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Improper colourings inspired by Hadwiger's conjecture

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Abstract

Hadwiger's conjecture asserts that every K_t-minor-free graph has a proper $(t-1)$-colouring. We relax the conclusion in Hadwiger's conjecture via improper colourings. We prove that every K_t-minor-free graph is $(2t-2)$-colourable with monochromatic components of order at most $\lceil \frac{1}{2}(t-2) \rceil$. This result has no more colours and much smaller monochromatic components than all previous results in this direction. We then prove that every K_t-minor-free graph is $(t-1)$-colourable with monochromatic degree at most $t-2$. This is the best known degree bound for such a result. Both these theorems are based on a decomposition method of independent interest. We give analogous results for $K_{s,t}$-minor-free graphs, which lead to improved bounds on generalised colouring numbers for these classes. Finally, we prove that graphs containing no K_t-immersion are 2-colourable with bounded monochromatic degree.

1. *Introduction*

Hadwiger's conjecture [25] asserts that every K_t-minor-free graph has a proper $(t-1)$-colouring. For $t \leqslant 3$ the conjecture is easy. Hadwiger [25] and Dirac [11] independently proved the conjecture for $t = 4$; while Wagner's result [50] means that the case $t = 5$ is equivalent to the Four Colour Theorem. Finally, Robertson *et al.* [44] proved Hadwiger's conjecture for $t = 6$. The conjecture remains open for $t \geqslant 7$. Hadwiger's conjecture is widely considered to be one of the most important open problems in graph theory. The best upper bound on the chromatic number of K_t-minor-free graphs is $\mathcal{O}(t\sqrt{\log t})$ independently due to Kostochka [34, 35] and Thomason [47, 48]. See the recent survey by Seymour [46] for more on Hadwiger's conjecture.

One possible way to approach Hadwiger's conjecture is to allow improper colourings. In a vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. A graph G is *k-colourable with clustering c* if each vertex can be assigned one of k colours such that each monochromatic subgraph has at most c vertices[†]. Kleinberg *et al.* [33] introduced this type of colouring, and now many results are known. The *clustered chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k for which there exists an integer c such that every graph in $\mathcal{G}$ is k-colourable with clustering c.

Kawarabayashi and Mohar [30] were the first to prove an $\mathcal{O}(t)$ upper bound on the clustered chromatic number of K_t-minor-free graphs. In particular, they proved that every K_t-minor-free graph is $\lceil \frac{31}{2} t \rceil$-colourable with clustering $f(t)$, for some function f. The number of colours in this result was improved to $\lceil \frac{1}{2}(7t-3) \rceil$ by Wood [52][‡], to $4t-4$ by Edwards *et al.* [16], and to $3t-3$ by Liu and Oum [38]. See [28, 29] for analogous results for graphs excluding odd minors. For all of these results, the function $f(t)$ is very large, often depending on constants from the Graph Minor Structure Theorem [45].

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[†]This type of colouring is sometimes called 'fragmented' in the literature, but we feel that increased fragmentation suggests smaller monochromatic components, hence we use the term 'clustered'.

[‡]This result depends on a result announced in 2008 which is not yet written.

Our first contribution is to prove an analogous theorem with the best known number of colours, and also with small clustering. The proof is simple, and does not depend on any deep theory.

THEOREM 1. *For $t \geqslant 4$, every K_t-minor-free graph is $(2t-2)$-colourable with clustering $\lceil \frac{1}{2}(t-2) \rceil$.*

Theorem 1 implies that the clustered chromatic number of K_t-minor-free graphs is at most $2t-2$. A construction of Edwards *et al.* [**16**] mentioned below implies that the clustered chromatic number of K_t-minor-free graphs is at least $t-1$.

The second way to relax the conclusion in Hadwiger's conjecture is to bound the maximum degree of monochromatic components. A graph G is *k-colourable with defect d* if each vertex can be assigned one of k colours such that each vertex is adjacent to at most d vertices of the same colour; that is, each monochromatic subgraph has maximum degree at most d. Cowen *et al.* [**6**] introduced the notion of defective graph colouring, and now many results for various graph classes are known. A graph class $\mathcal{G}$ is *defectively k-colourable* if there exists an integer d such that every graph in $\mathcal{G}$ is k-colourable with defect d. The *defective chromatic number* of $\mathcal{G}$ is the minimum integer k such that $\mathcal{G}$ is defectively k-colourable [**7**]. Edwards *et al.* [**16**] proved that every K_t-minor-free graph is $(t-1)$-colourable with defect $\mathcal{O}(t^2 \log t)$. Moreover, it is shown in [**16**] that the number of colours, $t-1$, is best possible in the following strong sense: for every integer d, there is a K_t-minor-free graph that is not $(t-2)$-colourable with defect d. Thus the defective chromatic number of K_t-minor-free graphs equals $t-1$. (This also shows that the clustered chromatic number of K_t-minor-free graphs is at least $t-1$.)

Our second contribution is an improved upper bound on the defect in the result of [**16**].

THEOREM 2. *For $t \geqslant 4$, every K_t-minor-free graph is $(t-1)$-colourable with defect $t-2$.*

Edwards *et al.* [**16**] wisely noted that their theorem mentioned earlier should not be considered evidence for the truth of Hadwiger's conjecture, since their method also proves that every K_t-topological-minor-free graph is $(t-1)$-colourable with defect $\mathcal{O}(t^4)$. It is not true that every K_t-topological-minor-free graph is properly $(t-1)$-colourable. This *last statement* is Hajós' conjecture, which is now known to be false [**4**, **49**]. On the other hand, our proof of Theorem 2 does not work for graphs excluding a topological minor.

Theorems 1 and 2 are corollaries of the following decomposition result of independent interest. A sequence $H_1, \ldots, H_\ell$ is a *connected partition* of a graph G if each H_i is a non-empty connected induced subgraph of G, the subgraphs $H_1, \ldots, H_\ell$ are pairwise disjoint, and $V(G) = V(H_1) \cup \cdots \cup V(H_\ell)$. Two disjoint subgraphs H and H' of a graph G are *adjacent* if there is an edge in G with one endpoint in H and one endpoint in H'. For a positive integers n, m, we use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n, m]$ to denote the set $\{n, \ldots, m\}$.

THEOREM 3. *For $t \geqslant 4$, every K_t-minor-free graph G has a connected partition $H_1, \ldots, H_\ell$ such that for $i \in [\ell]$:*
(1) *H_i is adjacent to at most $t-2$ of the subgraphs $H_1, \ldots, H_{i-1}$;*
(2) *H_i has maximum degree at most $t-2$; and*
(3) *H_i is 2-colourable with clustering $\lceil \frac{1}{2}(t-2) \rceil$.*

We actually prove a decomposition theorem with several further properties; see Theorem 11. It is easy to derive Theorems 1 and 2 from Theorem 3. Colour the subgraphs $H_1, \ldots, H_\ell$ greedily in this order, such that adjacent subgraphs receive distinct colours. By property (1), $t-1$ colours suffice. Theorem 2 follows from property (2) by colouring each vertex in H_i by the colour assigned to H_i. Theorem 1 follows from property (3) by taking the product of the $(t-1)$-colouring of $H_1, \dots, H_\ell$ with the given 2-colouring of each subgraph H_i.

Theorem 3 is an extension of a result by Van den Heuvel *et al.* [27] in which properties (2) and (3) are replaced by 'H_i has a Breadth-First Search (BFS) spanning tree with at most $t-3$ leaves'. Van den Heuvel *et al.* [27] were motivated by connections to generalised colouring numbers. Note that the result in [27] implies that H_i has at most $t-3$ vertices in each BFS layer. It follows that the maximum degree of H_i is at most $3t-10$. Alternately colouring the BFS layers shows that H_i is 2-colourable with clustering $t-3$. Constructing H_i more carefully, and choosing the 2-colouring more carefully, leads to the improved bounds in Theorem 3, which we prove in Section 3.

Our main decomposition theorem, Theorem 11, also has the following corollary, which might be of independent interest.

THEOREM 4. *For $t \geqslant 4$, every K_t-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ such that*

(1) *the quotient graph Q obtained by contracting each H_i to a single vertex is chordal with clique size at most $t-1$ (and hence has treewidth at most $t-2$); and*
(2) *each part H_i has bandwidth (and hence pathwidth and treewidth) at most $t-3$.*

Hadwiger's conjecture implies that for every graph H with t vertices, the maximum chromatic number of H-minor-free graphs equals $t-1$ (since K_{t-1} is H-minor-free). However, for clustered and defective colourings, fewer colours often suffice. For example, it follows from the main result by Ossona de Mendez *et al.* [40] that for every fixed non-complete graph H on t vertices, every H-minor-free graph is $(t-2)$-colourable with bounded defect, which is one fewer colour than in the complete graph case. More interestingly, Archdeacon [3] proved that graphs embeddable in a fixed surface are defectively 3-colourable (see also [5–7, 53]); while Dvořák and Norin [14] proved that such graphs are 4-colourable with bounded clustering. Ossona de Mendez *et al.* [40] conjectured that for every connected graph H, the defective chromatic number of H-minor-free graphs equals the treedepth of H minus 1. They proved this conjecture for $K_{s,t}$-minor-free graphs by showing that they are defectively s-colourable. Note that $K_{3,t}$-minor-free graphs are of particular interest since they include and generalise graphs embeddable in fixed surfaces. In the case $s \leqslant 3$ we prove decomposition results analogous to Theorem 3 that imply this result of [40] with much improved bounds on the defect. This direction is explored in Section 4.

In the same way as Van den Heuvel *et al.* [27] applied their decomposition result for K_t-minor-free graphs to the setting of generalised colouring numbers, we apply our decomposition results for $K_{s,t}$-minor-free graphs and $K^*_{s,t}$-minor-free graphs (where $K^*_{s,t}$ is the complete join of K_s and $\overline{K_t}$) to conclude new bounds on generalised colouring numbers. Our results when specialised for graphs of given genus are almost as strong as the best known bounds. These results on generalised colouring numbers are presented in Section 5.

The final section, Section 6, returns to the topic of defective graph colouring, but instead of excluding a K_t minor we exclude a K_t immersion. The analogue of Hadwiger's conjecture, that K_t-immersion-free graphs are properly $(t-1)$-colourable [1, 37], is open. For defective colouring, we show that only 2 colours suffice.

Before continuing, we mention an important connection between clustered and defective colourings, implicitly observed in [16]. We include the proof for completeness.

LEMMA 5 (Edwards *et al.* [16]). *For every minor-closed graph class $\mathcal{G}$, the clustered chromatic number of $\mathcal{G}$ is at most three times the defective chromatic number of $\mathcal{G}$.*

Proof. Liu and Oum [**38**] proved that for every minor-closed graph class $\mathcal{G}$ and integer d, there is an integer $c = c(\mathcal{G}, d)$ such that every graph in $\mathcal{G}$ with maximum degree d is 3-colourable with clustering c. (Esperet and Joret [**17**] previously proved an analogous result for graphs on surfaces.) Let k be the defective chromatic number of $\mathcal{G}$. Thus for some integer d, every graph G in $\mathcal{G}$ is k-colourable with defect d. Apply the result of Liu and Oum [**38**] to each monochromatic component of G, which has maximum degree at most d. Then G is $3k$-colourable with clustering c, and hence the clustered chromatic number of $\mathcal{G}$ is at most $3k$. □

2. *Preliminaries*

2.1. *Notation and definitions*

This subsection briefly states standard graph theoretic definitions probably familiar to most readers.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Equivalently, and often easier to use intuitively: a graph H with vertices $v_1, \dots, v_n$ is a minor of G if there exist pairwise disjoint connected subgraphs $H_1, \dots, H_n$ of G such that for every edge $v_i v_j$ in H, H_i and H_j are adjacent in G. We call H_i the *branch set corresponding to* v_i. A class of graphs $\mathcal{G}$ is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is also in $\mathcal{G}$. A graph H is a *topological minor* of a graph G if a graph isomorphic to a subdivision of H is a subgraph of G.

The *Euler genus* of an orientable surface with h handles is $2h$. The *Euler genus* of a non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds (with no crossing edges).

A *tree decomposition* of a graph G is given by a tree T whose nodes index a collection $(\, T_x \subseteq V(G) \mid x \in V(T) \,)$ of sets of vertices in G called *bags*, such that (1) for every edge vw of G, some bag T_x contains both v and w, and (2) for every vertex v of G, the set $\{\, x \in V(T) \mid v \in T_x \,\}$ induces a non-empty (connected) subtree of T. The *width* of a tree decomposition T is $\max\{\, |T_x \mid x \in V(T) \,\} - 1$, and the *treewidth* of a graph G is the minimum width of the tree decompositions of G. Note that the treewidth of G equals the minimum integer k such that G is a subgraph of a chordal graph with clique number $k+1$.

A *path decomposition* is a tree decomposition in which the underlying tree is a path. The *pathwidth* of a graph G is the minimum width of a path decomposition of G.

For a graph G and $A, B \subseteq V(G)$, an *AB-separator* is a set $S \subseteq V(G)$ such that every AB-path in G contains a vertex from S. (Note that we allow A and B to intersect and that all vertices in $A \cap B$ must be included in any AB-separator.) A pair (G_1, G_2) is a *k-separation* of a graph G if G_1 and G_2 are induced subgraphs of G such that $G = G_1 \cup G_2$, $G_1 \nsubseteq G_2$ and $G_2 \nsubseteq G_1$, and $|V(G_1) \cap V(G_2)| = k$.

2.2. *Connected induced subgraphs*

This subsection contains some elementary results about connected induced subgraphs containing a given set of vertices. We look in detail at so-called *lexicographic breadth-first search* (LexBFS) trees, since these form a key tool in our methods.

A *layering* of a graph G is a partition $(V_0, V_1, \dots, V_\ell)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. Each set V_i is called a *layer*.

Let r be a vertex in a connected graph G. Let $\ell = \max\{\, \mathrm{dist}_G(r, v) \mid v \in V(G) \,\}$, and for $i \in \{0, \dots, \ell\}$ define $V_i = \{\, v \in V(G) \mid \mathrm{dist}_G(r, v) = i \,\}$. Then $V_0, V_1, \dots, V_\ell$ is a layering of G, called the *BFS layering* of G starting from the *root* r; each V_i is called a *BFS layer* of G. A spanning tree T of G rooted at r is a *BFS spanning tree* if $\mathrm{dist}_G(v, r) = \mathrm{dist}_T(v, r)$ for every vertex v in G. A *BFS subtree* is a subtree of a BFS spanning tree that includes the root. Let S

be a BFS subtree rooted at r and consider a vertex $v \in V_i \cap V(S)$ for some $i \geqslant 1$. Let P_v be the vr-path in S. Then P_v has exactly one vertex in each of $V_0, \dots, V_i$. The *parent* of v is the neighbour of v (in S) in V_{i-1}. Every vertex x in G is adjacent to at most three vertices in P_v (since if $x \in V_j$, then $N_G(x) \subseteq V_{j-1} \cup V_j \cup V_{j+1}$). A *leaf* in a rooted tree is a non-root vertex of degree 1. If S has p leaves, then every vertex in G is adjacent to at most $3p$ vertices in S. This observation can be improved for a special type of BFS (sub)trees.

For our purposes, a BFS spanning tree T of G is a *LexBFS spanning tree* if each BFS layer V_i can be linearly ordered such that

(a) each vertex $v \in V_i$ with parent $w \in V_{i-1}$ in T has no neighbour in G that comes before w in the ordering of V_{i-1} (called the *priority rule*); and

(b) for every edge vw in T with $v \in V_i$ and $w \in V_{i-1}$, there is no edge xy in T with x before v in the ordering of V_i and y after w in the ordering of V_{i-1} (called the *non-crossing rule*).

It is easily seen that every connected graph has a LexBFS spanning tree rooted at any given vertex. A *LexBFS subtree* is a subtree of a LexBFS spanning tree that includes the root.

Throughout this paper we follow the convention that the root of a rooted tree (such as a BFS or LexBFS (sub)tree) is never a leaf.

LEMMA 6. *For $k \geqslant 1$, if S is a LexBFS subtree of a connected graph G and S has k leaves, then every vertex in G has at most $2k$ neighbours in $V(S)$.*

Proof. Let T be a LexBFS spanning tree of G, such that S is a subtree of T. Let $V_0, \dots, V_\ell$ be the BFS layers of T. Let v be a vertex in V_i (which may or may not be in S). If v is on some leaf-root path P of S, then $|N_G(v) \cap V(P)| \leqslant 2$. Now consider a leaf-root path P in S not containing v. Suppose on the contrary that there are distinct vertices $x, y, z \in N_G(v) \cap V(P)$, none of which are on a leaf-root path of T containing v. Without loss of generality, $x \in V_{i-1}$, $y \in V_i$ and $z \in V_{i+1}$. Let w be the parent of v in T. So $w \in V_{i-1}$, but $w \neq x$ (since x is not on a leaf-root path of T containing v). By the priority rule, w comes before x in the ordering of V_{i-1}. By the non-crossing rule, v comes before y in the ordering of V_i, which contradicts the priority rule for z. Thus $|N_G(v) \cap V(P)| \leqslant 2$. Since there are k leaf-root paths in S, in total this gives $|N_G(v) \cap V(S)| \leqslant 2k$. $\square$

A graph G has *bandwidth* at most k if there is a vertex ordering $v_1, \dots, v_n$ of $V(G)$, such that $|i - j| \leqslant k$ for each edge $v_i v_j$ of G.

LEMMA 7. *Every connected graph G that has a LexBFS spanning tree T with k leaves has bandwidth, pathwidth and treewidth at most k.*

Proof. Say T is rooted at r. Let $V_0, \dots, V_\ell$ be the BFS layers of T. Each V_i is linearly ordered by LexBFS. We claim that the vertex-ordering of $V(G)$ produced by using the orderings of $V_0, \dots, V_\ell$ in that order has bandwidth at most k. Consider an edge vw where $v \in V_i$ and $w \in V_i$. Since T has at most k leaves, $|V_i| \leqslant k$ and at most $k - 2$ vertices are between v and w in $V_0, \dots, V_\ell$. Now consider an edge vw where $v \in V_i$ and $w \in V_{i+1}$. Let X be the set of vertices that come after v in V_i or come before w in V_{i+1}. Then X is the set of vertices between v and w in the ordering of $V(G)$. Let p be the parent of w in T. By the priority rule, $p \notin X$. By the non-crossing rule, no vertex in $X \cup \{v\}$ is a descendent of another vertex in $X \cup \{v\}$. Hence, the number of leaves in T is at least $|X| + 1$, implying $|X| \leqslant k - 1$. Therefore G has bandwidth at most k.

It is well-known and easy to prove that the pathwidth of a graph is at most its bandwidth (and hence so is the treewidth). Take the vertex ordering $v_1, \dots, v_n$ of $V(H)$ that shows H has bandwidth at most k. For $i \in [n - k]$, let $T_i = \{v_i, \dots, v_{i+k}\}$. Then $T_1, T_2, \dots, T_{n-k}$ defines the desired path decomposition. $\square$

Lemma 8. *For every set A of $k \geqslant 2$ vertices in a connected graph G, every minimal induced connected subgraph H of G containing A satisfies the following properties:*

(1) *every (non-rooted) subtree of H has at most k leaves;*
(2) *H has maximum degree at most k;*
(3) *H has bandwidth (and hence pathwidth and treewidth) at most $k-1$;*
(4) *H can be 2-coloured with clustering $\lceil \frac{1}{2}k \rceil$; and*
(5) *H can be 2-coloured with $\{$red, blue$\}$ such that there are at most $k-2$ red vertices and the blue subgraph consists of at most $k-1$ pairwise disjoint paths.*

Proof. Let T be a spanning tree of H. By the minimality of H, every leaf of T is in A. Thus T has at most k leaves. Now let S be any tree in H. Extending S to a spanning tree of H cannot decrease the number of leaves, hence S also has at most k leaves.

The closed neighbourhood of a vertex $v \in V(H)$ contains a tree with $\deg_H(v)$ leaves, proving $\deg_H(v) \leqslant k$.

Let T be a LexBFS spanning tree of H rooted at a vertex r in A. By the minimality of H, every leaf of T is in A. Thus T has at most $k-1$ leaves (the root does not count as a leaf). By Lemma 7, H has bandwidth, pathwidth and treewidth at most $k-1$.

We now prove (4). We proceed by induction on $|V(H)|$. In the base case, $|V(H)| = |A| = k$ and the result is trivial. Now assume that $|V(H)| > k$. Thus $V(H-A) \neq \varnothing$, and by the minimality of H, every vertex in $H-A$ is a cut-vertex of H. Consider a leaf-block L of H. Every vertex in L, except the one cut-vertex in L, is in A. There are at least two leaf-blocks. Thus $|V(L-v)| \leqslant \frac{1}{2}k$ for some leaf block L, where v is the one cut-vertex of H in L. Let $H' = H - V(L-v)$ and $A' = (A \setminus V(L)) \cup \{v\}$. Then H' is a minimal induced connected subgraph of G containing A', and $|A'| \leqslant k$. By induction, H' has a 2-colouring with clustering $\lceil \frac{1}{2}k \rceil$. Colour every vertex in $L \setminus \{v\}$ by the colour not assigned to v in H'. Now H is 2-coloured with clustering $\lceil \frac{1}{2}k \rceil$.

It remains to prove (5). We proceed by induction on k. If $k = 2$, then H is a path between the two vertices in A. Colour every vertex in H blue, and we are done. So assume $k \geqslant 3$ and the result holds for $k-1$. Let x be a vertex in A. By induction, every minimal induced connected subgraph H' of H containing $A \setminus \{x\}$ can be 2-coloured with $\{\text{red}, \text{blue}\}$ such that there are at most $k-3$ red vertices and the blue subgraph consists of at most $k-2$ pairwise disjoint paths. If x is in H', then we are done. Otherwise, let P be a shortest path between x and H' in H. Say $P = x, \dots, u, v, w$, where w is in H'. Then v is the only vertex in $P - w$ adjacent to H'. Colour v red, and colour $x, \dots, u$ blue. (It is possible that $x = v$, in which case $\{x, \dots, u\} = \varnothing$.) Then $\{x, \dots, u\}$ induces a path in H that is not adjacent to H'. By the minimality of H, we have $V(H) = V(H') \cup \{x, \dots, u, v\}$. Thus H is 2-coloured with $\{\text{red}, \text{blue}\}$ such that there are at most $k-2$ red vertices and the blue subgraph consists of at most $k-1$ pairwise disjoint paths. □

We now prove the main result of this section.

Lemma 9. *For every set A of $k \geqslant 2$ vertices in a connected graph G, there is an induced connected subgraph H of G containing A, such that*

(1) *H has maximum degree at most k;*
(2) *H has bandwidth (and hence pathwidth and treewidth) at most $k-1$;*
(3) *H can be 2-coloured with clustering $\lceil \frac{1}{2}k \rceil$;*
(4) *H can be 2-coloured with $\{$red, blue$\}$ such that there are at most $k-2$ red vertices and the blue subgraph consists of at most $k-1$ pairwise disjoint paths; and*
(5) *every vertex in G has at most $2k-2$ neighbours in $V(H)$.*

Proof. Let T be a LexBFS spanning tree of G rooted at some vertex $r \in A$. Let S be the LexBFS subtree of T consisting of all ar-paths in T, where $a \in A$. Every leaf of S is in $A \setminus \{r\}$, implying that S has at most $k-1$ leaves. By Lemma 6, every vertex in G has at most $2k-2$ neighbours in $V(S)$. Let H be a minimal induced connected subgraph of $G[V(S)]$ containing A. The first four claims follow from Lemma 8. Since $V(H) \subseteq V(S)$, Lemma 6 means that every vertex in G has at most $2k-2$ neighbours in $V(H)$. □

3. *Decompositions of K_t-minor-free graphs*

Van den Heuvel *et al.* [27] introduced the following definition and proved the following decomposition theorem. A connected partition $H_1, \dots, H_\ell$ has *width* k if for each $i \in [\ell - 1]$, each component of $G - (V(H_1) \cup \dots \cup V(H_i))$ is adjacent to at most k of the subgraphs $H_1, \dots, H_i$. Note that this implies that H_{i+1} is adjacent to at most k of the subgraphs $H_1, \dots, H_i$ (since H_{i+1} is contained in some component of $G - (V(H_1) \cup \dots \cup V(H_i))$).

THEOREM 10 (Van den Heuvel *et al.* [27]). *Every K_t-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ with width $t-2$, such that each subgraph H_i is induced by a BFS subtree of $G - (V(H_1) \cup \dots \cup V(H_{i-1}))$ with at most $t-3$ leaves.*

The following similar decomposition theorem implies Theorem 3.

THEOREM 11. *For $t \geqslant 4$, every K_t-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ with width $t-2$, such that for $i \in [\ell]$ the following holds.*

(1) *The subgraph H_i has the following properties:*
 (a) *H_i has maximum degree at most $t-2$;*
 (b) *H_i has bandwidth, pathwidth and treewidth at most $t-3$;*
 (c) *H_i can be 2-coloured with clustering $\lceil \frac{1}{2}(t-2) \rceil$; and*
 (d) *H_i can be 2-coloured with $\{$red, blue$\}$ such that there are at most $t-4$ red vertices and the blue subgraph consists of at most $t-3$ pairwise disjoint paths.*

(2) *Each component C of $G - (V(H_1) \cup \dots \cup V(H_i))$ has the following properties.*
 (a) *At most $t-2$ subgraphs in $H_1, \dots, H_i$ are adjacent to C, and these subgraphs are pairwise adjacent. (This implies that at most $t-2$ subgraphs in $H_1, \dots, H_i$ are adjacent to H_{i+1}, and these subgraphs are pairwise adjacent.)*
 (b) *Every vertex in C is adjacent to at most $2t-6$ vertices in each of $H_1, \dots, H_i$. (This implies that every vertex in H_{i+1} is adjacent to at most $2t-6$ vertices in each of $H_1, \dots, H_i$.)*

Proof. We may assume that G is connected. We construct $H_1, \dots, H_\ell$ iteratively, maintaining properties (1) and (2). Let H_1 be the subgraph induced by a single vertex in G. Then (1) and (2) hold for $i = 1$.

Assume that $H_1, \dots, H_i$ satisfy (1) and (2) for some $i \geqslant 1$, but $V(H_1), \dots, V(H_i)$ do not partition $V(G)$. Let C be a component of $G - (V(H_1) \cup \dots \cup V(H_i))$. Let $Q_1, \dots, Q_k$ be the subgraphs in $H_1, \dots, H_i$ that are adjacent to C. By (2a), $Q_1, \dots, Q_k$ are pairwise adjacent and $k \leqslant t-2$. Since G is connected, $k \geqslant 1$.

For $j \in [k]$, let v_j be a vertex in C adjacent to Q_j. If $k = 1$, then let H_{i+1} be the subgraph induced by v_1. It is easily seen that (1) is satisfied. For $k \geqslant 2$, by Lemma 9 with $k \leqslant t-2$, there is an induced connected subgraph H_{i+1} of C containing $v_1, \dots, v_k$ that satisfies (1).

Consider a component C' of $G - (V(H_1) \cup \dots \cup V(H_{i+1}))$. Either C' is disjoint from C, or C' is contained in C. If C' is disjoint from C, then C' is a component of $G - (V(H_1) \cup \dots \cup V(H_i))$ and C' is not adjacent to H_{i+1}, implying (2) is maintained for C'.

Now assume C' is contained in C. Since every vertex in C has at most $2t-6$ neighbours in each of $H_1, \dots, H_i$, every vertex in C' has at most $2t-6$ neighbours in each of $H_1, \dots, H_i$. By Lemma 9 (5), every vertex in C' also has at most $2t-6$ neighbours in H_{i+1}. Thus (2b) is maintained for C'. The subgraphs in $H_1, \dots, H_{i+1}$ that are adjacent to C' are a subset of $Q_1, \dots, Q_k, H_{i+1}$, which are pairwise adjacent. Suppose that $k = t-2$ and C' is adjacent to all of $Q_1, \dots, Q_{t-2}, H_{i+1}$. Then C is adjacent to all of $Q_1, \dots, Q_{t-2}$. Contracting each of $Q_1, \dots, Q_{t-2}, H_{i+1}, C'$ into a single vertex gives K_t as a minor of G, a contradiction. Hence C' is adjacent to at most $t-2$ of $Q_1, \dots, Q_{t-2}, H_{i+1}$, and property (2a) is maintained for C'. □

Property (1d) in Theorem 11, along with a greedy $(t-1)$-colouring of the subgraphs $H_1, \dots, H_\ell$, gives the following results.

THEOREM 12. *For $t \geqslant 4$, every K_t-minor-free graph has a $(2t-2)$-colouring such that for $t-1$ colours each monochromatic component has at most $t-4$ vertices, and for the other $t-1$ colours each monochromatic component is a path.*

COROLLARY 13. *For $t \geqslant 4$, every K_t-minor-free graph has a $(3t-3)$-colouring such that for $t-1$ colours, each monochromatic component has at most $t-4$ vertices, and the other $2t-2$ colour classes are independent sets.*

The same greedy $(t-1)$-colouring of the subgraphs $H_1, \dots, H_\ell$, together with Theorem 11 (1b), gives the following result.

THEOREM 14. *For $t \geqslant 4$, every K_t-minor-free graph has a $(t-1)$-colouring such that each monochromatic component has treewidth at most $t-3$.*

Note that DeVos *et al.* [**8**] proved that for every proper minor-closed class of graphs, every graph in that class has a 2-colouring such that each monochromatic component has bounded treewidth. Their proof again uses the Graph Minor Structure Theorem, leading to a very large bound on the treewidth.

Property (2a) in Theorem 11 means that if Q is the graph obtained G by contracting each subgraph H_i to a single vertex, then Q is chordal with no K_t-subgraph, and thus with treewidth at most $t-2$. Indeed, $H_1, \dots, H_\ell$ defines an elimination ordering of Q. In the language of Reed and Seymour [**42**], $H_1, \dots, H_\ell$ is a *chordal decomposition* with *touching pattern* Q. We only need that Q is $(t-2)$-degenerate for Theorems 1 and 2, but it is interesting that, in fact, Q has treewidth at most $t-2$.

Even though we do not use it explicitly in this paper, it is an interesting aspect of our decomposition that the superstructure (that is, Q) has bounded treewidth, as does each piece of the decomposition. There are several other properties in Theorem 11 we do not use, but we mention them since they might be useful for other applications.

4. *Excluding a complete bipartite minor*

This section presents decomposition results analogous to Theorem 11 for $K_{s,t}$-minor-free graphs, leading to bounds on the defective and clustered chromatic number. Those decomposition results and the more technical proofs can be found towards the end of the section.

In fact, for most of this section we will consider the larger classes of $K^*_{s,t}$-minor-free graphs, where $K^*_{s,t}$ is the complete join of K_s and $\overline{K_t}$. We start with $s \in \{1,2,3\}$ before considering the general case. Graphs with no $K_{1,t}$ minor (note that $K_{1,t} = K^*_{1,t}$) are easily coloured. Every such graph has maximum degree at most $t-1$, and is therefore 1-colourable with defect $t-1$. Moreover, every BFS layer has at most $t-1$ vertices, so alternately colouring the BFS layers gives a 2-colouring with clustering $t-1$.

Next consider the $s=2$ case. Ossona de Mendez *et al.* [40] proved that every $K^*_{2,t}$-minor-free graph is 2-colourable with defect $\mathcal{O}(t^3)$. Our decomposition results imply the following improvement.

THEOREM 15. *Every $K^*_{2,t}$-minor-free graph is 2-colourable with defect $2t-2$.*

The decomposition results for $K^*_{2,t}$-minor-free graphs also imply that every is 4-colourable with clustering $t-1$. This result can be improved as follows. The proof is inspired by a method of Gonçalves [22].

THEOREM 16. *Every $K^*_{2,t}$-minor-free graph G is 3-colourable with clustering $t-1$. Moreover, for each edge vw of G, there is such a 3-colouring in which v and w are both isolated in their respective monochromatic subgraphs.*

Now consider $K^*_{3,t}$-minor-free graphs. Ossona de Mendez *et al.* [40] proved that the defective chromatic number of $K^*_{3,t}$-minor-free graphs equals 3. In particular, every $K^*_{3,t}$-minor-free graph is 3-colourable with defect $\mathcal{O}(t^4)$. Our decomposition results again imply an improvement.

THEOREM 17. *Every $K^*_{3,t}$-minor-free graph is 3-colourable with defect $4t$, and is 6-colourable with clustering $2t$.*

It follows from Euler's Formula that graphs with Euler genus g exclude $K_{3,2g+3}$ as a minor. Thus the second part of Theorem 17 is related to the results of Esperet and Ochem [18] and Kawarabayashi and Thomassen [31] that every graph of Euler genus g can be 5-coloured with clustering $\mathcal{O}(g)$. Kleinberg *et al.* [33] constructed planar graphs that cannot be 3-coloured with bounded clustering. We conjecture that every $K_{3,t}$-minor-free graph is 4-colourable with clustering $f(t)$, for some function f.

It is possible to improve the bound on the cluster size for the 6-colouring result in Theorem 17. In a $K^*_{3,t}$-minor-free graph, every BFS layer induces a $K^*_{2,t}$-minor-free graph, which is 3-colourable with clustering $t-1$ by Theorem 16. Using disjoint sets of three colours for alternate BFS layers gives a 6-colouring with clustering $t-1$.

Finally, in this section we consider general $K_{s,t}$-minor-free graphs. Ossona de Mendez *et al.* [40] proved that the defective chromatic number of $K_{s,t}$-minor-free graphs equals s. We show that the clustered chromatic number of $K_{s,t}$-minor-free graphs is at least $s+1$, thus generalising the above-mentioned lower bound of Kleinberg *et al.* [33].

PROPOSITION 18. *For every $s \geqslant 1$, $t \geqslant \max\{s,3\}$ and $c \geqslant 1$, there is a $K_{s,t}$-minor-free graph G_s such that every s-colouring of G_s has a monochromatic component of order greater than c.*

Proof. Define G_s recursively as follows. Let G_1 be the path on $c+1$ vertices. For $s \geqslant 2$, let G_s be the graph obtained from c disjoint copies of G_{s-1} by adding one dominant vertex.

We claim that G_s is not s-colourable with clustering c. We prove this claim by induction on $s \geqslant 1$. Obviously, G_1 is not 1-colourable with clustering c. Now assume that $s \geqslant 2$ and G_{s-1} is not $(s-1)$-colourable with clustering c. Suppose that G_s has an s-colouring with clustering c. Say the dominant vertex in G_s is coloured black. At most $c-1$ copies of G_{s-1} contain a black vertex, which implies that at least one copy has no black vertex. Thus G_{s-1} has an

$(s-1)$-colouring with clustering c, which is a contradiction. Hence G_s is not s-colourable with clustering c, as claimed.

It remains to show that G_s is $K_{s,t}$-minor-free with $t \geqslant \max\{s, 3\}$. We do so by induction on $s \geqslant 1$. G_1 is a path, and therefore contains no $K_{1,3}$ minor. G_2 is outerplanar, and therefore contains no $K_{2,3}$ minor. G_3 is planar, and therefore contains no $K_{3,3}$ minor.

Now assume that $s \geqslant 4$ and G_{s-1} contains no $K_{s-1,s-1}$ minor, but G_s contains a $K_{s,s}$ minor. Let v be the dominant vertex in G_s. We may assume that v is the entire image of one vertex in the $K_{s,s}$ minor in G. Since $K_{s,s}$ is 2-connected, the $K_{s,s}$ minor is contained in one copy of G_{s-1} plus v. Deleting any one vertex from $K_{s,s}$ gives a subgraph that contains a $K_{s-1,s-1}$ subgraph. Thus G_{s-1} contains a $K_{s-1,s-1}$ minor, which is a contradiction. We conclude that for $s \geqslant 4$, G_s has no $K_{s,s}$ minor, so certainly no $K_{s,t}$ minor with $t \geqslant s$ $(= \max\{s, 3\})$. □

Determining the clustered chromatic number of $K_{s,t}$-minor-free graphs is an open problem. Proposition 18 provides a lower bound of $s+1$. Since $K_{s,t}$-minor-free graphs are defectively s-colourable [**40**], Lemma 5 implies an upper bound of $3s$. In general, for every graph H, it is possible that the clustered chromatic number of H-minor-free graphs is at most one more than the defective chromatic number of H-minor-free graphs.

We now give the structural results and proofs of the above statements in this section. All the results in this section are based on LexBFS, so we first present the following general lemma. Recall the definition of the width of a connected partition from the beginning of Section 3.

LEMMA 19. *Suppose that a graph G has a connected partition $H_1, \dots, H_\ell$ with width k. If each subgraph H_i is induced by a BFS subtree of $G - V(V(H_1) \cup \dots \cup V(H_{i-1}))$ with at most p leaves, then G is $(k+1)$-colourable with defect $3p-1$, and G is $(2k+2)$-colourable with clustering p.*

If, in addition, each subgraph H_i is induced by a LexBFS subtree of $G - (V(H_1) \cup \dots \cup V(H_{i-1}))$ with at most p leaves, then G is $(k+1)$-colourable with defect $2p$.

Proof. Colour the subgraphs $H_1, \dots, H_\ell$ greedily in this order, such that adjacent subgraphs receive distinct colours. Since the partition has width k, $k+1$ colours suffice. Colour each vertex in H_i by the colour assigned to H_i. In each subgraph H_i each BFS layer has at most p vertices. Since a vertex in a BFS subtree has neighbours in its own layer and in the two layers below and above its own layer only, H_i has maximum degree at most $3p-1$. Hence G is $(k+1)$-colourable with defect $3p-1$. Moreover, if each subgraph H_i is induced by a LexBFS subtree of $G - (V(H_1) \cup \dots \cup V(H_{i-1}))$ with at most p leaves, then by Lemma 6, H_i has maximum degree $2p$.

For the clustering claim, alternately 2-colour the BFS layers in each H_i, and take the product with the $(k+1)$-colouring of $H_1, \dots, H_\ell$ to produce a $(2k+2)$-colouring of G with clustering p. □

As an aside, note that Van den Heuvel *et al.* [**27**] proved that every planar graph has a connected partition $H_1, \dots, H_n$ with width 2, such that each subgraph H_i is a shortest path in $G - (V(H_1) \cup \dots \cup V(H_{i-1}))$. Thus, Lemma 19 with $k = 2$ and $p = 1$ implies that planar graphs are 3-colourable with defect 2, which is the best possible result for defective 3-colouring of planar graphs, first proved by Cowen *et al.* [**6**]. In fact, each monochromatic component is a path, which was previously proved by Goddard [**21**] and Poh [**41**].

For $K^*_{2,t}$-minor-free graphs we have the following.

LEMMA 20. *Every $K^*_{2,t}$-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ with width 1, such that each subgraph H_i is induced by a LexBFS subtree of $G - (V(H_1) \cup \dots \cup V(H_{i-1}))$ with at most $t-1$ leaves.*

Proof. We may assume that G is connected. We construct $H_1, \dots, H_\ell$ iteratively. Let H_1 be the subgraph induced by a single vertex in G.

Assume that $H_1, \dots, H_i$ are defined for some $i \geqslant 1$, and C is a component of $G - (V(H_1) \cup \cdots \cup V(H_i))$ adjacent to one of $H_1, \dots, H_i$. (Since G is connected, C is adjacent to at least one of those subgraphs.) So C is adjacent to H_a, for some $a \in [i]$, and to no other subgraph in $H_1, \dots, H_i$; let A be the set of vertices in C adjacent to H_a, and let r be a vertex in A. Let S be a LexBFS subtree of C rooted at r, such that every vertex in A is in S, and subject to this property, $|V(S)|$ is minimal. Thus every leaf of S is in A. Let S_0 be the subtree of S obtained by deleting the leaves. If S has at least t leaves, then a $K^*_{2,t}$ minor is obtained by contracting H_a to a vertex and contracting S_0 to a vertex. Thus S has at most $t-1$ leaves. Let H_{i+1} be the subgraph of C induced by $V(S)$. Since every vertex in A is in S, every component of $G - (V(H_1) \cup \cdots \cup V(H_{i+1}))$ is adjacent to at most one of $H_1, \dots, H_{i+1}$. Iterating this process gives the desired partition. ☐

Lemmas 19 and 20 immediately imply Theorem 15 and show that $K^*_{2,t}$-minor-free graphs are 4-colourable with clustering $t-1$. As expressed in Theorem 16, this can be improved to a 3-colouring with the same clustering bounds, as we now prove.

Proof of Theorem 16. We proceed by induction on $|V(G)|$. The claim is trivial if $|V(G)| \leqslant t+1$. Now assume that vw is an edge in a $K^*_{2,t}$-minor-free graph G, and the result holds for $K^*_{2,t}$-minor-free graphs with fewer vertices than G. If $\deg_G(v) = 1$, then by induction $G - v$ has a 3-colouring in which w is isolated in its monochromatic subgraph. Assign v a colour not assigned to w. We obtain the desired colouring of G.

Now assume that $\deg_G(v) \geqslant 2$ and, similarly, $\deg_G(w) \geqslant 2$. Let A and B be disjoint sets of vertices in G such that $v \in A$ and $w \in B$, $G[A]$ and $G[B]$ are connected, and vw is the only edge between A and B, and subject to these properties, $|A \cup B|$ is maximum. The sets A and B are well-defined, since $A = \{v\}$ and $B = \{w\}$ satisfy the conditions. Let Z be the set of vertices in $V(G) \setminus (A \cup B)$ adjacent to both A and B, and let $Y = V(G) \setminus (A \cup B \cup Z)$.

If $|Z| \geqslant t$, then contracting A and B into single vertices gives a $K^*_{2,t}$ minor. Thus $|Z| \leqslant t-1$. Since $G[A]$ is connected and every vertex in Z is adjacent to A, $G[A \cup Z]$ is connected. Similarly, $G[B \cup Z]$ is connected.

Let G_1 be obtained from G by contracting $G[B \cup Z]$ into a single vertex x. Note that vx is an edge of G_1. Let G_2 be obtained from G by contracting $G[A \cup Z]$ into a single vertex y. Note that wy is an edge of G_2. Since G_1 and G_2 are minors of G, they both contain no $K^*_{2,t}$ minor. Since $\deg_G(v) \geqslant 2$ and $\deg_G(w) \geqslant 2$, both G_1 and G_2 have fewer vertices than G.

By induction, G_1 is 3-colourable with clustering $t-1$ such that v and x are both isolated in their respective monochromatic subgraphs, and G_2 is 3-colourable with clustering $t-1$ such that w and y are both isolated in their respective monochromatic subgraphs. Permute the colours in G_2 so that $x \in V(G_1)$ and $y \in V(G_2)$ receive the same colour, and $v \in V(G_1)$ and $w \in V(G_2)$ receive distinct colours.

Let G_3 be obtained from G by contracting $G[A \cup B \cup Z]$ into a single vertex z. Note that $V(G_3) = Y \cup \{z\}$. By induction, G_3 is 3-colourable with clustering $t-1$ such that z is isolated in its colour class. Permute the colours in G_3 so that z receives the same colour as $x \in V(G_1)$, which is the same colour assigned to $y \in V(G_2)$.

Colour each vertex in Z by the colour assigned to x and y. Colour each vertex in A by its colour in G_1. Colour each vertex in B by its colour in G_2. Finally, colour each vertex in Y by its colour in G_3.

Since x is isolated in its monochromatic subgraph in G_1, y is isolated in its monochromatic subgraph in G_2, and z is isolated in its monochromatic subgraph in G_3, every monochromatic component intersecting Z is contained in Z, and thus has at most $t-1$ vertices. Since vw is the only edge between A and B, and v and w are assigned distinct colours, every monochromatic component that intersects A is contained in A, and therefore by induction has at most $t-1$ vertices. Similarly, every monochromatic component that intersects B is contained in B, and therefore by induction has at most $t-1$ vertices. □

The following lemma is used in our decomposition result for $K^*_{3,t}$-minor-free graphs.

LEMMA 21. *For every connected graph G, non-empty sets $A, B \subseteq V(G)$, and integer $t \geqslant 1$,*
(1) *G has a LexBFS subtree T with at most $2t+1$ leaves, such that T intersects both A and B, and $V(T)$ separates A and B; or*
(2) *G has a $K_{1,t}$ minor with every branch set intersecting both A and B.*

Proof. Let r be a vertex in A. Let X be a LexBFS spanning tree of G rooted at r. For a set $L \subseteq V(G)$, let T_L be the subtree of X consisting of the union of all paths in X between L and r. Choose $L \subseteq V(G)$ so that $V(T_L)$ is an AB-separator, and subject to this property, $|L|$ is minimum. This is well defined since if $L = V(G)$, then $V(T_L) = V(G)$. By the minimality of $|L|$, every vertex in L is a leaf of T_L. And by the definition of T_L, every leaf of T_L is in L.

For each $x \in L$, let p_x be the vertex closest to x in T_L, such that $\deg_T(p_x) \geqslant 3$ or $p_x = r$. Let Q_x be the path in T_L between x and p_x not including p_x. We call Q_x the *leaf path at* x. Let $T_0 = T_L - \bigcup_{x\in L} V(Q_x)$. Let H be the graph obtained from $G - V(T_0)$ by contracting the leaf path Q_x corresponding to each $x \in L$ into a single vertex y_x. We consider A to also be a set of vertices in H, where a vertex y_x is in A if any vertex of Q_x is in A, and similarly for B. Let S be a minimum AB-separator in H.

First suppose that $|S| \geqslant t+1$. By Menger's Theorem, there are $t+1$ pairwise disjoint AB-paths $Z_1, \dots, Z_{t+1}$ in H. Since $V(T_L)$ is an AB-separator in G, each Z_i contains y_x for some $x \in L$, and each vertex y_x is on at most one path Z_i. For $i \in [t]$, if Z_i contains y_x, then contract $Z_i \cup Q_x$ into a single vertex. If Z_{t+1} contains y_x, then contract $Z_{t+1} \cup Q_x \cup T_0$ into a single vertex. We obtain a $K_{1,t}$ minor with every branch set intersecting both A and B (since each Q_x is adjacent to T_0), and (2) holds.

Now assume that $|S| \leqslant t$. Let S_1 be the set of vertices $x \in L$ such that y_x is in S. Let S_2 be the set of vertices in $G - S_1$ that correspond to vertices in S. Thus $|S| = |S_1| + |S_2|$. Let Z be the set of vertices $z \in V(T_0)$ such that $z = p_x$ for some $x \in L \setminus S_1$, and $z \neq p_x$ for all $x \in S_1$. Let $T' = T_{L'}$, where $L' = S_1 \cup S_2 \cup Z$. Since S separates A and B in H, and T' contains $T_0 \cup S_1 \cup S_2$ along with Q_x for each $x \in S_1$, it follows that $V(T')$ separates A and B in G.

By the definition of p_x, for each vertex $z \in Z$ there are at least two vertices x and x' in $L \setminus S_1$ for which $z = p_x = p_{x'}$. Thus $|L \setminus S_1| \geqslant 2|Z|$. By the choice of L, we can argue

$$|L| \leqslant |L'| = |S_1| + |S_2| + |Z| \leqslant |S| + \tfrac{1}{2}|L \setminus S_1| \leqslant t + \tfrac{1}{2}|L|.$$

Hence $|L| \leqslant 2t$. Thus T_L is a LexBFS subtree with at most $2t$ leaves, such that $V(T_L)$ separates A and B. Let T be obtained from T_L by adding a shortest path in X from T_L to B. Then T is a LexBFS subtree with at most $2t+1$ leaves, such that T intersects both A and B, and $V(T)$ separates A and B. □

We are now ready to prove the following structural lemma.

LEMMA 22. *Every $K^*_{3,t}$-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ with width 2, such that each subgraph H_i is induced by a LexBFS subtree of $G - (V(H_1) \cup \cdots \cup V(H_{i-1}))$ with at most $2t+1$ leaves.*

Proof. We again may assume that G is connected. We construct $H_1, \ldots, H_\ell$ iteratively, maintaining the property that for each $i \in [\ell - 1]$, each component C of $G - (V(H_1) \cup \cdots \cup V(H_i))$ is adjacent to at most two of $H_1, \ldots, H_i$, and if C is adjacent to H_a and H_b, for some distinct $a, b \in [i]$, then H_a and H_b are adjacent. Call this property $(\star)$.

Assume that $H_1, \ldots, H_i$ is defined, and C is a component of $G - (V(H_1) \cup \cdots \cup V(H_i))$. (Hence C satisfies property $(\star)$.)

Suppose C is adjacent to H_a, for some $a \in [i]$, and to no other subgraph in $H_1, \ldots, H_i$. Let H_{i+1} be a subgraph of C induced by one vertex adjacent to H_a. Let C' be a component of $G - (V(H_1) \cup \cdots \cup V(H_{i+1}))$. If C' is a component of $G - (V(H_1) \cup \cdots \cup V(H_i))$, then C' is not adjacent to H_i, and $(\star)$ is maintained for C'. Otherwise C' is a component of $C - V(H_{i+1})$, and C' is adjacent to H_{i+1} and possibly H_a. Since H_{i+1} and H_a are adjacent, $(\star)$ holds for C'.

Now assume that C is adjacent to H_a and H_b, for some distinct $a, b \in [i]$, and to no other subgraph in $H_1, \ldots, H_i$. Let A be the set of vertices in C adjacent to H_a, and let B be the set of vertices in C adjacent to H_b. By Lemma 21 above we have the following: (1) C has a LexBFS subtree T separating A and B, such that T intersects both A and B, and T has at most $2t + 1$ leaves, or (2) C has a $K_{1,t}$ minor with every branch set intersecting both A and B. In case (1), let H_{i+1} be the subgraph of C induced by $V(T)$. Since T intersects both A and B, the subgraph H_{i+1} is adjacent to both H_a and H_b. Let C' be a component of $G - (V(H_1) \cup \cdots \cup V(H_{i+1}))$. If C' is a component of $G - (V(H_1) \cup \cdots \cup V(H_i))$, then C' is not adjacent to H_{i+1}, and $(\star)$ is maintained for C'. Otherwise, C' is a component of $C - V(H_{i+1})$. Then C' is adjacent to H_{i+1} and at most one of H_a and H_b (since $V(T)$ separates A and B). Thus property $(\star)$ holds for C' (since H_{i+1} is adjacent to both H_a and H_b).

In Case (2), with H_a and H_b we obtain a $K^*_{3,t}$ minor in G, which is a contradiction. $\square$

5. *Generalised colouring numbers*

This section presents bounds on generalised colouring numbers, first introduced by Kierstead and Yang [**32**]. Generalised colouring numbers are important because they characterise bounded expansion classes [**54**], they characterise nowhere dense classes [**23**], and have several algorithmic applications such as the constant-factor approximation algorithm for domination number by Dvořák [**12**], and the almost linear-time model-checking algorithm of Grohe *et al.* [**24**]. They also interpolate between degeneracy and treewidth (strong colouring numbers) and between degeneracy and treedepth (weak colouring numbers). See [**27**, **36**, **39**] for more details.

For a graph G, linear ordering $\preccurlyeq$ of $V(G)$, vertex $v \in V(G)$, and integer $r \geqslant 1$, let $S_r(G, \preccurlyeq, v)$ be the set of vertices $x \in V(G)$ for which there is a path $v = w_0, w_1, \ldots, w_{r'} = x$ of length $r' \in [0, r]$ such that $x \preccurlyeq v$ and $v \prec w_i$ for all $i \in [r - 1]$. Similarly, let $W_r(G, \preccurlyeq, v)$ be the set of vertices $x \in V(G)$ for which there is a path $v = w_0, w_1, \ldots, w_{r'} = x$ of length $r' \in [0, r]$ such that $x \preccurlyeq v$ and $x \prec w_i$ for all $i \in [r' - 1]$. For a graph G and integer $r \geqslant 1$, the *r-strong colouring number* $\operatorname{scol}_r(G)$ of G is the minimum integer k such that there is a linear ordering $\preccurlyeq$ of $V(G)$ with $|S_r(G, \preccurlyeq, v)| \leqslant k$ for each vertex v of G. Similarly, the *r-weak colouring number* $\operatorname{wcol}_r(G)$ is the minimum integer k such that there is a linear ordering $\preccurlyeq$ of $V(G)$ with $|W_r(G, \preccurlyeq, v)| \leqslant k$ for each vertex v of G.

The following lemma is implicitly proved by Van den Heuvel *et al.* [**27**].

LEMMA 23 (Van den Heuvel *et al.* [**27**]). *Let $H_1, \ldots, H_\ell$ be a connected partition of a graph G with width k, such that there exists p such that for $i \in [\ell]$, $V(H_i) = V(P_{i,1}) \cup \cdots \cup V(P_{i,p_i})$, where $p_i \leqslant p$ and each $P_{i,j}$ is a shortest path in $G - ((V(H_1) \cup \cdots \cup V(H_{i-1})) \cup (V(P_{i,1}) \cup \cdots \cup V(P_{i,j-1})))$. Then the generalised colouring numbers of G satisfy for every $r \geqslant 1$:*

$$\operatorname{scol}_r(G) \leqslant p(k+1)(2r+1) \quad \textit{and} \quad \operatorname{wcol}_r(G) \leqslant p\binom{r+k}{k}(2r+1).$$

Note that the conditions on the paths $P_{i,j}$ in the lemma are implied if H_i is induced by a BFS subtree with at most p leaves in $G - (V(H_1) \cup \cdots \cup V(H_{i-1}))$.

For example, combining Lemma 23 with a variant of Theorem 10, Van den Heuvel *et al.* [**27**] proved that every K_t-minor-free graph G satisfies:

$$\operatorname{scol}_r(G) \leqslant \binom{t-1}{2}(2r+1) \quad \text{and} \quad \operatorname{wcol}_r(G) \leqslant (t-3)\binom{r+t-2}{t-2}(2r+1).$$

Lemmas 20 and 23 imply:

Theorem 24. *For every $K^*_{2,t}$-minor-free graph G and every $r \geqslant 1$,*

$$\operatorname{scol}_r(G) \leqslant 2(t-1)(2r+1) \quad \textit{and} \quad \operatorname{wcol}_r(G) \leqslant (t-1)(r+1)(2r+1).$$

And Lemmas 22 and 23 imply:

Theorem 25. *For every $K^*_{3,t}$-minor-free graph G and every $r \geqslant 1$,*

$$\operatorname{scol}_r(G) \leqslant 3(2t+1)(2r+1) \quad \textit{and} \quad \operatorname{wcol}_r(G) \leqslant (2t+1)\binom{r+2}{2}(2r+1).$$

Since graphs with Euler genus g exclude $K_{3,2g+3}$ as a minor, Theorem 25 implies that for every graph G with Euler genus g,

$$\operatorname{scol}_r(G) \leqslant 3(4g+7)(2r+1) \quad \text{and} \quad \operatorname{wcol}_r(G) \leqslant (4g+7)\binom{r+2}{2}(2r+1).$$

These result are within a constant factor of the best known bounds for graphs of Euler genus g, due to Van den Heuvel *et al.* [**27**]. Note that Theorem 25 applies to a broader class of graphs than those with bounded Euler genus. For example, the disjoint union of $g+1$ copies of K_5 has Euler genus $g+1$, but contains no $K_{3,3}$ minor. It is easy to construct 3-connected examples as well.

We conjecture that Theorems 24 and 25 can be generalised as follows:

Conjecture 26. There exists a function f such that for every $K^*_{s,t}$-minor-free graph G and every $r \geqslant 1$,

$$\operatorname{wcol}_r(G) \leqslant f(s,t)\, r^s.$$

Conjecture 26 would be implied by Lemma 23 and the following conjecture.

Conjecture 27. For all $t \geqslant s \geqslant 1$, there exists an integer p, such that every $K^*_{s,t}$-minor-free graph G has a connected partition $H_1, \dots, H_\ell$ with width $s-1$, such that for $i \in [\ell]$, $V(H_i) = V(P_{i,1}) \cup \cdots \cup V(P_{i,p_i})$, where $p_i \leqslant p$ and each $P_{i,j}$ is a shortest path in $G - ((V(H_1) \cup \cdots \cup V(H_{i-1})) \cup (V(P_{i,1}) \cup \cdots \cup V(P_{i,j-1})))$.

We now show that Conjecture 26 is true with r^s replaced by r^{s+1}.

Proposition 28. *For every $K^*_{s,t}$-minor-free graph G and every $r \geqslant 1$, we have*

$$\operatorname{scol}_r(G) \leqslant s(s+1)(t-1)(2r+1) \quad \textit{and} \quad \operatorname{wcol}_r(G) \leqslant s(t-1)\binom{r+s}{s}(2r+1).$$

Proposition 28 follows from Lemma 23 and the next lemma.

LEMMA 29. *Every $K^*_{s,t}$-minor-free graph has a connected partition $H_1, \dots, H_\ell$ with width s, such that for $i \in [\ell]$, $V(H_i) = V(P_{i,1}) \cup \dots \cup V(P_{i,p_i})$, where $p_i \leqslant s(t-1)$ and each $P_{i,j}$ is a shortest path in $G - ((V(H_1) \cup \dots \cup V(H_{i-1})) \cup (V(P_{i,1}) \cup \dots \cup V(P_{i,j-1})))$.*

Proof. Once more we may assume that G is connected. We construct $H_1, \dots, H_\ell$, maintaining the property that for each $i \in [\ell - 1]$, each component C of $G - (V(H_1) \cup \dots \cup V(H_i))$ is adjacent to at most s subgraphs in $H_1, \dots, H_i$, and that the subgraphs C is adjacent to are also pairwise adjacent. Call this property $(\star)$.

Assume that $H_1, \dots, H_i$ is defined, and C is a component of $G - (V(H_1) \cup \dots \cup V(H_i))$. (Hence C satisfies property $(\star)$.) Let $Q_1, \dots, Q_k$ be the subgraphs in $H_1, \dots, H_i$ that are adjacent to C. Thus $Q_1, \dots, Q_k$ are pairwise adjacent and $k \leqslant s$.

Since G is connected, $k \geqslant 1$. For $j \in [k]$, let A_j be the set of vertices in C adjacent to Q_j. Each A_j is non-empty. Let $\{F_1, \dots, F_m\}$ be a maximal set of pairwise disjoint connected subgraphs constructed as follows. The subgraph F_1 is induced by a minimal BFS subtree S_1 in C rooted at some vertex $v \in V(C)$ and with S_1 intersecting all of $A_1, \dots, A_k$. For $j \geqslant 1$, F_{j+1} is induced by a minimal BFS subtree S_{j+1} in $C - (V(F_1) \cup \dots \cup V(F_j))$ rooted at some vertex v that is adjacent to $F_1 \cup \dots \cup F_j$, and with S_{j+1} intersecting all of $A_1, \dots, A_k$. By minimality, each S_j has at most $k \leqslant s$ leaves. Thus each S_j is the union of at most s shortest paths in $C - (V(F_1) \cup \dots \cup V(F_{j-1}))$.

Suppose that $k \leqslant s - 1$. Let $H_{i+1} = F_1$. Then H_{i+1} satisfies the claim. Consider a component C' of $G - (V(H_1) \cup \dots \cup V(H_{i+1}))$. If C' is disjoint from C, then C' is a component of $G - (V(H_1) \cup \dots \cup V(H_i))$ and C' is not adjacent to H_{i+1}. Otherwise, C' is contained in C, and the subgraphs in $H_1, \dots, H_{i+1}$ that are adjacent to C' are a subset of the at most s subgraphs $Q_1, \dots, Q_k, H_{i+1}$, which are pairwise adjacent since F_1 intersects all of $A_1, \dots, A_k$. In both cases, property $(\star)$ is maintained.

Now assume that $k = s$. If $m \geqslant t$, then contracting each of $Q_1, \dots, Q_s, F_1, \dots, F_t$ to a single vertex gives a $K^*_{s,t}$ minor. So we are left with the case $m \leqslant t - 1$. Let H_{i+1} be the subgraph of C induced by $V(F_1) \cup \dots \cup V(F_m)$. Hence H_{i+1} is induced by the union of p_{i+1} paths $P_1, \dots, P_{p_{i+1}}$, where $p_{i+1} \leqslant ms \leqslant s(t-1)$ and each P_j is a shortest path in $G - ((V(H_1) \cup \dots \cup V(H_i)) \cup (V(P_1) \cup \dots \cup V(P_{j-1})))$. Consider a component C' of $G - (V(H_1) \cup \dots \cup V(H_{i+1}))$. If C' is disjoint from C, then C' is a component of $G - (V(H_1) \cup \dots \cup V(H_i))$ and C' is not adjacent to H_{i+1}. Otherwise, C' is contained in C, and C' does not intersect some A_j by the maximality of m. Thus C' is adjacent to a subset of at most s subgraphs in $Q_1, \dots, Q_s, H_{i+1}$, which are pairwise adjacent (since H_{i+1} intersects all of $A_1, \dots, A_s$). In both cases, property $(\star)$ is maintained. □

6. *Excluded immersions*

This section studies the defective chromatic number of graphs excluding a fixed immersion. A graph G contains a graph H as an *immersion* (also called a *weak immersion*) if the vertices of H can be mapped to distinct vertices of G, and the edges of H can be mapped to pairwise edge-disjoint paths in G, such that each edge vw of H is mapped to a path in G whose endpoints are the images of v and w. The image in G of each vertex in H is called a *branch vertex*. A graph G contains a graph H as a *strong immersion* if G contains H as an immersion such that for each edge vw of H, no internal vertex of the path in G corresponding to vw is a branch vertex.

Inspired no doubt by Hadwiger's conjecture, Lescure and Meyniel [**37**] and Abu-Khzam and Langston [**1**] independently conjectured that every K_t-immersion-free graph is properly $(t-1)$-colourable. Often motivated by this question, structural and colouring properties of graphs excluding a fixed immersion have recently been widely studied [**9**, **10**, **13**, **15**, **20**, **43**, **51**]. The best upper bound, due to Gauthier *et al.* [**19**], says that every K_t-immersion-free graph is properly $(3.54t + 3)$-colourable.

We prove that the defective chromatic number of K_t-immersion-free graphs equals 2.

THEOREM 30. *Every graph not containing K_t as an immersion is 2-colourable with defect $(t-1)^3$.*

THEOREM 31. *For every integer t, there is an integer d such that every graph not containing K_t as a strong immersion is 2-colourable with defect d.*

Notice that immersions naturally also appear in the setting of *multigraphs*, allowing multiple edges but no loops. It is obvious that if G is a multigraph with edge multiplicity at most m, then the results of the theorems above hold with defect $m(t-1)^3$ and md, respectively. On the other hand, if every edge in a multigraph has multiplicity $m+1$, then no two adjacent vertices get the same colour in a colouring with defect m. In particular, the graph obtained by replacing the edges in the complete graph K_{t-1} by $m+1$ parallel edges does not have K_t as an immersion, but is also not $(t-2)$-colourable with defect m.

We leave as an open problem to determine the clustered chromatic number of graphs excluding a (strong or weak) K_t immersion. It was observed by both Haxell *et al.* [**26**] and Liu and Oum [**38**] that the results in Alon *et al.* [**2**] prove that for every k, N, there exists a $(4k-2)$-regular graph G such that every k-colouring of G has a monochromatic component of size at least N. In other words, the clustered chromatic number of graphs with maximum degree Δ is at least $\lfloor \frac{1}{4}(\Delta+6) \rfloor$. Since every graph with maximum degree at most $t-2$ contains no (strong or weak) K_t immersion, the clustered chromatic number of graphs excluding a (strong or weak) K_t immersion is at least $\lfloor \frac{1}{4}(t+4) \rfloor$.

The proof of Theorem 30 uses the following structure theorem from DeVos *et al.* [**10**]. The theorem is not explicitly proved in [**10**], but can be derived easily from the proof of [**10**, Theorem 1] on page 4 of that paper.

For each edge xy of a tree T, let $T(xy)$ and $T(yx)$ be the components of $T-xy$, where x is in $T(xy)$ and y is in $T(yx)$. For a tree T and graph G, a *T-partition* of G is a partition $(T_x \subseteq V(G) : x \in V(T))$ of $V(G)$ indexed by the nodes of T. As before, each set T_x is called a *bag*. Note that a bag may be empty. For each edge $xy \in E(T)$, let $G(T,xy) = \bigcup_{z \in V(T(xy))} T_z$ and $G(T,yx) = \bigcup_{z \in V(T(yx))} T_z$. Let $E(T,xy)$ $(= E(T,yx))$ be the set of edges in G between $G(T,xy)$ and $G(T,yx)$. The *adhesion* of a T-partition is the maximum, taken over all edges xy of T, of $|E(T,xy)|$. For each node x of T, the *torso of x* (with respect to a T-partition) is the graph obtained from G by identifying $G(T,yx)$ into a single vertex for each edge xy incident to x, deleting resulting parallel edges and loops.

THEOREM 32 (following DeVos *et al.* [**10**]). *For every graph H with t vertices and every graph G that does not contain H as an immersion, there is a tree T and a T-partition of G with adhesion less than $(t-1)^2$, such that each bag has at most $t-1$ vertices.*

A structural result similar to this theorem was proved by Wollan [**51**]. We also need the following lemma.

LEMMA 33. *Let G be a graph such that for some tree T with vertex set $V(G)$, for each edge xy of T, the number of edges of G between $V(T(xy))$ and $V(T(yx))$ is at most k. Then G is 2-colourable with defect k.*

Proof. We use induction on $|V(G)|$, noting that there is nothing to prove if $|V(G)| \leqslant 2$. So assume $|V(G)| \geqslant 3$. Call a vertex v of G *large* if $\deg_G(v) \geqslant k+1$; otherwise v is *small*.

If G has no large vertices, then every 2-colouring of G has defect k. Now assume that G has some large vertex. Thus there is an edge uv of T such that u is large and u is the only large vertex in $V(T(uv))$. Set $a = |V(T(uv))|$. Suppose that every vertex in $V(T(uv)) \setminus \{u\}$ has a neighbour in G in $V(T(vu))$. Since u has at least $k+1-(a-1)$ neighbours outside $V(T(uv))$, the number of edges between $V(T(uv))$ and $V(T(vu))$ is at least $(k+1-(a-1))+(a-1) = k+1$, a contradiction.

So there is a vertex $w \in V(T(uv)) \setminus \{u\}$ with $N_G(w) \subseteq V(T(uv))$. Note that w is small. Let wz be an edge in T. Form the graphs G' and T', respectively, from G and T by identifying w and z (deleting loops and parallel edges). For each edge xy of T', the number of edges of G' between $V(T'(xy))$ and $V(T'(yx))$ is still at most k. Hence by induction, G' has a 2-colouring with defect k. This colouring gives a 2-colouring with defect k of all vertices of G except w. Since all vertices in $V(T(uv))$ except u are small, u is the only possible large neighbour of w. Give w the colour different from u. As all other neighbours of w are small, the monochromatic degree can increase only for small vertices. Thus the defect is at most k, as required. □

Now we are ready to prove our 2-colour result for graphs excluding an immersion.

Proof of Theorem 30. By Theorem 32, there is a tree T and a T-partition of G with adhesion at most $(t-1)^2-1$, such that each bag has at most $t-1$ vertices. Let Q be the graph with vertex set $V(T)$, where $xy \in E(Q)$ whenever there is an edge of G between T_x and T_y. Any one edge of Q corresponds to at most $t-1$ edges in G. By Lemma 33, the graph Q is 2-colourable with defect $(t-1)^2-1$. Assign to each vertex v in G the colour assigned to the vertex x in Q with $v \in T_x$. Since at most $t-1$ vertices of G are in each bag, G is 2-coloured with defect at most $(t-1)\cdot((t-1)^2-1)+(t-2) < (t-1)^3$. □

To prove our result for strong immersions, we employ the following more involved structure theorem of Dvořák and Wollan [**15**].

Theorem 34 (Dvořák and Wollan [**15**]). *For every integer t, there is an integer α such that for every graph G that does not contain K_t as a strong immersion, there is a tree T and a T-partition of G with adhesion at most α such that the following holds. For each node x of T with torso S_x, if W_x is the set of vertices in S_x with degree at least α, then there is a subset $A_x \subseteq W_x$ of size at most α such that $W_x \setminus A_x$ can be enumerated $\{x_1,\dots,x_p\}$ and $V(S_x - W_x)$ can be partitioned $B_0, B_1, \dots, B_p$ (allowing $B_j = \varnothing$), such that*

(1) *each vertex $v \in A_x$ is adjacent to at most α of $B_0, B_1, \dots, B_p$ and adjacent to at most α vertices in $W_x \setminus A_x$; and*
(2) *for each $i \in [p]$, there are at most α edges between $B_0 \cup \dots \cup B_{i-1} \cup \{x_1,\dots,x_{i-1}\}$ and $B_i \cup \dots \cup B_p \cup \{x_{i+1},\dots,x_p\}$.*

We actually only need the following corollary of Theorem 34.

Corollary 35. *For every integer t, there is an integer α such that for every graph G that does not contain K_t as a strong immersion, there is a tree T and T-partition of G with adhesion at most α^2 such that for each node x of T with torso S_x, $S_x[T_x]$ has degree at most $3\alpha+2$.*

Proof. Consider a tree T and T-partition of G in accordance with Theorem 34. Consider a node x of T with torso S_x. We use the notation from the theorem.

Consider a vertex $v \in W_x$. If $v \in A_x$, then v has at most $|A_x|-1 < \alpha$ neighbours in A_x and at most α neighbours in $W_x \setminus A_x$, and thus has less than 2α neighbours in W_x. If $v \in W_x \setminus A_x$, then $v = x_i$ for some $i \in [p]$. Then v has at most $|A_x| \leqslant \alpha$ neighbours in A_x. Furthermore, there are at most α edges between $\{x_1,\dots,x_{i-2}\}$ and $\{x_i,\dots,x_p\}$, at most α edges between $\{x_1,\dots,x_i\}$ and $\{x_{i+2},\dots,x_p\}$, and at most 2 edges between x_i and $\{x_{i-1},x_{i+1}\}$. Thus v has at most $3\alpha+2$ neighbours in W_x. Hence $S_x[W_x]$ has maximum degree at most $3\alpha+2$.

Apply the following operation for each vertex $v\in T_x\setminus W_x$, for each node x of T with $|T_x|\geqslant 2$. Since $v\notin W_x$, the degree of v in S_x is at most $\alpha-1$. Since there are at most α edges from G between T_x and each contracted vertex in S_x, v has degree at most $(\alpha-1)\alpha<\alpha^2$ in G. Now delete v from T_x, add a new node y in T adjacent only to x, and define $T_y=\{v\}$. Note that the number of edges between T_y and $G-T_y$ is less than α^2, and the torso of y is isomorphic to K_2 (hence has degree one). Finally, the torso of x hasn't changed, since the contraction of the single-vertex node T_y just gives the vertex v again. In particular, the degree of v in the torso of x is still at most $\alpha-1$, and hence W_x also has not changed.

After having applied the operation from the previous paragraph as long a possible, we obtain a tree-partition of G on a tree T' with adhesion at most α^2. Moreover, for each node x of T' we have $|T'_x|=1$, and then $S'_x[T'_x]$ has degree zero, or $T'_x\subseteq W'_x$ and $S'_x[W'_x]$ has degree at most $3\alpha+2$. We immediately get that $S'_x[T'_x]$ has degree at most $3\alpha+2$ as well. ☐

Now we are ready to prove our 2-colouring result for graphs excluding a strong immersion.

Proof of Theorem 31. By Corollary 35, there is an integer α, a tree T and a T-partition of G with adhesion at most α^2, such that for each node x of T with torso S_x, $S_x[T_x]$ has degree at most $3\alpha+2$. Let Q be the graph with vertex set $V(T)$, where $xy\in E(Q)$ whenever there is an edge of G between T_x and T_y. By Lemma 33 with $k=\alpha^2$, the graph Q is 2-colourable with defect α^2. Assign to each vertex v in G the colour assigned to the vertex x in Q with $v\in T_x$.

If $v\in V(T_x)$, then every edge vw in G with $w\notin T_x$ gives rise to an edge in Q. Since the adhesion is at most α^2, any one edge of Q corresponds to at most α^2 edges in G. As the monochromatic degree of x in Q is at most α^2, this means that v has at most α^4 neighbours outside T_x with the same colour. Adding the at most $3\alpha+2$ neighbours of v in T_x, we obtain that the monochromatic degree of v in G is at most $\alpha^4+3\alpha+2$. ☐

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