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Grid drawings of k-colourable graphs

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Abstract

It is proved that every k-colourable graph on n vertices has a grid drawing with $\mathcal{O}(kn)$ area, and that this bound is best possible. This result can be viewed as a generalisation of the no-three-in-line problem. A further area bound is established that includes the aspect ratio as a parameter.

Keywords: Graph drawing; Grid drawing; Area; Aspect ratio; No-three-in-line problem

1. Introduction

Let $G = (V, E)$ be a graph. All graphs considered are simple, finite and undirected. A *grid drawing* of G is an injective mapping $\theta : V \to \mathbb{Z}^2$ such that for all edges $vw \in E$ and vertices $x \in V$, $\theta(x) \in \overline{\theta(v)\theta(w)}$ implies that $x = v$ or $x = w$, where $\overline{ab}$ denotes the line-segment with endpoints a and b. That is, a grid drawing of a graph represents each vertex by a distinct gridpoint in the plane, and each edge by a line-segment between its endpoints, such that the only vertices an edge intersects are its own endpoints. Let θ be a grid drawing of a graph $G = (V, E)$ such that $\theta(v) = (X(v), Y(v))$ for all vertices $v \in V$. If for some $w, h \in \mathbb{Z}^+$, we have $|X(u) - X(v)| < w$ and $|Y(u) - Y(v)| < h$ for all vertices $u, v \in V$, then θ is said to be a $w \times h$ grid drawing with *area* wh and *aspect ratio* $\max\{w, h\}/\min\{w, h\}$.

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This paper studies grid drawings with small area, and with small aspect ratio as a secondary criterion. In applications such as graph visualisation [2], minimising the area and the aspect ratio are important considerations. Obviously to view a graph drawing with good resolution on a computer screen (which itself has fixed aspect ratio) requires the area and the aspect ratio to be small.

A *k-colouring* of a graph $G=(V,E)$ is a partition of V into *colour classes* $V_0, V_1, \dots, V_{k-1}$ such that for every edge $vw \in E$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. A graph admitting a k-colouring is *k-colourable*. A *complete k-partite* graph is a k-colourable graph such that there is an edge between any two vertices from distinct colour classes. A complete k-partite graph is *balanced* if every colour class has the same number of vertices. Let $K(t,k)$ denote the balanced complete k-partite graph with t vertices in each colour class.

2. Results

Theorem 1. *For all $k \geqslant 1$ and $t \geqslant 1$, the balanced complete k-partite graph $K(t,k)$ has a $k \times pt$ grid drawing, where p is the minimum prime such that $p \geqslant k$.*

Proof. Let $V_0, V_1, \dots, V_{k-1}$ be the k-colouring of $K(t,k)$. For each $0 \leqslant i \leqslant k-1$, let $V_i = \{v_{i,0}, v_{i,1}, \dots, v_{i,t-1}\}$, and for each $0 \leqslant j \leqslant t-1$, let $\theta(v_{i,j}) = (i, pj + (i^2 \bmod p))$. If an edge intersects a vertex other than its endpoints then the three vertices are collinear. Since the vertices in each V_i are positioned in the $X = i$ line, to prove that θ is a valid grid drawing, it is sufficient to prove that any three vertices from distinct colour classes are not collinear. Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0.$$

For vertices v_{i_1,j_1}, v_{i_2,j_2} and v_{i_3,j_3} from distinct colour classes we have

$$\begin{vmatrix} 1 & i_1 & pj_1 + (i_1^2 \bmod p) \\ 1 & i_2 & pj_2 + (i_2^2 \bmod p) \\ 1 & i_3 & pj_3 + (i_3^2 \bmod p) \end{vmatrix} \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} \equiv (i_1 - i_2)(i_1 - i_3)(i_2 - i_3) \pmod p,$$

which is nonzero since p is a prime and $1 \leqslant i_\alpha - i_\beta \leqslant k-1 \leqslant p-1$ for all $1 \leqslant \alpha < \beta \leqslant 3$. Thus v_{i_1,j_1}, v_{i_2,j_2} and v_{i_3,j_3} are not collinear. Therefore the only vertices an edge intersects are its own endpoints, and θ is a valid grid drawing of $K(t,k)$. For every vertex v, $0 \leqslant X(v) \leqslant k-1$ and $0 \leqslant Y(v) \leqslant p(t-1) + (p-1)$. Thus the drawing is a $k \times tp$ grid drawing. □

An example of a grid drawing produced by Theorem 1 is shown in Fig. 1. By Bertrand's Postulate and the Prime Number Theorem we have the following corollary of Theorem 1.

Corollary 2. *For all $k \geqslant 1$ and $t \geqslant 1$, the balanced complete k-partite graph $K(t,k)$ on $n = kt$ vertices has a $k \times 2n$ grid drawing. For all $\varepsilon > 0$, there exists k_ε such that for all $k \geqslant k_\varepsilon$ and $t \geqslant 1$, $K(t,k)$ has a $k \times (1+\varepsilon)n$ grid drawing.*

We now prove that the upper bound in Theorem 1 is asymptotically optimal.

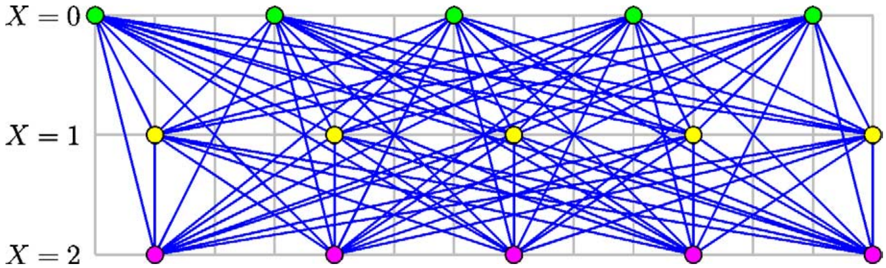

Fig. 1. The (rotated and scaled) grid drawing of $K(5,3)$ produced by Theorem 1.

Theorem 3. *Every grid drawing of $K(k,t)$ has area at least $\frac{1}{4}k^2t = \frac{1}{4}kn$.*

Proof. Consider a $w \times h$ grid drawing of $K(t,k)$. Let the *y-row* be the set of vertices with a Y-coordinate of y, and the *x-column* be the set of vertices with an X-coordinate of x. For each colour $0 \leqslant i \leqslant k-1$, let r_i be the number of rows containing a vertex coloured i, and let c_i be the number of columns containing a vertex coloured i. Then the arithmetic and harmonic means of $\{c_i\colon 0 \leqslant i \leqslant k-1\}$ satisfy the following (see [1] for example):

$$\left(\frac{1}{k}\sum_i c_i\right)\left(\frac{1}{k}\sum_i \frac{1}{c_i}\right) \geqslant 1.$$

Clearly $t \leqslant c_i r_i$ for each $0 \leqslant i \leqslant k-1$. Thus $\frac{1}{c_i} \leqslant \frac{r_i}{t}$ and

$$\left(\sum_i c_i\right)\left(\sum_i r_i\right) \geqslant k^2 t.$$

In each row and column there is at most two distinct colours, as otherwise there would be a 3-cycle contained in that row or column. Hence $\sum_i c_i \leqslant 2w$ and $\sum_i r_i \leqslant 2h$, which implies that $4wh \geqslant k^2t$. Thus $wh \geqslant \frac{1}{4}k^2t$. □

In the following result we generalise Theorem 1 for arbitrary k-colourable graphs, and introduce the aspect ratio as a parameter. This result suggests a trade-off between small area and small aspect ratio.

Theorem 4. *Let G be a k-colourable graph with n vertices. For every integer r such that $1 \leqslant r \leqslant \frac{n}{k}$, G has a $\frac{2n}{r} \times 4n$ grid drawing, which has area $\frac{8n^2}{r}$ and aspect ratio $2r$.*

Proof. Consider a k-colouring of G. Partition each colour class into sets each with exactly r vertices except for one set with at most r vertices. There are at most $\frac{n}{r}$ sets of size r, and at most k smaller sets, one for each colour class. Since $r \leqslant \frac{n}{k}$, the total number of sets is at most $\frac{2n}{r}$. Thus we have a $\lfloor\frac{2n}{r}\rfloor$-colouring of G such that each colour class has at most r vertices. Hence G is a subgraph of $K(r, \lfloor\frac{2n}{r}\rfloor)$. By Corollary 2, G has a $\frac{2n}{r} \times 4n$ grid drawing. □

Observe that with $r = \lfloor\frac{n}{k}\rfloor$ the drawing in Theorem 4 is $\mathcal{O}(k) \times \mathcal{O}(n)$ with area $\mathcal{O}(kn)$.

3. Conclusion

We conclude with some bibliographic remarks and conjectures. Note that a number of ideas in the proofs of Theorems 1 and 4 are from results by Pach et al. [6] and Dujmović et al. [4] regarding three-dimensional grid drawings (with no crossings). In turn, these proofs date to the seminal construction by Erdős [5] for the no-three-in-line problem. This problem introduced in 1917 by Dudeney [3] asks, what is the maximum number of points in the $n \times n$ grid with no three points collinear? Clearly θ is a grid drawing of a complete graph $K_n = (V, E)$ if and only if $\{\theta(v)\colon\ v \in V\}$ is a set of gridpoints with no three collinear. Thus the problem of producing a grid drawing with small area for any given graph can be viewed as a generalisation of the no-three-in-line problem. Note that Theorem 1 applied to a complete graph produces the no-three-in-line construction of Erdős [5].

Conjecture 5. *The lower bound in Theorem 3 can be improved to* $\frac{1}{2}kn$. (*This is clearly the minimum area for a grid drawing of the balanced complete bipartite graph* $K(\frac{n}{2}, 2)$.)

Conjecture 6. *Every grid drawing of* any *complete* k*-partite graph with* n *vertices has* $\Omega(kn)$ *area.*

Conjecture 7. *Every grid drawing of an* n*-vertex* $K(k, t)$ *with aspect ratio* r *has* $\Omega(\frac{n^2}{r})$ *area.*

Conjecture 7 would establish a trade-off between small area and small aspect ratio.

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