maximum number of cliques; see Proposition 3.) □

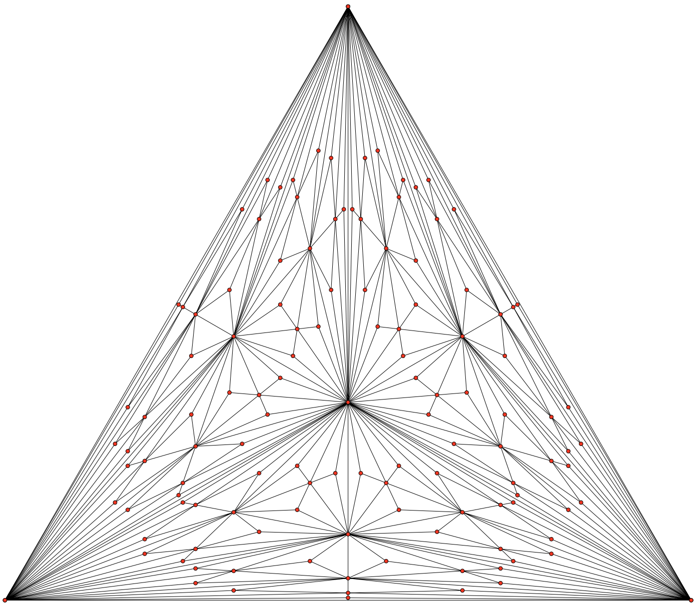

Fig. 2. A planar graph with 124 vertices, 366 edges, 364 triangles, 121 4-cliques, and 976 cliques. It is obtained by repeatedly adding one degree-3 vertex inside each internal face (starting from K_3)

Proposition 7. *For all* $n \geq 3$ *and* $m \in \{3, 6, \ldots, 3n-6\}$*, there is a planar* (n, m)*-graph* G *with* $c(G) = n + \frac{7}{3}m - 2$.

Proof. Let $n' := \frac{m}{3} + 2$. Let G consist of a maximal planar graph on n' vertices with $8(n'-2)$ cliques (from Proposition 6), plus $n - n'$ isolated vertices. Then G has n vertices and m edges, and $c(G) = 8(n'-2) + n - n' = n + 7n' - 16 = n + 7(\frac{m}{3} + 2) - 16 = n + \frac{7}{3}m - 2$. □

7. Graphs with no K_5-Minor

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. The graphs with no K_3-minor are the forests, which have at most $2n$ cliques, and this bound is tight. The graphs with no K_4-minor (called *series-parallel*) are 2-degenerate, and thus have at most $4(n-1)$ cliques, and this bound is tight. The Kuratowski-Wagner Theorem characterises planar graphs as those with no K_5-minor and no $K_{3,3}$-minor. We now extend Corollary 2 for graphs with no K_5-minor (but possibly a $K_{3,3}$-minor).

Theorem 6. *Every graph* G *with* $n \geq 3$ *vertices and no* K_5*-minor has at most* $8(n-2)$ *cliques.*

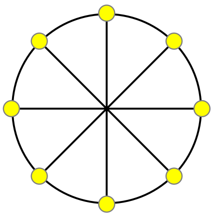

Fig. 3. The graph V_8

Proof. Let V_8 be the graph obtained from the 8-cycle by adding an edge between each pair of antipodal vertices; see Figure 3. Let G be a minimum counterexample to the theorem. We can assume that G is edge-maximal with no K_5-minor. Wagner [48] proved that (a) G is a maximal planar graph, (b) $G = V_8$, or (c) G is obtained by pasting two smaller graphs (that are thus not counterexamples), each with no K_5-minor, on an edge or a triangle T. In case (a) the result is Corollary 2. In case (b), since V_8 is triangle-free, $c(V_8) = 1 + |V(V_8)| + |E(V_8)| = 21 < 8(|V(V_8)| - 2)$ by Equation (2). In case (c), if T is an edge, we have $c(G) \leq 8(n-2)$ by Corollary 1 with $k = 2$, $x = 8$, $y = 0$ and $z = -16$ (since $8 \cdot 2 + 0 - 16 < 2^2$). In case (c), if T is a triangle, we have $c(G) \leq 8(n-2)$ by Corollary 1 with $k = 3$, $x = 8$, $y = 0$ and $z = -16$ (since $8 \cdot 3 + 0 - 16 = 2^3$). □

A similar result is obtained for graphs with no $K_{3,3}$-minor.

Theorem 7. *Every graph G with $n \geq 3$ vertices and no $K_{3,3}$-minor has at most $\frac{4}{3}(7n - 11)$ cliques. Conversely, for all $n \equiv 2 \pmod 3$ with $n \geq 5$ there is an n-vertex graph with no $K_{3,3}$-minor and $c(G) = \frac{4}{3}(7n - 11)$.*

Proof. Let G be a minimum counterexample. We can assume that G is edge-maximal with no $K_{3,3}$-minor. Wagner [48] proved that (a) G is a maximal planar graph, (b) $G = K_5$, or (c) G is obtained by pasting two smaller graphs (that are thus not counterexamples), each with no $K_{3,3}$-minor, on an edge. In case (a) the result follows from Corollary 2 since $8n - 16 < \frac{4}{3}(7n - 11)$. In case (b), $c(K_5) = 32 = \frac{4}{3}(7 \cdot 5 - 11)$. In case (c), we have $c(G) \leq \frac{4}{3}(7n - 11)$ by Corollary 1 with $k = 2$, $x = \frac{28}{3}$, $y = 0$ and $z = -\frac{44}{3}$ (since $\frac{28}{3} \cdot 2 + 0 - \frac{44}{3} = 2^2$). By the same analysis, the graph obtained from K_5 by repeatedly pasting copies of K_5 on an edge has no $K_{3,3}$-minor and $\frac{4}{3}(7n - 11)$ cliques. □

We finish with an open problem: What is the maximum number of cliques in an n-vertex graph G with no K_t-minor? Kostochka [21] and Thomason [45] independently proved that G is $\mathcal{O}(t\sqrt{\log t})$-degenerate.[2] Thus Proposition 2 implies that G has at most $2^{\mathcal{O}(t\sqrt{\log t})}n$ cliques; similar bounds can be found in [28, 32]. It is unknown whether this bound can be improved to $c^t n$ for some constant c (possibly for sufficiently large n).

[2] Moreover, this bound is best possible; Thomason [46] even determined the asymptotic constant.

We have proved that $c(G) \leq 2^{t-2}(n-t+3)$ whenever $t \leq 5$. Moreover, the graph G in Proposition 3 (with $t = d + 2$) has no K_t-minor and $c(G) = 2^{t-2}(n - t + 3)$. However, for large values of t this upper bound does not hold for the complete k-partite graph $K_{2,2,\dots,2}$. By Theorem 8 in Appendix B, the maximum order of a clique minor in $K_{2,2,\dots,2}$ is $\lfloor \frac{3}{2}k \rfloor$. But by Proposition 10, $c(K_{2,2,\dots,2}) = 3^k > 2^{\lfloor 3k/2 \rfloor -1}(2k - \lfloor \frac{3}{2}k \rfloor + 2)$ for all $k \geq 42$.

Acknowledgement. Thanks to a referee for pointing out reference [43].

A. Graphs with Bounded Cliques

In this appendix we give a simple inductive proof of Theorem 1.

Proposition 8. *For all integers $k \geq \ell \geq 0$, every graph G with $n \geq \ell$ vertices and no $(k+1)$-clique has at most $\binom{k}{\ell}\left(\frac{n}{k}\right)^{\ell}$ ℓ-cliques.*

Proof. We proceed by induction on n. For the base case, suppose that $n \leq k$. Trivially $c_\ell(G) \leq \binom{n}{\ell}$, which is at most $\binom{k}{\ell}\left(\frac{n}{k}\right)^{\ell}$ by Lemma 3 below. Now assume that the result holds for graphs with less than n vertices, and $n > k$. Let G be a graph with n vertices, no $(k+1)$-clique, and with $c_\ell(G)$ maximum. We can add edges to G until it contains a k-clique X. Every ℓ-clique of G is the union of some i-clique of $G \setminus X$ and some $(\ell - i)$-clique of $G[X]$, for some $0 \leq i \leq \ell$. Moreover, the vertices in each i-clique of $G \setminus X$ have at most $k - i$ common neighbours in X (since X is a clique and G has no $(k+1)$-clique). Thus from each i-clique of $G \setminus X$, we obtain at most $\binom{k-i}{\ell-i}$ ℓ-cliques of G. By induction, $c_i(G \setminus X) \leq \binom{k}{i}\left(\frac{n-k}{k}\right)^{i}$. Thus

$$c_\ell(G) \leq \sum_{i=0}^{\ell} \binom{k}{i}\left(\frac{n-k}{k}\right)^{i}\binom{k-i}{\ell-i} = \binom{k}{\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\left(\frac{n}{k}-1\right)^{i} = \binom{k}{\ell}\left(\frac{n}{k}\right)^{\ell} ,$$

by the binomial theorem.[3] □

Lemma 3. $\binom{n}{\ell}k^{\ell} \leq \binom{k}{\ell}n^{\ell}$ *for all integers $k \geq n \geq \ell \geq 0$.*

Proof. We proceed by induction on ℓ. The claim is trivial with $\ell = 0$. Now assume that $\ell \geq 1$. Thus $k - n \leq \ell(k - n)$, implying $kn + k - n \leq kn + \ell(k - n)$. That is, $k(n - \ell + 1) \leq n(k - \ell + 1)$. By induction,

$$\binom{n}{\ell-1}k^{\ell-1} \cdot k(n - \ell + 1) \leq \binom{k}{\ell-1}n^{\ell-1} \cdot n(k - \ell + 1).$$

[3] Twice we use that $x^t = \sum_{j=0}^{t}\binom{t}{j}(x-1)^j$ for all real x.

That is,

$$\frac{n!\,k^{\ell}(n-\ell+1)}{(n-\ell+1)!\,(\ell-1)!} \leq \frac{k!\,n^{\ell}(k-\ell+1)}{(k-\ell+1)!\,(\ell-1)!}.$$

Hence

$$\frac{n!\,k^{\ell}}{(n-\ell)!\,\ell!} \leq \frac{k!\,n^{\ell}}{(k-\ell)!\,\ell!},$$

as desired. □

Proposition 9. *Every graph G with n vertices and no $(k+1)$-clique has at most $\left(\frac{n}{k}+1\right)^{k}$ cliques.*

Proof. By Proposition 8 and the binomial theorem,

$$c(G) \leq \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{n}{k}\right)^{\ell} = \left(\frac{n}{k}+1\right)^{k}.$$

□

We now prove that Propositions 8 and 9 are tight.

Proposition 10. *For every complete k-partite graph $G = K_{n_1,n_2,\dots,n_k}$,*

$$c(G) = \prod_{i=1}^{k}(n_i+1).$$

In particular, if every $n_i = \frac{n}{k}$ then $c(G) = (\frac{n}{k}+1)^k$ and $c_\ell(G) = \binom{k}{\ell}(\frac{n}{k})^\ell$ whenever $0 \leq \ell \leq k$.

Proof. Every clique consists of at most one vertex from each of the k colour classes. There are n_i+1 ways to choose at most one vertex from the i-th colour class. Thus $c(G) = \prod_i (n_i+1)$. (This result can also be proved using Equation (3).) Now assume that every $n_i = \frac{n}{k}$. Every ℓ-clique consists of exactly one vertex from each of ℓ colour classes. There are $\binom{k}{\ell}$ ways to choose ℓ colour classes and $\frac{n}{k}$ ways to choose exactly one vertex from each colour class. Each combination gives a distinct ℓ-clique. The result follows. □

It is interesting to note that the extremal examples in Proposition 1 for graphs of bounded degree (disjoint copies of cliques) are the complements of the extremal examples in Proposition 10 for graphs with bounded cliques (complete multipartite graphs).

B. Clique Minors in a Complete Multipartite Graph

The *Hadwiger number* of a graph G, denoted by $\eta(G)$, is the maximum order of a clique minor in G. Stiebitz [43] proved that $\eta(G) \leq \frac{1}{2}(n+k)$ for every n-vertex graph G with no $(k+1)$-clique. We now prove that this bound is tight for every complete k-partite graph if the largest colour class is not too large.

Theorem 8. *Let G be a complete k-partite graph on n vertices with n' vertices in the largest colour class. Then* $\eta(G) = \min\left\{\frac{1}{2}(n+k), n - n' + 1\right\}$.

The proof of Theorem 8 is based on the following lemma.

Lemma 4. *Let G be the complete k-partite graph $K_{n_1,n_2,\dots,n_k}$ with each $n_i \geq 1$. Then $\eta(G)$ equals k plus the size of the largest matching in $G' := K_{n_1-1,n_2-1,\dots,n_k-1}$.*

Proof. Consider G' to be a subgraph of G, so that $S := V(G)\setminus V(G')$ is a k-clique of G. Let M be a matching of G'. If v is a vertex and e is an edge of G', then v is adjacent to at least one endpoint of e. Thus every vertex in S is adjacent to at least one endpoint of every edge in M, and for all edges e and f in M, at least one endpoint of e is adjacent to at least one endpoint of f. Thus by contracting each edge of M within G, we obtain a $K_{k+|M|}$-minor in G.

Now suppose that K_t is a minor of G with t maximum. Then G has disjoint vertex sets $X_1, X_2, \dots, X_t$, such that each X_i induces a connected subgraph of G, and for all $i \neq j$, some vertex in X_i is adjacent to some vertex in X_j.

Suppose that some X_i contains two vertices v and w in the same colour class of G. Since v and w have the same neighbourhood, we can delete w from X_i and still have a K_t-minor. Now assume that the vertices in each set X_i are from distinct colour classes.

Suppose that some X_i contains at least three vertices u, v, w. Since the neighbourhood of u is contained in the union of the neighbourhoods of v and w, we can delete u from X_i and still have a K_t-minor. Now assume that each set X_i has cardinality 1 or 2.

Suppose that for some colour class ℓ, no set X_i contains a vertex coloured ℓ. Then $X_1, \dots, X_t$ along with a set consisting of one vertex coloured ℓ forms a K_{t+1}-minor, which is a contradiction. Now assume that for every colour class ℓ, there is some set X_i that contains a vertex coloured ℓ.

Suppose that for some colour class ℓ, every set X_i that contains some vertex coloured ℓ has cardinality 2. Let $X_i = \{v, w\}$ be such a set, where v is coloured ℓ. Thus v is adjacent to some vertex in every set X_j. Thus we can delete w from X_i and still have a K_t-minor. Now assume that for each colour class ℓ, some set X_i consists of one vertex coloured ℓ. No two singleton sets X_i and X_j contain vertices of the same colour. Thus there are k singleton sets X_i, one for each colour class. The remaining sets X_i thus form a matching in G'. □

Proof of Theorem 8. Sitton [41] proved that the size of the largest matching in a complete multipartite graph on n vertices with n' vertices in the largest colour class is $\min\left\{\left\lfloor\frac{n}{2}\right\rfloor, n-n'\right\}$. Applying this result to the graph G' in Lemma 4,

$$\eta(G) \;=\; k+\min\left\{\tfrac{1}{2}(n-k),\,(n-k)-(n'-1)\right\} \;=\; \min\left\{\tfrac{1}{2}(n+k),\,n-n'+1\right\}.$$

□

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