On Pseudo Vertex-Magic and Edge-Magic Total-Labellings

David R. Wood*

Basser Department of Computer Science
The University of Sydney
Sydney NSW 2006
Australia
davidw@cs.usyd.edu.au

Abstract. To date the study of graph labellings has focused on finding classes of graphs which admit a particular type of labelling. Here we consider variations of the well-known edge-magic and vertex-magic labellings for which all graphs admit such a labelling. In particular we consider two types of labellings of the vertices and edges of a graph with distinct positive integers: (1) for every edge the sum of its label and those of its end-vertices is some constant (*pseudo edge-magic*); and (2) for every vertex the sum of its label and those of the edges incident to it is some constant (*pseudo vertex-magic*). Our aim is to minimise the constant, called the *magic* number, associated with the labelling. We present lower and upper bounds on the magic number in pseudo edge-magic and pseudo vertex-magic labellings of complete graphs, trees and arbitrary graphs. In a number of cases these bounds are within a constant factor.

1 Introduction

To date the study of graph labellings has focused on finding classes of graphs which admit a particular type of labelling, the more well-known including graceful, harmonious and various types of magic labellings (see Gallian [18]). In this paper we relax the definition of the well-known edge-magic and vertex-magic labelling schemes so that every graph admits such a labelling. Our aim is to then minimise the magic number associated with such a labelling.

We consider simple connected finite graphs $G = (V, E)$ with $n = |V|$ vertices, $m = |E|$ edges and with maximum degree Δ. The set of edges $\{vw \in E\}$ incident to a vertex v is denoted by $E(v)$. We consider labellings of the vertices and/or edges of a graph with positive integers such that distinct graph elements receive distinct labels. More formally, a *vertex-labelling* of a graph $G = (V, E)$ is an injective function $\lambda : V \to \mathbb{N}$; an *edge-labelling* of G is an injective function $\lambda : E \to \mathbb{N}$; and a *total-labelling* of G is an injective function $\lambda : V \cup E \to \mathbb{N}$.

* Supported by the Australian Research Council Large Grant A49906214. Partially completed while visiting the School of Computer Science, McGill University, Montréal, Canada.

Pseudo Magic Labellings: A total-labelling λ of G is *pseudo edge-magic* if for every edge $vw \in E$,

$$\lambda(v) + \lambda(vw) + \lambda(w) \ = \ \kappa \ ,$$

for some constant κ. A total-labelling λ of G is *pseudo vertex-magic* if for every vertex $v \in V$,

$$\lambda(v) + \sum_{vw \in E(v)} \lambda(vw) \ = \ \kappa \ ,$$

for some constant κ. The constant κ associated with a pseudo edge-magic or pseudo vertex-magic total-labelling is called the *magic number* of the labelling.

Clearly K_2 does not have a pseudo vertex-magic total-labelling. In fact it is the only such graph. Therefore all statements in this paper concerning vertex-magic total-labellings of a graph G implicitly assume $G \neq K_2$.

If a pseudo edge-magic (respectively, vertex-magic) total-labelling does not have a vertex or edge labelled 1 then reducing each label by one less than the minimum label produces a pseudo edge-magic (vertex-magic) total-labelling. A pseudo edge-magic (vertex-magic) total-labelling with a vertex or edge labelled 1 is called *minimal*.

In this paper we show that all graphs admit pseudo edge-magic and pseudo vertex-magic labellings. We consider the problems of minimising the magic number in pseudo edge-magic and pseudo vertex-magic labellings, which we conjecture are NP-hard problems.

(Non-pseudo) Magic Labellings: A pseudo edge-magic (respectively, pseudo vertex-magic) total-labelling

$$\lambda : V \cup E \rightarrow \{1, 2, \ldots, m+n\}$$

is called *edge-magic* (*vertex-magic*). Here the labels are restricted to be consecutive positive integers. A graph admitting an edge-magic (vertex-magic) total-labelling is called *edge-magic* (*vertex-magic*). Numerous classes of graphs have been identified as been edge-magic or not edge-magic [13, 20, 28, 35, 58], and vertex-magic or not vertex-magic [5, 6]. The following results are known about edge-magic total-labellings.

- The n-vertex complete graph K_n is edge-magic iff $n \in \{1, 2, \ldots, 6\}$ [29] (see also [12]).
- The complete bipartite graph $K_{a,b}$ is edge-magic [28].
- The n-vertex cycle C_n is edge-magic [28] (see also [20]).
- All caterpillars are edge-magic [28]. It is conjectured that every tree is edge-magic [28] (see also [42]).
- The n-vertex path P_n is edge-magic [58].
- The n-vertex sun S_n (consisting of an n-cycle with an additional vertex of degree 1 adjacent to each vertex in the cycle) is edge-magic [58].

- The n-spoke wheel W_n is edge-magic for $n \equiv 0 \pmod 4$ [58], $n \equiv 6 \pmod 8$ [49] and $n \equiv 1 \pmod 4$ [58], W_n is not edge-magic for $n \equiv 3 \pmod 4$ [58], and it is conjectured that W_n is edge-magic for $n \equiv 2 \pmod 4$ [58].
- The Petersen graph is edge-magic [58].
- The union of disjoint copies of n-vertex cycles C_n is edge-magic [59].
- The union of disjoint copies of n-vertex paths P_n is edge-magic [59].
- A Δ-regular n-vertex graph with $n \equiv 4 \pmod 8$ and Δ odd is not edge-magic [12].
- A Δ-regular n-vertex m-edge graph with $n + m \equiv 2 \pmod 4$, m even and Δ odd is not edge-magic [42].

Enomoto, Llado, Nakamigawa, and Ringel [13] and Enomoto, Masuda, and Nakamigawa [14] studied a more restrictive form of edge-magic total-labelling called *super edge-magic*. Here the vertices are labelled $1, 2, \dots, n$ and the edges are labelled $n+1, n+2, \dots, n+m$. The following results are known about super edge-magic total-labellings.

- A super edge-magic graph has $m \leq 2n - 3$ [13].
- The n-vertex complete graph K_n is super edge-magic iff $n \in \{1, 2, 3\}$ [13].
- The n-vertex cycle C_n is super edge-magic iff n is odd [13].
- The complete bipartite graph $K_{a,b}$ is super edge-magic iff $a = 1$ or $b = 1$; that is, $K_{a,b}$ is a star [13].
- The n-vertex wheel W_n is not super edge-magic if $n \equiv 0 \pmod 4$ [13].
- Every n-vertex tree with $n \leq 15$ is super edge-magic [13]; it is conjectured that every tree is super edge-magic [13]. (The edge-magic labellings of caterpillars in [28] are actually super edge-magic.)
- Every graph can be embedded in a connected super edge-magic graph as an induced subgraph [14].

The following results are known about vertex-magic total-labellings (also see [37–39, 50]).

- Every generalised Petersen graph $P(n, m)$, $n \geq 3, 1 \leq m \leq \lfloor \frac{1}{2}(n-1) \rfloor$ has a vertex-magic total-labelling [5].

Antimagic Labellings: There are important relationships between magic total-labellings and so-called antimagic labellings, which are described in Section 2.3 and Section 2.2. A vertex-labelling λ of a graph $G = (V, E)$ is *pseudo edge-antimagic* if for all distinct edges $vw, xy \in E$,

$$\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y) \ .$$

A pseudo edge-antimagic vertex-labelling $\lambda : V \to \{1, 2, \dots, n\}$ is called *edge-antimagic*. We prove that every tree has an edge-antimagic vertex-labelling.

An edge-labelling λ of a graph $G = (V, E)$ is *pseudo vertex-antimagic* if for all distinct vertices $v, w \in V$,

$$\sum_{vx \in E(v)} \lambda(vx) \neq \sum_{wx \in E(w)} \lambda(wx) \ .$$

A pseudo vertex-antimagic edge-labelling $\lambda : E \to \{1, 2, \ldots, m\}$ is called *vertex-antimagic*. We show that every graph has a pseudo vertex-antimagic edge-labelling with polynomial sized labels. The best present result, due to Bodendiek and Walther [6], used exponential sized labels.

This paper: This paper is organised as follows. In Section 2 we give some introductory results including a duality structure and a method for determining pseudo magic total-labellings from a given antimagic labelling. Section 3 describes lower bounds for the magic number in pseudo edge-magic and vertex-magic total-labellings. We use these lower bounds to prove that the constructions in the latter sections are often within a constant factor of optimal. In Section 4 we establish a parallel between pseudo edge-antimagic vertex-labellings of complete graphs and so-called Sidon sequences; we derive pseudo edge-magic total-labellings of complete graphs whose magic number is within a constant factor of optimal. In Section 5 we present the first known algorithm for constructing a (non-pseudo) edge-antimagic vertex-labelling of a tree. We then give an algorithm for determining a pseudo vertex-antimagic edge-labelling of a tree. We derive pseudo edge-magic and pseudo vertex-magic total-labellings of trees. Section 6 describes greedy algorithms for determining pseudo edge-magic and pseudo vertex-magic total-labellings of arbitrary graphs. The latter is based on our corresponding algorithm for trees and the bound on the magic number is within a constant factor of optimal.

2 Basic Results

2.1 Duality

There is a well-known duality structure for edge-magic total-labellings which can be extended to pseudo edge-magic total-labellings. Given a total-labelling λ of a graph $G = (V, E)$, let $\hat{\lambda}$ be the maximum label assigned to a vertex or edge of G. The *dual* labelling λ' is given by

$$\lambda'(x) = 1 + \hat{\lambda} - \lambda(x) \ ,$$

for all vertices and edges $x \in V \cup E$.

Lemma 1. *Let λ be a minimal pseudo edge-magic total-labelling of a graph $G = (V, E)$ with magic number κ. Then λ' is also a minimal pseudo edge-magic total-labelling of a G, and $\hat{\lambda} = \hat{\lambda}'$. If κ' is the magic number of λ' then* $\min\{\kappa, \kappa'\} \leq \frac{3}{2}(\hat{\lambda} + 1)$.

Proof. Clearly the labels of λ' are all positive integers. For all edges $vw \in E$, $\lambda'(v) + \lambda'(vw) + \lambda'(w) = 3(\hat{\lambda} + 1) - (\lambda(v) + \lambda(vw) + \lambda(w))$. Since $\lambda(v) + \lambda(vw) + \lambda(w)$ is constant, λ' is pseudo edge-magic with magic number $\kappa' = 3(\hat{\lambda} + 1) - \kappa$. The maximum label of λ' is

$$\hat{\lambda}' \ = \ \hat{\lambda} + 1 - \min_{x \in V \cup E} \lambda(x) \ .$$

Since λ is minimal, $\hat{\lambda}' = \hat{\lambda}$. Suppose $\kappa > \frac{3}{2}(\hat{\lambda}+1)$. Then $\kappa' < 3(\hat{\lambda}+1) - \frac{3}{2}(\hat{\lambda}+1) = \frac{3}{2}(\hat{\lambda}+1)$. Thus $\min\{\kappa, \kappa'\} \leq \frac{3}{2}(\hat{\lambda}+1)$. □

There is a similar notion of duality for pseudo vertex-magic total-labellings of regular graphs.

Lemma 2. *Let λ be a minimal pseudo vertex-magic total-labelling of a Δ-regular graph $G = (V, E)$ with magic number κ. Then λ' is also a minimal pseudo vertex-magic total-labelling of G, and $\hat{\lambda} = \hat{\lambda}'$. If κ' is the magic number of λ' then* $\min\{\kappa, \kappa'\} \leq \frac{1}{2}(\Delta+1)(\hat{\lambda}+1)$.

Proof. Clearly the labels of λ' are all positive integers. For all nodes $v \in V$,

$$\lambda'(v) + \sum_{vw \in E(v)} \lambda'(vw) \;=\; (\deg(v)+1)(\hat{\lambda}+1) - \left(\lambda(v) + \sum_{vw \in E(v)} \lambda(vw)\right) \;.$$

Since $\lambda(v) + \sum_{vw \in E(v)} \lambda(vw)$ is constant, and $\deg(v) = \Delta$ for all vertices $v \in V$, λ' is pseudo vertex-magic with magic number $\kappa' = (\Delta+1)(\hat{\lambda}+1) - \kappa$. The maximum label of λ' is

$$\hat{\lambda}' \;=\; \hat{\lambda} + 1 - \min_{x \in V \cup E} \lambda(x) \;.$$

Since λ is minimal, $\hat{\lambda}' = \hat{\lambda}$. Suppose $\kappa > \frac{1}{2}(\Delta+1)(\hat{\lambda}+1)$. Then $\kappa' < (\Delta+1)(\hat{\lambda}+1) - \frac{1}{2}(\Delta+1)(\hat{\lambda}+1) = \frac{1}{2}(\Delta+1)(\hat{\lambda}+1)$. Thus $\min\{\kappa, \kappa'\} \leq \frac{1}{2}(\Delta+1)(\hat{\lambda}+1)$. □

2.2 Pseudo Edge-Antimagic Vertex-Labellings

There are important relationships between magic total-labellings and antimagic labellings, which we now explore. Recall that a vertex-labelling λ of a graph $G = (V, E)$ is pseudo edge-antimagic if for all distinct edges $vw, xy \in E$, $\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y)$.

Observation 1. *In a pseudo edge-magic total-labelling the induced vertex-labelling is pseudo edge-antimagic.*

Proof. Let λ be a pseudo edge-magic total-labelling of a graph $G = (V, E)$. Then for all distinct edges $vw, xy \in E$, $\lambda(v) + \lambda(w) + \lambda(vw) = \lambda(x) + \lambda(y) + \lambda(xy)$. Since $\lambda(vw) \neq \lambda(xy)$, we have $\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y)$; that is, λ is a pseudo edge-antimagic vertex-labelling. □

Moreover, a pseudo edge-antimagic vertex-labelling can be extended to produce a pseudo edge-magic total-labelling by the following algorithm.

Algorithm 1. Extend Pseudo Edge-Antimagic Vertex-Labelling
Input: pseudo edge-antimagic vertex-labelling λ of a graph $G = (V, E)$
Output: pseudo edge-magic total-labelling λ of G

---

1. Set $\Lambda_V \leftarrow \max_{x \in V} \lambda(x)$
2. Set $\Lambda_E \leftarrow \max_{xy \in E} \lambda(x) + \lambda(y)$
3. Set $\kappa \leftarrow \min \left\{ \begin{array}{l} i \ : i \neq \lambda(x) + \lambda(v) + \lambda(w), \\ \qquad x \in V, vw \in E, \Lambda_E + 1 \leq i \leq \Lambda_E + \Lambda_V + 1 \end{array} \right\}$
4. For each edge $vw \in E$, set $\lambda(vw) \leftarrow \kappa - \lambda(v) - \lambda(w)$

Lemma 3. *The algorithm* Extend Pseudo Edge-Antimagic Vertex-Labelling *determines a pseudo edge-magic total-labelling λ of G which preserves the given vertex-labelling. Of all such total-labellings, λ has the minimum magic number, which is at most $\Lambda_E + \Lambda_V + 2 \leq 3\Lambda_V$.*

Proof. For every edge $vw \in E$, $\Lambda_E \geq \lambda(v) + \lambda(w)$, and for every vertex $x \in V$, $\Lambda_V \geq \lambda(x)$. Hence $\Lambda_E + \Lambda_V + 1 > \lambda(x) + \lambda(v) + \lambda(w)$. Therefore with $i = \Lambda_E + \Lambda_V + 1$, the conditions of Step 3 are satisfied; thus Step 3 always assigns a value to κ.

We now show that λ is a pseudo edge-magic total-labelling with magic number κ. For all edges $vw \in E$, $\lambda(vw) = \kappa - \lambda(v) - \lambda(w)$ with κ constant. Since $\lambda(v) + \lambda(w)$ is unique in a pseudo edge-antimagic labelling, edge-labels are distinct. Now, κ is chosen so that for all edges $vw \in E$ and for all vertices $x \in V$, $\kappa \neq \lambda(x) + \lambda(v) + \lambda(w)$. Hence $\kappa - \lambda(v) - \lambda(w) \neq \lambda(x)$. Thus $\lambda(vw) \neq \lambda(x)$, and therefore all labels are distinct. For each edge $vw \in E$, $\lambda(v) + \lambda(vw) + \lambda(w) = \kappa$. Hence λ is a pseudo edge-magic total-labelling with magic number κ.

Suppose, to the contrary, that there exists a pseudo edge-magic total-labelling λ of G which preserves the given vertex-labelling and with magic number $\kappa' < \kappa$. Let vw be the edge with $\lambda(v) + \lambda(w) = \Lambda_E$. Then $\lambda(v) + \lambda(w) + \lambda(vw) = \kappa'$, thus $\Lambda_E + \lambda(vw) = \kappa'$, and since $\lambda(vw) \geq 1$ we have $\kappa' - \Lambda_E \geq 1$. Now, for all edges $vw \in E$ and for all vertices $x \in V$, $\lambda(vw) \neq \lambda(x)$. Thus $\kappa' - \lambda(v) - \lambda(w) \neq \lambda(x)$, and $\kappa' \neq \lambda(x) + \lambda(v) + \lambda(w)$. Thus $\kappa' < \kappa$ satisfies the conditions of Step 3 of the algorithm, which is a contradiction, since κ is chosen to be minimum. Therefore κ is the minimum magic number in a pseudo edge-magic total-labelling of G which preserves the given pseudo edge-antimagic vertex-labelling.

Now, $\kappa \leq \Lambda_E + \Lambda_V + 1$. Since edge-labels are distinct, $\Lambda_E \leq 2\Lambda_V - 1$. Thus $\kappa \leq 2\Lambda_V - 1 + \Lambda_V + 1 = 3\Lambda_V$. □

2.3 Pseudo Vertex-Antimagic Edge-Labellings

Recall that an edge-labelling λ of a graph $G = (V, E)$ is pseudo vertex-antimagic if for all distinct vertices $v, w \in V$,

$$\sum_{vx \in E(v)} \lambda(vx) \neq \sum_{wx \in E(w)} \lambda(wx) \ .$$

Observation 2. *In a pseudo vertex-magic total-labelling the induced edge-labelling is pseudo vertex-antimagic.*

Proof. Let λ be a pseudo vertex-magic total-labelling of a graph $G = (V, E)$. Then for all distinct vertices $v, w \in V$.

$$\lambda(v) + \sum_{vx \in E(v)} \lambda(vx) \;=\; \lambda(w) + \sum_{wx \in E(w)} \lambda(wx) \;.$$

Since $\lambda(v) \neq \lambda(w)$, λ is pseudo vertex-antimagic. □

A pseudo vertex-antimagic edge-labelling can be extended to produce a vertex-magic total-labelling by the following algorithm.

Algorithm 2. Extend Pseudo Vertex-Antimagic Edge-Labelling
Input: pseudo vertex-antimagic edge-labelling λ of a graph $G = (V, E)$
Output: pseudo vertex-magic total-labelling λ of G

---

1. Set $\Lambda_E \leftarrow \max\{\lambda(vw) : vw \in E\}$.
2. For each vertex $v \in V$, set $S_v \leftarrow \sum_{vx \in E(v)} \lambda(vx)$.
3. Set $\Lambda_V \leftarrow \max\{S_v : v \in V\}$.
4. Set $\kappa \;\leftarrow\; \min \left\{ \begin{array}{l} i \;: \Lambda_V + 1 \leq i \leq \Lambda_V + \Lambda_E + 1 \\ \quad\; i \neq \lambda(vw) + S_x, x \in V, vw \in E \end{array} \right\}$
5. For each vertex $v \in V$, set $\lambda(v) \;\leftarrow\; \kappa - S_v$

Lemma 4. *The algorithm* Extend Pseudo Vertex-Antimagic Edge-Labelling *determines a pseudo vertex-magic total-labelling λ of G which preserves the given edge-labelling. Of all such total-labellings, λ has the minimum magic number, which is at most $(\Delta + 1)\Lambda_E$.*

Proof. For all vertices $x \in V$, $S_x \leq \Lambda_V$, and for all edges $vw \in E$, $\lambda(vw) \leq \Lambda_E$; thus $\lambda(vw) + S_x < \Lambda_E + \Lambda_V + 1$. Thus the condition of Step 4 of the algorithm is always meet with $i = \Lambda_E + \Lambda_V + 1$, and therefore κ is set to some value.

We now show that λ is pseudo vertex-magic. For all vertices $v \in V$, $\lambda(v) = \kappa - S_v$, where κ is a constant. Since S_v is unique in a pseudo vertex-antimagic labelling, $\lambda(v)$ is unique. Now, κ is chosen so that for all vertices $x \in V$ and for all edges $vw \in E$, $\kappa - S_x \neq \lambda(vw)$. Hence $\lambda(x) \neq \lambda(vw)$. Thus vertex-labels are distinct from edge-labels. Thus all labels are distinct. For all vertices $v \in V$, $\lambda(v) + S_v = \kappa$. Therefore λ is pseudo vertex-magic total-labelling of G.

We now show that κ is the minimum magic number of a pseudo vertex-magic total-labelling which preserves the given edge-labelling. Suppose, to the contrary, that there exists a pseudo vertex-magic total-labelling of G which preserves the give edge-labelling and with magic number $\kappa' < \kappa$. Let v be the vertex with $S_v = \Lambda_V$. Then $\lambda(v) + \Lambda_V = \kappa'$. Since $\lambda(v) \geq 1$, $\kappa \geq \Lambda_V + 1$. Now, for all edges $vw \in E$ and for all vertices $x \in V$, $\lambda(vw) \neq \lambda(x)$. Hence $\lambda(vw) \neq \kappa' - S_x$, and thus $\kappa' \neq \lambda(vw) + S_x$. Therefore $\kappa' < \kappa$ satisfies the conditions of Step 4 of the algorithm, which is a contradiction since κ is chosen to be minimum.κ is the minimum magic number of a pseudo vertex-magic total-labelling which preserves the given edge-labelling. Therefore κ is the minimum magic number of a pseudo vertex-magic total-labelling which preserves the given edge-labelling.

We now prove the upper bound on κ. κ is chosen so that $\kappa \leq \Lambda_V + \Lambda_E + 1$. Let v be the vertex with $S_v = \Lambda_V$. If $\deg(v) = 1$ then $\kappa \leq 2\Lambda_E + 1 \leq 3\Lambda_E \leq (\Delta + 1)\Lambda_E$ since $G \neq K_2$. If $\deg(v) \geq 2$ then

$$\Lambda_V = S_v \leq \sum_{i=1}^{\deg(v)} (\Lambda_E - i + 1) \leq \Delta\Lambda_E - 1 \enspace .$$

Therefore $\kappa \leq (\Delta + 1)\Lambda_E$. □

Bodendiek and Walther [6] show that every graph has a pseudo vertex-antimagic edge-labelling. We repeat their proof for completeness. Label the edges $\lambda(e_i) \leftarrow 2^{i-1}$, $i = 1, 2, \ldots, m$. The sum of the labels of the edges incident to a particular vertex v can then be interpreted as the binary representation of a natural number. For distinct vertices v and w of a connected graph with at least two edges, the set of edges incident to v is different to the set of edges incident to w. Hence the sum of the labels of edges incident to v is different to the sum of the labels of edges incident to w. Thus the defined edge-labelling is pseudo vertex-antimagic with maximum label 2^{m-1}. By Lemma 4, a pseudo vertex-magic total labelling can be then be constructed with magic number at most $(\Delta + 1)2^{m-1}$.

In this paper we show that every graph has a pseudo vertex-antimagic edge-labelling with polynomial sized labels (see Section 6). In particular, the maximum label is at most $m + 2n - 4$, and hence, by Lemma 4, every graph has a pseudo vertex-magic total-labelling with magic number at most $(\Delta+1)(m+2n-4)$. It is expected that better upper bounds are achievable for the maximum label in a pseudo vertex-antimagic edge-labelling. In fact, Ringel and Llado [42] (see also Hartsfield and Ringel [22]) conjecture that every graph has a (non-pseudo) vertex-antimagic edge-labelling; that is, a vertex-antimagic edge-labelling with labels $1, 2, \ldots, m$. By Lemma 4, this would imply that every graph has a pseudo vertex-magic total-labelling with magic number at most $(\Delta + 1)m$.

3 Lower Bounds

The following lower bounds for the magic number in pseudo edge-magic and pseudo vertex-magic total-labellings are proved using a double-counting argument.

Theorem 1. *Let $G = (V, E)$ be an n-vertex m-edge graph with maximum degree Δ and no isolated vertices. Let n_i be the number of vertices with degree i, and let N_i be the number of vertices with degree at least i. The magic number in a pseudo edge-magic total-labelling of G is at least*

$$n + \frac{1}{2}(m + 1) \;+\; \frac{1}{m} \sum_{i=1}^{\Delta} n_i i \left(N_{i+1} + \frac{1}{2}(n_i + 1) \right) \enspace .$$

Proof. Consider a pseudo edge-magic total-labelling λ of G with magic number κ. For every edge $vw \in E$, $\lambda(v) + \lambda(vw) + \lambda(w) = \kappa$. Hence,

$$\kappa \cdot m \;=\; \sum_{vw \in E} \lambda(v) + \lambda(vw) + \lambda(w) = \sum_{vw \in E} \lambda(vw) \;+\; \sum_{v \in V} \deg(v) \cdot \lambda(v) \;\; . \quad (1)$$

Since $\deg(v) \geq 1$ for all vertices v, (1) is minimised when edges are labelled $\{n+1, n+2, \ldots, n+m\}$, and vertices are labelled $\{1, 2, \ldots, n\}$ with small labels are applied to vertices of high degree, and high labels are applied to vertices of low degree. That is, the vertices of degree i, $1 \leq i \leq \Delta$, are labelled $\{N_{i+1} + 1, N_{i+1} + 2, \ldots, N_{i+1} + n_i (= N_i)\}$. Thus,

$$\begin{aligned}
\kappa \cdot m \;&\geq\; \sum_{j=1}^{m} (n+j) \;+\; \sum_{i=1}^{\Delta} i \sum_{k=1}^{n_i} (N_{i+1} + k) \\
\kappa \cdot m \;&\geq\; nm + \frac{1}{2} m(m+1) \;+\; \sum_{i=1}^{\Delta} i \, \left(N_{i+1} \cdot n_i + \frac{1}{2} n_i (n_i + 1) \right) \\
\kappa \;&\geq\; n + \frac{1}{2}(m+1) \;+\; \frac{1}{m} \sum_{i=1}^{\Delta} n_i i \, \left(N_{i+1} + \frac{1}{2}(n_i + 1) \right) \;\; .
\end{aligned}$$

□

Theorem 1 enables us to prove a simpler lower bound for regular graphs.

Corollary 1. *The magic number of a pseudo edge-magic total-labelling of an n-vertex m-edge regular graph is at least $\frac{1}{2}(m+3) + 2n$.*

Proof. Suppose there exists a pseudo edge-magic total-labelling of an n-vertex m-edge Δ-regular graph with magic number κ. By Theorem 1, $\kappa \geq n + \frac{1}{2}(m + 1) + \frac{n\Delta}{m} \left(0 + \frac{1}{2}(n+1)\right) \;=\; n + \frac{1}{2}(m+1) + (n+1) \;=\; \frac{1}{2}(m+3) + 2n$. □

A similar lower bound to Theorem 1 can be proved for pseudo vertex-magic total-labellings.

Theorem 2. *The magic number in a pseudo vertex-magic total-labelling of an n-vertex m-edge graph is at least $\frac{m}{n}(m+1) + m + \frac{1}{2}(n+1)$.*

Proof. Suppose there exists a pseudo vertex-magic total-labelling λ of an n-vertex m-edge graph $G = (V, E)$ with magic number κ. For all vertices $v \in V$,

$$\lambda(v) \;+\; \sum_{vx \in E(v)} \lambda(vx) \;=\; \kappa \;\; .$$

Hence,

$$\kappa \cdot n \;=\; \sum_{v \in V} \left(\lambda(v) \;+\; \sum_{vx \in E(v)} \lambda(vx) \right) \;=\; 2 \sum_{vw \in E} \lambda(vw) \;+\; \sum_{v \in V} \lambda(v) \;\; . \quad (2)$$

(2) is minimised if edges are assigned small labels and vertices are assigned large labels. Since all labels are distinct positive integers,

$$\kappa \cdot n \;\geq\; 2\sum_{i=1}^{m} i \;+\; \sum_{i=1}^{n}(m+i) \;=\; m(m+1)+nm+\frac{1}{2}n(n+1)$$

$$\kappa \geq \frac{m(m+1)}{n}+m+\frac{1}{2}(n+1)\ . \qquad \square$$

Corollary 2. *The magic number in a pseudo vertex-magic total-labelling of an n-vertex m-edge Δ-regular graph is at least $(\frac{1}{2}\Delta+1)m+\frac{1}{2}(n+1+\Delta)$.*

Proof. Suppose a Δ-regular graph has a pseudo vertex-magic total-labelling with magic number κ. By Theorem 2, $\kappa \geq \frac{m}{n}(m+1)+m+\frac{1}{2}(n+1)=\frac{1}{2}\Delta(m+1)+m+\frac{1}{2}(n+1)=(\frac{1}{2}\Delta+1)m+\frac{1}{2}(n+1+\Delta)$. $\square$

4 Labellings of Complete Graphs

In this section we show that pseudo edge-magic total-labellings of complete graphs are closely related to the so-called Sidon sequences [47, 51], which have recently been rediscovered under the guise of *well-spread* sequences [41]. The positive integers $a_1, a_2, \dots, a_n$ are called a *Sidon* sequence (or a B_2 sequence[1]), if $a_i + a_j \neq a_k + a_l$ for all $i, j, k, l \in \{1, 2, \dots, n\}$; see Halberstam and Roth [21] for an overview. The next result follows immediately from the definitions of a Sidon sequence and a pseudo edge-antimagic vertex-labelling.

Observation 3. *There exists a pseudo edge-antimagic vertex-labelling of the complete graph K_n with maximum label M if and only if there exists a Sidon sequence $a_1 < a_2 < \cdots < a_n = M$.*

Theorem 3. *For all n, the complete graph K_n has a pseudo edge-magic total-labelling with magic number at most $3n^2 + o(n^2)$.*

Proof. Define $f(x)$ to be the maximum number of elements from $\{1, 2, \dots, x\}$ which form a Sidon sequence, and $g(n)$ to be the minimum maximum element taken over all Sidon sequences of length n; that is, g is the inverse function of f, and a lower bound for f or an upper bound for g corresponds to the construction of a Sidon sequence. An elementary result of Sidon [47] is that $f(x) = \Omega(x^{1/4})$. A much stronger result is that $f(x) = \sqrt{x} + o(\sqrt{x})$ (see Halberstam and Roth [21, Theorem 7]; in particular, Erdös and Turán [15] show that $f(x) < \sqrt{x} + O(x^{1/4})$ and Bose and Chowla [10] show that $f(x) > \sqrt{x} - O(x^{5/16})$). Since f and g are inverse functions, $g(n) = n^2 + o(n^2)$; that is, there exists a Sidon sequence of length n with a maximum element of $n^2 + o(n^2)$. By Observation 3 there exists a pseudo edge-antimagic vertex-labelling of K_n with a maximum label of $n^2 + o(n^2)$. By Lemma 3, the algorithm Extend Pseudo Edge-Antimagic Vertex-Labelling determines a pseudo edge-magic total-labelling with magic number at most $3n^2 + o(n^2)$. $\square$

[1] $(a_1, a_2, \dots, a_n)$ is a B_h sequence ($h \geq 2$) if every integer x has at most one representation of the form $x = b_1 + b_2 + \cdots + b_h$ ($b_i \in \{a_1, a_2, \dots, a_n\}$) with $b_1 \leq b_2 \leq \cdots \leq b_h$.

By the lower bound in Corollary 1, the above upper bound for the magic number in a pseudo edge-magic total-labelling of a complete graph is within a constant factor of being optimal.

Given an n-vertex graph G, by deleting the appropriate edges from K_n we obtain the following result.

Corollary 3. *All n-vertex graphs have a pseudo edge-magic total-labelling with maximum label and magic number at most $3n^2 + o(n^2)$.* □

5 Labellings of Trees

In this section we present algorithms for constructing edge-antimagic vertex-labellings and pseudo vertex-antimagic edge-labellings of a tree. Given a tree $T = (V, E)$, let $\mathrm{dist}(v, w)$ be the graph-theoretic distance between vertices $v, w \in V$. Given a specified *root* vertex r, for each vertex $v \neq r$, let $\mathrm{parent}(v)$ be the unique vertex adjacent to v with $\mathrm{dist}(\mathrm{parent}(v), r) = \mathrm{dist}(v, r) - 1$.

5.1 Edge-Antimagic Vertex-Labellings

We now describe our algorithm for finding a (non-pseudo) edge-antimagic vertex-labelling of a tree $T = (V, E)$ rooted at some vertex $r \in V$. Define V_i, $i \geq 0$, to be the set of vertices $\{v \in V : \mathrm{dist}(v, r) = i\}$. Let

$$n_i = |V_i| \text{ and } S_i = \sum_{0 \leq j \leq i} |V_j| .$$

We construct an edge-antimagic vertex-labelling $\lambda : V \to \{1, 2, \ldots, n\}$ using the following breadth-first search strategy.

Algorithm 3. EDGE-ANTIMAGIC TREE VERTEX-LABELLING

Input: tree $T = (V, E)$ rooted at $r \in V$
Output: edge-antimagic vertex-labelling of T

---

set $\lambda(r) \leftarrow 1$
for $i = 1, 2, \ldots, \max\{\mathrm{dist}(v, r) : v \in V\}$ **do**
  let V_i be ordered $(v_{i,1}, v_{i,2}, \ldots, v_{i,n_i})$ by non-increasing $\lambda(\mathrm{parent}(v_{i,j}))$
  (that is, $1 \leq j < k \leq n_i \Rightarrow \lambda(\mathrm{parent}(v_{i,j})) \geq \lambda(\mathrm{parent}(v_{i,k}))$)
  set $\lambda(v_{i,j}) \leftarrow S_{i-1} + j$, $1 \leq j \leq n_i$
end-for

Lemma 5. *Given an n-vertex tree, the algorithm* EDGE-ANTIMAGIC TREE LABEL *determines an edge-antimagic vertex-labelling with labels $\{1, 2, \ldots, n\}$.*

Proof. The parents of vertices in V_i are in V_{i-1}. Hence the parents are labelled before their children, and thus the vertex-labelling is well-defined. Clearly the vertices are labelled $\{1, 2, \dots, n\}$.

We now show that the sums of the labels of the end-vertices of edges are distinct. Let $vw, xy \in E$ with $v = \text{parent}(w)$ and $x = \text{parent}(y)$. Suppose $v \in V_i$ and $x \in V_j$. Thus $w \in V_{i+1}$ and $y \in V_{j+1}$.

Consider the case that $i = j$. If $v = x$ then $\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y)$ since w and y receive different labels. Otherwise $v \neq x$. Assume without loss of generality that $\lambda(v) < \lambda(x)$. Then by the choice of labels for vertices in V_{i+1}, we have $\lambda(w) < \lambda(y)$. Hence $\lambda(v) + \lambda(w) < \lambda(x) + \lambda(y)$.

Now suppose that $i \neq j$ and assume without loss of generality that $i < j$. Then $\lambda(v)+\lambda(w) \leq S_i+S_{i+1}$ and $S_{j-1}+S_j+2 \leq \lambda(x)+\lambda(y)$. Clearly $S_i \leq S_{j-1}$ and $S_{i+1} \leq S_j$. Hence $\lambda(v) + \lambda(w) + 2 \leq \lambda(x) + \lambda(y)$, and $\lambda(v) + \lambda(w) < \lambda(x) + \lambda(y)$. Therefore the sum of the labels of the end-vertices of each edge is unique, and vertices are labelled $\{1, 2, \dots, n\}$; that is, the vertex-labelling is edge-antimagic. □

By Lemma 3 and Lemma 5 we have the following result.

Corollary 4. *Given an n-vertex tree T, the algorithms* Edge-Antimagic Tree Label *and* Extend Pseudo Edge-Antimagic Vertex-Labelling *determine a pseudo edge-magic total-labelling of T with magic number at most $3n$.*

The upper bound in Corollary 4 for arbitrary trees can be improved in the case of complete d-ary trees to a little more than $2n$.

Lemma 6. *An n-vertex complete d-ary tree has a pseudo edge-magic total-labelling with magic number $\frac{1}{d}\big(2(d+1)n-1\big)+2$.*

Proof. Let $T = (V, E)$ be the complete d-ary tree of height h for some $d \geq 2$ and $h \geq 1$. It is well-known that T has $n = \frac{1}{d-1}(d^{h+1}-1)$ vertices. Let r be the root-vertex of T, and apply the Edge-Antimagic Tree Vertex-Labelling algorithm to determine an edge-antimagic vertex-labelling of T.

Let $v_{i,j}$ be the jth vertex at depth i ($0 \leq i \leq h$, $1 \leq j \leq d^i$), where vertices within depth i are ordered by increasing label. The label of $v_{i,j}$ is $\lambda(v_{i,j}) = \frac{1}{d-1}(d^i-1)+j$. As proved in Lemma 5, for each edge of T, the sum of the labels of its end-vertices is unique. Clearly the edge from v_{h,d^h} to $v_{h-1,d^{h-1}}$ has the maximum sum of the labels of the end-vertices of an edge. Hence,

$$
\begin{aligned}
\Lambda_E &= \max\{\lambda(x)+\lambda(y) : xy \in E\} \\
&= \lambda(v_{h,d^h}) + \lambda(v_{h-1,d^{h-1}}) \\
&= \frac{d^h-1}{d-1} + d^h + \frac{d^{h-1}-1}{d-1} + d^{h-1} \\
&= \frac{d^h - 1 + (d^{h+1}-d^h) + d^{h-1} - 1 + (d^h - d^{h-1})}{d-1} \\
&= \frac{d^{h+1} - 1 + d^h - 1}{d-1} \;=\; \frac{d^{h+2} - d + d^{h+1} - d}{d(d-1)} \\
&= \frac{(d+1)(d^{h+1}-1) - (d-1)}{d(d-1)} \;=\; \frac{(d+1)n-1}{d} \quad .
\end{aligned}
$$

Applying Lemma 3 we obtain a pseudo edge-magic total-labelling with magic number at most $\Lambda_V + \Lambda_E + 2 = n + \frac{(d+1)n-1}{d} + 2 = \frac{(2d+1)n-1}{d} + 2$. □

We now evaluate the lower bound in Theorem 1 for the magic number in a pseudo edge-magic total-labelling of a complete d-ary tree.

Lemma 7. *In a pseudo edge-magic total-labelling of a complete d-ary tree the magic number is at least $\frac{(4d+1)n}{2d} - O(\frac{n}{d^2})$*

Proof. Let T be the complete d-ary tree of height h. T has $n = \frac{1}{d-1}(d^{h+1}-1)$ vertices, of which one has degree d, $d^h = \frac{d^{h+1}-1+1}{d} = \frac{(d-1)n+1}{d}$ have degree one, and the remainder have degree $d+1$; that is $n - d^h - 1 = n - \frac{1}{d}((d-1)n+1) - 1 = \frac{1}{d}(nd - (d-1)n - 1 - d) = \frac{1}{d}(n-1-d)$ vertices. By Theorem 1 (with $i=1$ and $i = \Delta = d+1$ only),

$$
\begin{aligned}
\kappa \;\geq\;& n + \frac{1}{2}(m+1) + \\
&+ \frac{1}{m}\left(\frac{(d-1)n+1}{d} \cdot \left(\left(\frac{n-d-1}{d}+1\right) + \frac{1}{2}\left(\frac{(d-1)n+1}{d}+1\right)\right)\right) \\
&+ \frac{1}{m}\left(\frac{n-d-1}{d} \cdot (d+1)\left(0 + \frac{1}{2}\left(\frac{n-d-1}{d}+1\right)\right)\right) \\
\geq\;& \frac{3n}{2} + \frac{1}{n-1} \cdot \frac{(d-1)n}{d} \cdot \left(\frac{n-1}{d} + \frac{dn-n+d+1}{2d}\right) \\
&+ \frac{1}{n-1}\left(n-(d+1)\right)\left(\frac{n-1}{2d}\right) \\
\geq\;& \frac{3n}{2} + \frac{1}{n-1} \cdot \frac{(d-1)n}{d} \cdot \frac{(d+1)n}{2d} + \frac{n-(d+1)}{2d} \\
\geq\;& \frac{3n}{2} + \frac{(d^2-1)(n+1)}{2d^2} + \frac{n-(d+1)}{2d} \\
\geq\;& \frac{3d^2n + d^2n + d^2 - n - 1 + dn - d^2 - d}{2d^2}
\end{aligned}
$$

$$\geq \frac{(4d+1)n}{2d} - O\Big(\frac{n}{d^2}\Big) \quad . \qquad \square$$

It follows from Lemma 6 and Lemma 7 that the ratio between our lower and upper bounds for the magic number in a pseudo edge-magic total-labelling of a complete d-ary tree is $1 + O(\frac{1}{d})$.

5.2 Pseudo Vertex-Antimagic Edge-Labellings

We now describe a breadth-first search algorithm for determining a pseudo vertex-antimagic edge-labelling of an arbitrary tree. This algorithm is used as the first step in Algorithm Greedy Pseudo Vertex-Antimagic Edge-Labelling in Section 6.2, and applying Algorithm Extend Pseudo Vertex-Antimagic Edge-Labelling we obtain a pseudo vertex-magic total-labelling of a tree.

Suppose λ is a pseudo vertex-antimagic partial edge-labelling of a graph $G = (V, E)$; that is, there is some set $E' \subseteq E$ such that λ restricted to E' is vertex-antimagic. Unlabelled edges $vw \in E \backslash E'$ are considered to have $\lambda(vw) = 0$. For each vertex $v \in V$, let

$$S_v = \sum_{vx \in E(v)} \lambda(vx) \quad .$$

Algorithm 4. Pseudo Vertex-Antimagic Tree Edge-Labelling

Input: tree $T = (V, E) \neq K_2$
Output: pseudo vertex-antimagic edge-labelling of T

---

choose a *root* vertex $r \in V$ with $\deg(r) \geq 2$
for $i = 1, 2, \ldots, \max\{\mathrm{dist}(v, r) : v \in V\}$ **do**
  for all edges $vw \in E$ with $v = \mathrm{parent}(w)$ and $\mathrm{dist}(w, r) = i$ **do**
    set $\lambda(vw)$ to be the minimum positive integer l such that:
      (1) for all labelled edges $xy \in E$, $l \neq \lambda(xy)$, and
      (2) for all nodes $x \in V \setminus \{v, w\}$, $l \neq S_x - S_v$ and $l \neq S_x$
  end-for
end-for

Lemma 8. *The Algorithm* Pseudo Vertex-Antimagic Tree Edge-Labelling *determines a pseudo vertex-antimagic edge-labelling of the given n-vertex tree T with maximum label at most $3n - 5$.*

Proof. A tree which is not a single edge has a vertex r with $\deg(r) \geq 2$. Suppose rx and ry are the first and second edges to be labelled by the algorithm. Then $\lambda(rx) = 1$ and $\lambda(ry) = 2$, and $S_r = 3$, $S_x = 1$ and $S_y = 2$. All other vertices v

have $S_v = 0$. We shall prove that the algorithm maintains the following property, which as we have just shown holds after two edges are labelled:

$$\text{For distinct vertices } v, w \in V \text{, if } S_v = S_w \text{ then } S_v = 0 = S_w. \tag{3}$$

Consider the iteration of the algorithm where the label l is chosen for the edge vw. Since $v = \text{parent}(w)$, all other edges incident to w are unlabelled, and hence $S_w = 0$. By (3), $S_v > 0$. The new value S'_v of S_v is $S_v + l$, the new value S'_w of S_w is l, and S_x does not change for any other vertex x. Since $S_v > 0$, we have $S'_v = S_v + l \neq l = S'_w$. For all nodes $x \in V \setminus \{v, w\}$, by the choice of l, we have $S'_v = S_v + l \neq S_x$ and $S'_w = l \neq S_x$. Hence (3) holds after labelling vw. Therefore (3) holds after all edges are labelled, in which case $S_v > 0$ for all vertices v; that is, for all distinct vertices $v, w \in V$, $S_v \neq S_w$. Hence the labelling is vertex-antimagic.

For each edge vw, there are at most $(m-1) + 2(n-2) = 3n - 6$ values which the label $\lambda(vw)$ cannot take on. Hence the maximum label is at most $3n - 5$. □

It is expected that better upper bounds are achievable for the maximum label in a pseudo vertex-antimagic edge-labelling. In fact, Hartsfield and Ringel [22] conjecture that every *graph* has a (non-pseudo) vertex-antimagic edge-labelling; that is, a vertex-antimagic edge-labelling with labels $1, 2, \ldots, m$.

By Lemma 4 and Lemma 8 we have the following result.

Corollary 5. *Given an n-vertex tree T with maximum degree Δ, the algorithms* Pseudo Vertex-Antimagic Tree Edge-Labelling *and* Extend Pseudo Vertex-Antimagic Edge-Labelling *determine a pseudo vertex-magic total-labelling of T with magic number at most $(\Delta + 1)(3n - 5)$.* □

6 Greedy Algorithms for Arbitrary Graphs

In this section we describe and analyse greedy algorithms for producing pseudo edge-magic and vertex-magic total-labellings. These algorithms sequentially choose the minimum possible label for the vertices or edges of the given graph.

6.1 Pseudo Edge-Magic Total-Labellings

Consider the following algorithm for determining an edge-antimagic vertex-labelling of an arbitrary graph.

Algorithm 5. Greedy Edge-Antimagic Vertex-Labelling

Input: graph $G = (V, E)$
Output: edge-antimagic vertex-labelling of G

---

suppose $V = \{v_1, v_2, \ldots, v_n\}$ with $\deg(v_i) \leq \deg(v_{i+1})$, $1 \leq i \leq n - 1$
for $i = 1, 2, \ldots, n$ **do**

set $\lambda(v_i)$ to be the minimum positive integer l such that:

(1) for all $j < i$, $l \neq \lambda(v_j)$, and

(2) for all edges $v_iv_j, v_pv_q \in E$ with $j, p, q < i$ and $p \neq j \neq q$, $l \neq \lambda(v_p) + \lambda(v_q) - \lambda(v_j)$

end-for

Theorem 4. *Given an n-vertex m-edge graph G with maximum degree Δ, the algorithm* Greedy Edge-Antimagic Vertex-Labelling *will determine an edge-antimagic vertex-labelling of G with maximum label at most $\Delta(m-\Delta)+n$.*

Proof. At each step of the algorithm, by the choice of $\lambda(v_i)$, for all edges xy with both end-vertices labelled, the sum $\lambda(x) + \lambda(y)$ is unique. Thus at the end of the algorithm, λ is a pseudo edge-antimagic vertex-labelling of G. When choosing $\lambda(v_i)$, there are at most $(i-1) + \text{pred}(v_i) \cdot m_i$ positive integers which $\lambda(v_i)$ cannot become, where $\text{pred}(v_i)$ is the number of vertices v_j adjacent to v_i with $j < i$, and m_i is the number of edges $v_pv_q \in E$ with $p, q < i$. Hence $\lambda(v_i) \leq i + \text{pred}(v_i) \cdot m_i$. Now $\text{pred}(v_i) \leq \Delta$ and $m_i \leq m - \deg(v_n) = m - \Delta$ since vertices are ordered by non-decreasing degree. Hence the maximum label is at most $n + \Delta(m-\Delta)$. □

By Lemma 3 and Theorem 4 we have the following result.

Corollary 6. *Given an n-vertex m-edge graph G with maximum degree Δ, algorithms* Greedy Edge-Antimagic Vertex-Labelling *and* Extend Pseudo Edge-Antimagic Vertex-Labelling *determine a pseudo edge-magic total-labelling of G with magic number at most* $3(\Delta(m-\Delta)+n)$. □

6.2 Pseudo Vertex-Magic Total-Labellings

Suppose λ is a pseudo vertex-antimagic partial edge-labelling of a graph $G = (V, E)$; that is, there is some set $E' \subseteq E$ such that λ restricted to E' is vertex-antimagic. Edges $vw \in E \setminus E'$ are considered to have $\lambda(vw) = 0$. For each vertex $v \in V$, let

$$S_v = \sum_{vx \in E(v)} \lambda(vx) \ .$$

Consider the following greedy algorithm.

Algorithm 6. Greedy Pseudo Vertex-Antimagic Edge-Labelling

Input: graph $G = (V, E)$

Output: pseudo vertex-antimagic edge-labelling of G

---

1. Find a spanning tree $T = (V, E')$ of G.

2. Determine a pseudo vertex-antimagic edge-labelling λ of T using algorithm PSEUDO VERTEX-ANTIMAGIC TREE EDGE-LABELLING in Section 5.
3. For each edge $vw \in E \setminus E'$, set $\lambda(vw)$ to be the minimum positive integer l such that:
   (1) for all labelled edges $xy \in E$, $l \neq \lambda(xy)$.
   (2) for all nodes $x \in V \setminus \{v, w\}$, $l \neq S_x - S_v$ and $l \neq S_x - S_w$.

Theorem 5. *Given an n-vertex m-edge connected graph G, the algorithm* GREEDY PSEUDO VERTEX-ANTIMAGIC EDGE-LABELLING *determines a pseudo vertex-antimagic edge-labelling of G with maximum label at most $m + 2n - 4$.*

Proof. After Step 2, λ is a pseudo vertex-antimagic partial edge-labelling such that every vertex $v \in V$ has $S_v > 0$ (since T is a spanning tree). Thus, for all distinct vertices $v, w \in V$, $S_v \neq S_w$. Suppose, at some point during Step 3, λ is still pseudo vertex-antimagic on the set of labelled edges, and the label l is chosen for some edge vw. The new value S'_v of S_v is $S_v + l$, the new value S'_w of S_w is $S_w + l$, and S_x does not change for any other vertex x. Since λ is vertex-antimagic, $S_v \neq S_w$, and hence $S'_v = S_v + l \neq S_w + l = S'_w$. By the choice of l, for all other vertices $x \in V$, $S'_v = S_v + l \neq S_x$ and $S'_w = S_w + l \neq S_x$. Hence λ remains a pseudo vertex-antimagic partial edge-labelling throughout Step 3, and after Step 3, λ is a pseudo vertex-antimagic edge-labelling of G.

By Lemma 8 the maximum label in the labelling of T is $3n - 5$. For each edge $vw \in E \setminus E'$, there are at most $(m - 1) + 2(n - 2) = m + 2n - 5$ values which the label $\lambda(vw)$ cannot take on. Hence the maximum label is at most $\max\{3n - 5, m + 2n - 4\} = m + 2n - 4$ since G is connected. □

By Lemma 4 and Theorem 5 we have the following result.

Corollary 7. *Given an n-vertex m-edge graph G with maximum degree Δ, algorithms* GREEDY PSEUDO VERTEX-ANTIMAGIC EDGE-LABELLING *and* EXTEND PSEUDO VERTEX-ANTIMAGIC EDGE-LABELLING *determine a pseudo vertex-magic total-labelling of G with magic number at most $(\Delta + 1)(m + 2n - 4)$.* □

By Corollary 2 the above upper bound on the magic number is within a constant factor of being optimal for regular graphs.

7 Conclusion

In this paper we have studied variations of the well-known edge-magic and vertex-magic labellings in which labels are arbitrary positive integers. All graphs admit such pseudo magic labellings. We have presented lower and upper bounds on the magic number for complete graphs, trees and arbitrary graphs, which in a number of cases are within a constant factor. We expect that better upper bounds are possible for the magic number in pseudo edge-magic total-labellings. In particular, we conjecture that every n-vertex m-edge graph has a pseudo edge-magic total-labelling with magic number $O(m + n)$. This conjecture has been verified in the case of complete graphs, complete bipartite graphs, trees and cycles. Finally, for the sake of completeness, we draw the readers attention to a number of other publications in the field of magic graph labellings [1–4, 7–9, 16, 17, 19, 23–27, 30–36, 39, 40, 43–46, 48, 50, 52–57, 59].

Acknowledgements

Thanks to Paul Manuel for introducing the author to pseudo total-labellings, and to David Streader for stimulating discussions on the topic

Bibliography

[1] M. Arisoy, Antimagic labellings of the complete graphs with a common vertex. *Internat. J. Math. Ed. Sci. Tech.*, **29(2)**:301–305, 1998.

[2] S. Avadayappan and R. Vasuki, Magic strength of a graph. *Indian J. Pure Appl. Math.*, **31(7)**:873–883, 2000.

[3] B. Bača, I. Holländer, and K.-W. Lih, Two classes of super-magic quartic graphs. *J. Combin. Math. Combin. Comput.*, **23**:113–120, 1997.

[4] M. Bača, On certain properties of magic graphs. *Utilitas Math.*, **37**:259–264, 1990.

[5] M. Bača, M. Miller, and Slamin, Every generalized Petersen graph has a vertex-magic total labeling. In [11], pp. 1–6.

[6] R. Bodendiek and G. Walther, On the relations between certain graph labelings. *Discrete Math.*, **134(1-3)**:9–16, 1994.

[7] R. Bodendiek and G. Walther, On (a, d)-antimagic parachutes. *Ars Combin.*, **42**:129–149, 1996.

[8] R. Bodendiek and G. Walther, (a, d) antimagic parachutes. II. *Ars Combin.*, **46**:33–63, 1997.

[9] R. Bodendiek and G. Walther, On arithmetic antimagic edge labelings of graphs. *Mitt. Math. Ges. Hamburg*, **17**:85–99, 1998.

[10] R. C. Bose and S. Chowla, Theorems in the additive theory of numbers. *Comment. Math. Helv.*, **37**:141–147, 1962/1963.

[11] L. Brankovic and J. Ryan, eds., *Proc. 11th Australasian Workshop on Combinatorial Algorithms (AWOCA'00)*, The University of Newcastle, Australia, 2000.

[12] D. Craft and E. H. Tesar, On a question by Erdős about edge-magic graphs. *Discrete Math.*, **207(1-3)**:271–276, 1999.

[13] H. Enomoto, A. S. Llado, T. Nakamigawa, and G. Ringel, Super edge-magic graphs. *SUT J. Math.*, **34(2)**:105–109, 1998.

[14] H. Enomoto, K. Masuda, and T. Nakamigawa, Induced graph theorem on magic valuations. *Ars Combin.*, **56**:25–32, 2000.

[15] P. Erdös and P. Turán, On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, **16**:212–215, 1941.

[16] R. M. Figueroa-Centeno, R. Ichishima, and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings. *Discrete Math.*, **231(1-3)**:153–168, 2001.

[17] Y. Fukuchi, A recursive theorem for super-edge-magic labelings of trees. *SUT J. Math.*, **36(2)**:279–285, 2000.

[18] J. A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.*, **#DS6**, 2000.

[19] F. Göbel and C. Hoede, Magic labelings of graphs. *Ars Combin.*, **51**:3–19, 1999.

[20] R. D. Godbold and P. J. Slater, All cycles are edge-magic. *Bull. Inst. Combin. Appl.*, **22**:93–97, 1998.

[21] H. Halberstam and K. F. Roth, *Sequences. Vol. I.* Clarendon Press, Oxford, 1966.

[22] N. Hartsfield and G. Ringel, *Pearls in graph theory: a comprehensive introduction.* Academic Press, 1994.

[23] J. Ivančo, On supermagic regular graphs. *Math. Bohem.*, **125(1)**:99–114, 2000.

[24] R. H. Jeurissen, Disconnected graphs with magic labelings. *Discrete Math.*, **43(1)**:47–53, 1983.

[25] R. H. Jeurissen, Pseudomagic graphs. *Discrete Math.*, **43(2-3)**:207–214, 1983.

[26] B. Kalantari and G. B. Khosrovshahi, Magic labeling in graphs: bounds, complexity, and an application to a variant of TSP. *Networks*, **28(4)**:211–219, 1996.

[27] M. C. Kong, S.-M. Lee, and H. S. H. Sun, On magic strength of graph. *Ars Combin.*, **45**:193–200, 1997.

[28] A. Kotzig and A. Rosa, Magic valuations of finite graphs. *Canad. Math. Bull.*, **13**:451–461, 1970.

[29] A. Kotzig and A. Rosa, Magic valuations of complete graphs. *Publications du Centre de Recherches Mathématiques Université de Montréal*, **175**, 1972.

[30] S.-M. Lee and A. N.-T. Lee, A construction of super edge-magic graphs. In *Proc. 14th Mid-West Conference on Combinatorics, Cryptography, and Computing*, Wichita State University, Kansas, 2000.

[31] S.-M. Lee, F. Saba, and G. C. Sun, Magic strength of the kth power of paths. In *Proc. 22nd Manitoba Conference on Numerical Mathematics and Computing*, vol. 92 of *Congr. Numer.*, pp. 177–184, 1993.

[32] S.-M. Lee, F. Saba, and G. C. Sun, Magic strength of the kth power of paths. In *Proc. of 22nd Manitoba Conf. on Numerical Mathematics and Computing*, vol. 92 of *Congr. Numer.*, pp. 177–184, 1993.

[33] S.-M. Lee, E. Seah, and S.-K. Tan, On edge-magic graphs. In *Proc. of Twenty-first Manitoba Conference on Numerical Mathematics and Computing*, vol. 86 of *Congr. Numer.*, pp. 179–191, 1992.

[34] S.-M. Lee and S.-M. Tong, On super edge-magic deficiencies of graphs. In *Proc. 14th Mid-West Conference on Combinatorics, Cryptography, and Computing*, Wichita State University, Kansas, 2000.

[35] S.-M. Lee, S.-M. Tong, and E. Seah, On the edge-magic and edge-graceful total graphs conjecture. In *Proc. of 30th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 141 of *Congr. Numer.*, pp. 37–48, 1999.

[36] Y. Q. Lin, M. Miller, R. Simanjuntak, and Slamin, Magic and antimagic labelings of wheels. In [11], pp. 67–74.

[37] J. MacDougall, M. Miller, Slamin, and W. D. Wallis, Vertex-magic total labelings. *Utilitas Math.*, to appear.

[38] J. MacDougall, M. Miller, and W. D. Wallis, Vertex-magic total labelings of wheels and related graphs. *Utilitas Math.*, to appear.

[39] M. Miller, M. Bača, and J. MacDougall, Vertex-magic total labeling of generalized Petersen graphs. In R. Raman and J. Simpson, eds., *Proc. 10th Australasian Workshop on Combinatorial Algorithms (AWOCA'99)*, pp. 19–25, Curtin University of Technology, Australia, 1999.

[40] N. C. K. Phillips, R. S. Rees, and W. D. Wallis, Edge-magic total labelings of wheels. *Bull. Inst. Combin. Appl.*, to appear.

[41] N. C. K. Phillips and W. D. Wallis, Well-spread sequences. *J. Combin. Math. Combin. Comput.*, **31**:91–96, 1999.

[42] G. Ringel and A. S. Llado, Another tree conjecture. *Bull. Inst. Combin. Appl.*, **18**:83–85, 1996.

[43] Y. Roditty and T. Bachar, A note on edge-magic cycles. *Bull. Inst. Combin. Appl.*, **29**:94–96, 2000.

[44] K. Schaffer and S. M. Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs. In *Proc. of 30th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 141 of *Congr. Numer.*, pp. 119–134, 1999.

[45] W. C. Shiu, P. C. B. Lam, and H. L. Cheng, Supermagic labeling of an s-duplicate of $K_{n,n}$. In *Proc. of 31st Southeastern International Conference on Combinatorics, Graph Theory and Computing*, vol. 146 of *Congr. Numer.*, pp. 119–124, 2000.

[46] W. C. Shiu, P. C. B. Lam, and S.-M. Lee, Edge-magic index sets of (p, p)-graphs. In *Proc. of 30th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 137 of *Congr. Numer.*, pp. 97–107, 1999.

[47] S. Sidon, Ein Satz űber trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen. *Math. Ann.*, **106**:536–539, 1932.

[48] R. Simanjuntak, M. Miller, and F. Bertault, Two new (a, d)-antimagic graph labelings. In [11], pp. 179–190.

[49] Slamin, M. Bača, Y. Lin, M. Miller, and R. Simanjuntak, Edge-magic total labelings of wheels, fans and friendship graphs. *Bulletin of ICA*, to appear.

[50] Slamin and M. Miller, On two conjectures concerning vertex-magic total labelings of generalized Petersen graphs. *Bulletin of ICA*, **32**:9–16, 2001.

[51] N. J. A. Sloane, Sequence A011185. *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

[52] R. P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters, and Cohen-Macaulay rings. *Duke Math. J.*, **43(3)**:511–531, 1976.

[53] B. M. Stewart, Supermagic complete graphs. *Canad. J. Math.*, **19**:427–438, 1967.

[54] G. C. Sun, J. Guan, and T. L. Holloman, Factorization by label of magic graphs. In *Proc. of 26th Southeastern International Conf. on Combinatorics, Graph Theory and Computing*, vol. 112 of *Congr. Numer.*, pp. 37–43, 1995.

[55] G. C. Sun, J. Guan, and S.-M. Lee, A labeling algorithm for magic graph. In *Proc. of 25th Southeastern International Conf. on Combinatorics, Graph Theory and Computing*, vol. 102 of *Congr. Numer.*, pp. 129–137, 1994.

[56] G. C. Sun and S.-M. Lee, Construction of magic graphs. In *Proc. of 25th Southeastern International Conf. on Combinatorics, Graph Theory and Computing*, vol. 103 of *Congr. Numer.*, pp. 243–251, 1994.

[57] G. C. Sun, Y.-N. Yang, and Y.-H. Liao, Algorithm of edge labeling to a magic graph by factorization. In *Proc. of 30th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 137 of *Congr. Numer.*, pp. 197–206, 1999.

[58] W. D. Wallis, E. T. Baskoro, M. Miller, and Slamin, Edge-magic total labelings. *Austral. J. Combin.*, **22**:177–190, 2000.

[59] K. Wijaya and E. T. Baskoro, Edge-magic total labelling on disconnected graphs. In [11], pp. 139–144.