Tutte-Whitney polynomials: some history and generalisations

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26 March 2006

Abstract

The Tutte-Whitney polynomials play a central role in the study of counting problems on graphs, and have close connections with statistical mechanics and knot theory. We briefly review their history and a number of generalisations. We then describe some generalised Tutte-Whitney functions introduced by the author, extending some of them further. These functions are defined for arbitrary collections of sets, and for arbitrary binary functions, and have some of the characteristic elements of Tutte-Whitney theory, including interesting partial evaluations and deletion-contraction relations.

1 Introduction

A central object in the study of counting problems on graphs is the two-variable polynomial known as the *Tutte polynomial* or (after a translation of its domain) the *(Whitney) rank generating function* of a graph. It is essentially (after a simple change of variables) a generating function for the number of sets of edges of a given size and rank in a graph. (The *rank* $\rho(X)$ of a set of edges $X \subseteq E(G)$ in a graph G is the number of vertices it meets minus the number of connected components[1] of the subgraph it defines. It is also the rank of the submatrix indexed by X in the incidence matrix over $GF(2)$ of G.) Formally, the Whitney rank generating function $R(G;x,y)$ of a graph G is

$$R(G;x,y) = \sum_{X \subseteq E(G)} x^{\rho(E)-\rho(X)} y^{|X|-\rho(X)}. \qquad (1)$$

The Tutte polynomial [69] is given for graphs by

$$T(G;x,y) = R(G;x-1,y-1). \qquad (2)$$

[1] not counting isolated vertices

We use the term *Tutte-Whitney polynomials* when referring to these polynomials in general, in contexts where it does not matter whether we speak of $R(G;x,y)$ or $T(G;x,y)$. We give a short history of the polynomials and their names in §3.

Much of the importance of these polynomials comes from the amount of information they contain. By substituting appropriate values for the variables (and doing some trivial calculations), one can obtain the numbers of colourings, flows, spanning trees, and forests for the graph in question. These two-variable polynomials also have a number of famous and interesting single-variable polynomials as specialisations, each obtained from $R(G;x,y)$ by simple algebraic transformations. These include the chromatic and flow polynomials, all-terminal reliability, partition functions of the Ising and Potts models from statistical mechanics, and the Jones polynomial of an alternating link. The definition (1) can be extended immediately to matroids, as was done by Tutte [68] and Crapo [20] (see also [60]), and for representable matroids it contains, as a specialisation, the weight enumerator of a linear code.

Furthermore, it has been shown that *any* function on a graph that is invariant under isomorphism and multiplicative over blocks (or connected components in the matroid)[2] and which obeys a *deletion-contraction relation* (a simple type of recursive rule) can be obtained from an appropriate evaluation of $R(G;x,y)$. This was shown for matroids by Tutte [68] and Brylawski [13], independently; see also [70]. Oxley and Welsh extended this matroid result to deletion-contraction relations of more general form [51]. This became known as the 'Recipe Theorem'. Its hypotheses are sufficiently broad and natural that they are met by a huge variety of interesting functions.

Dominic Welsh has played a leading role in the study and promotion of these polynomials over many years. His published results include: his famous 'Recipe Theorem' [51], mentioned above; identification of interesting evaluations of the polynomials (e.g., [51, 79]); seminal work with Jaeger and Vertigan [33] on the complexity of computing the polynomials, and a series of subsequent papers on this topic (e.g., [53, 72, 75]); investigation with Oxley of connections with clutter reliability (or percolation) [51]; and some generalisations (e.g., [48, 83, 84]). However, his contribution goes far beyond a collection of theorems. He wrote what became the leading reference works on the topic. His 1976 book *Matroid Theory* [73], for many years the standard reference on that field, contains a chapter on the Tutte and Whitney polynomials which helped to make them widely known. He also wrote a book, published in 1993, on the polynomials and closely related topics in the computational complexity of counting problems [74]. He wrote a number of important surveys [76, 78, 80, 81], and indeed many of his research papers in the field contain updated survey material on the polynomials and their evaluations. He jointly organised the first Workshop on the Tutte Polynomial and its Applications (Barcelona, 2001), and jointly edited the resulting special journal issue [41].

It is a characteristic of Dominic's work in general that his informal promotion of topics and problems, via discussions with students and colleagues, has a big influence. This has certainly been the case for the Tutte-Whitney polynomials for decades. I

[2] I.e., its value on a graph G, consisting of two graphs G_1 and G_2 that share at most one vertex, is the product of its values on G_1 and G_2.

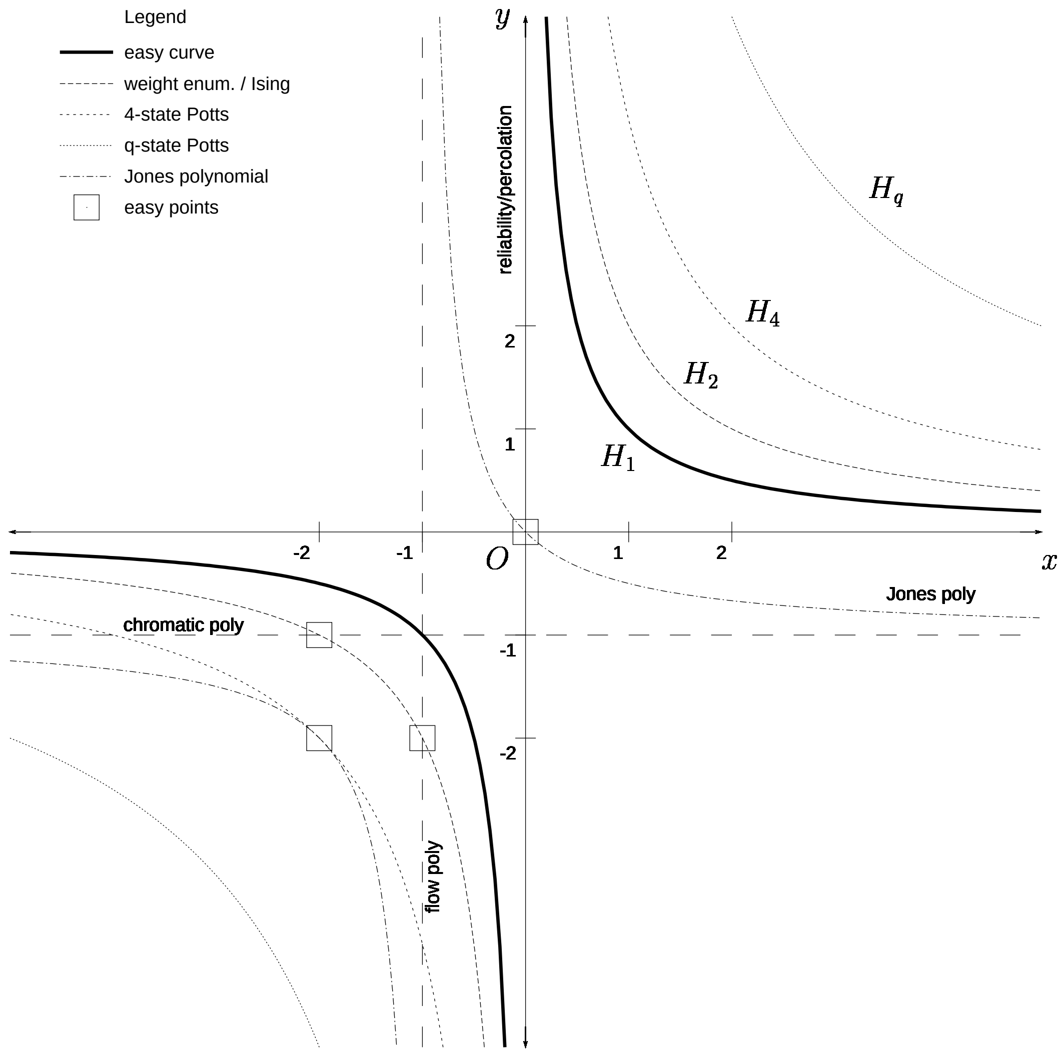

Figure 1: The Tutte-Whitney plane. The solid axes are the Whitney axes, and the dashed axes are the Tutte axes.

can recall him drawing diagrams of the "Tutte plane" when I was one of his doctoral students in the early 1980s. I was, like many others, surprised and intrigued by the amount and variety of information contained in this one bivariate polynomial. Many who have met him will recall similar diagrams, drawn at coffee tables or displayed in lectures. One of these is published as [74, Fig. 8.2]. We give such a diagram here, in Figure 1, though we use the co-ordinate axes of $R(G; x, y)$ rather than those of $T(G; x, y)$, since the extensions we concentrate on later are based on the former rather than the latter. The diagram shows a number of curves in the domain $\mathbb{R}^2$ of $R(G; x, y)$, each such curve labelled by the information contained in it (which can be obtained by evaluating $R(G; x, y)$ along an appropriate parameterisation of this curve, possibly with some trivial algebraic modification). Later, we will compare this diagram with an analogous diagram for a generalisation of $R(G; x, y)$.

The next section gives some definitions and notation. In §3 we trace the history of the Tutte-Whitney polynomials and their names, something that has not always been clear in the literature, though Dominic Welsh's writings have been important in clarifying it. In §4 we give a short and incomplete survey of some of the many generalisations of the Tutte-Whitney polynomials, concentrating on those closest to the work of Welsh. In the remaining sections, we discuss in greater detail a generalisation introduced by the author, in which a theory of Tutte-Whitney functions (alas, no longer strictly polynomials) is given for a very general class of combinatorial objects.

Further information on the Tutte-Whitney polynomials, including more complete lists of invariants obtainable from them, can be found in many of Welsh's works cited above, or [6, 15, 26]. For recent surveys on the complexity of computing these polynomials exactly or approximately, see [47] and [34], respectively.

2 Definitions

Throughout this article, $G = (V, E)$ is a graph on n vertices with vertex set V and edge set E. Loops and multiple edges are allowed. The number of connected components of G is $k(G)$. The rank $\rho(X)$ of $X \subseteq E$ satisfies $\rho(X) = n - k((V, X))$, as is easily shown from the definition given at the start of §1.

Let $e = uv$ be any edge of G. The graph $(V, E \setminus \{e\})$, formed by *deletion* of e, is denoted by $G \setminus e$. The operation of deleting e and then identifying u and v is called *contraction* of e, and the resulting graph is denoted by G/e. Contraction may create loops and/or multiple edges.

The *matrix representation* A of G is defined as follows. It has n rows and m columns indexed by vertices and edges, respectively. The column $\mathbf{a}^{(e)}$ corresponding to edge $e = uv$ (with $u \neq v$) has entries 1 and -1 in the two positions indexed by the endpoints u and v (it does not matter which way round), and all other entries 0. If $u = v$, so that e is a loop, the column $\mathbf{a}^{(e)}$ has all entries 0: as far as Tutte-Whitney polynomials are concerned, it does not matter which vertex a loop is attached to. This matrix representation can be done over any field, and of course the -1 entries will just be 1 if the field has characteristic two. In that case, we have the usual incidence matrix of G.

It is convenient at times to move between a subset X of E and its incidence vector $\mathbf{x} = (x_e)_{e \in E}$ in $GF(2)^E$, where $x_e = 1$ if $e \in X$ and $x_e = 0$ otherwise.

The *cutset space* of G is just the rowspace of the matrix representation of G over $GF(2)$. Equivalently, it consists of those edge sets whose incidence vectors belong to this rowspace (under the above correspondence between sets of edges and their incidence vectors).

A *matroid* is a pair (S, ρ) where S is a set (the *ground set*) and $\rho : 2^S \to \mathbb{N} \cup \{0\}$ (the *rank function*) satisfies: (i) $\forall X : \rho(X) \leq |X|$; (ii) $\forall X, Y : X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y)$; (iii) $\forall X, Y : \rho(X \cap Y) + \rho(X \cup Y) \leq \rho(X) + \rho(Y)$. The matroid is *graphic* if it is isomorphic[3] to (E, ρ) for some graph $G = (V, E)$, where ρ is the rank function of G as defined earlier.

Let M be a matrix over a field $\mathbb{F}$, with columns indexed by S. The *rank function* ρ assigns to each $X \subseteq S$ the rank $\rho(X)$ of the submatrix $M[X]$ of M consisting of those columns indexed by X. If a function $\rho : 2^S \to \mathbb{N} \cup \{0\}$ can be realised by a matrix over $\mathbb{F}$ in this way, then the matroid (S, ρ) is *representable* over $\mathbb{F}$. If $\mathbb{F} \cong GF(2)$ here,

[3]For matroids, an *isomorphism* is a rank-preserving bijection between ground sets.

then (S, ρ) is a *binary matroid*. Every graphic matroid is binary, via the incidence matrix, but not every binary matroid is graphic. For further information on matroids, see [49, 50, 73].

Note that the Tutte-Whitney polynomial of a graph depends only on its rank function or, in other words, on its graphic matroid.

Any function $r : 2^S \to \mathbb{R}$ has a *dual* $r^* : 2^S \to \mathbb{R}$ defined by $r^*(X) = |X| + r(S \setminus X) - r(S) + r(\emptyset)$. It is easily seen that $r^{**} = r$, and that if r is a matroid rank function then so is r^*.

This duality allows the following representation of Tutte-Whitney polynomials:

$$R(G; x, y) = \sum_{X \subseteq E} x^{\rho(E)-\rho(X)} y^{\rho^*(E)-\rho^*(E \setminus X)}.$$

If $f : 2^S \to \mathbb{R}$ then its *Hadamard transform* (or $GF(2)^{|S|}$ *Fourier transform*) $\hat{f} : 2^S \to \mathbb{R}$ is given by $\hat{f}(X) = 2^{-|S|} \sum_{Y \subseteq S} (-1)^{|X \cap Y|} f(Y)$ (see, e.g., [44, pp. 53–54]). It is well known that this transform maps the indicator function of a linear subspace of $GF(2)^S$ to a scalar multiple of the indicator function of its orthogonal subspace, and so describes the same duality as the transform $r \mapsto r^*$ for binary matroid rank functions. The correspondence is in fact more general, as shown in [22]: $f \mapsto \hat{f}$ and $r \mapsto r^*$ are always just different expressions of the same underlying duality. For other applications of Fourier transforms in graph theory, see [27].

A *2-polymatroid* is a pair (S, ν) where S is a set and $\nu : 2^S \to \mathbb{N} \cup \{0\}$ satisfies all the rank function axioms *except* that the upper bound (i) is relaxed: we require just that $\nu(X) \leq 2|X|$, rather than the ordinary rank function upper bound of $|X|$. An important class is the class of *graphic 2-polymatroids*, where $S = E$ is the edge set of a graph G and $\nu(X)$ is the number of vertices incident with edges in X.

3 History

As Welsh writes, "there is some obscurity in the literature on the history and nomenclature of these polynomials" [74, p. 44]. The polynomial $R(G; x, y)$ is often credited to one or other of two papers published by Whitney in 1932 [85, 86]. The first of these, titled 'A logical expansion in mathematics' [85], is essentially an exposition of the inclusion-exclusion principle and several applications of it, including to counting colourings. He discusses the chromatic polynomial (citing Birkhoff [7]) and its relationship with the numbers m_{ij} that give the number of sets of edges in G of rank i and nullity j. (The *nullity* of X is $|X| - \rho(X)$.) These are (after reversal of order on the first subscript) the coefficients of $R(G; x, y)$, but the paper does not give the two-variable polynomial itself.

In the second of these papers, titled 'The coloring of graphs' [86], Whitney studies the chromatic polynomial at greater length, making considerable use of the m_{ij}. In [86, §11, p. 205] he mentions a two-variable polynomial with coefficients denoted by m_{ij}, though at this stage the discussion is more general: the relationship between the coefficients of a generating function and those of its logarithm. This is his only mention, in these papers, of a two-variable polynomial. He does introduce a multivariate graph polynomial, with one variable x_T for each isomorphism type T of subgraphs of G. Each term is just a single x_T with a coefficient giving the number of subgraphs of G of type T.

In summary, it seems fairly likely that Whitney thought of his numbers m_{ij} as coefficients of a two-variable polynomial, though he does not appear to have done so explicitly in his papers on the topic.

Tutte [67] appears to be the first to define $R(G; x, y)$ explicitly, though he uses different notation and (in effect) includes an extra factor of $x^{k(G)}$. He denotes the result by $Q(G; x, y)$, and calls it the *dichromatic polynomial* in [70]. In [69] he introduced another two-variable polynomial, the *dichromate*, which is now known as the *Tutte polynomial* and denoted by $T(G; x, y)$. He defined it in terms of internal and external activities of spanning trees of G, which we shall not discuss here, except to note that this gave a combinatorial interpretation of the coefficients of $T(G; x, y)$ and a proof that they are non-negative. Welsh observes that there is "notable confusion between the dichromate and dichromatic polynomial" [74, p. 53], and comments briefly on Tutte's own use of the terms in [82].

When noting that his dichromate is now called the Tutte polynomial, Tutte writes: "This may be unfair to Hassler Whitney who knew and used analogous coefficients without bothering to affix them to two variables" [71, p. 8]. (Tutte cites [86].)

The relation (2) was first shown by Tutte in [70, p. 306] (note especially (18)–(20) there), using [69, (18)–(20)].

Among many other results, Tutte showed that these polynomials give the numbers $P_k(G; \lambda)$ of colour assignments $V(G) \to \{1, \dots, \lambda\}$ in which exactly k edges of G are monochromatic (i.e., both endpoints receive the same colour) [67, p. 35] (see also [15, §6.3] and [74, pp. 63–64]):

$$\sum_{k=0}^{m} P_k(G; \lambda) s^k = \lambda^{k(G)} (s-1)^{\rho(E)} R(G; \lambda/(s-1), s-1). \tag{3}$$

This polynomial was called the *coboundary polynomial* by Crapo and studied by him in [19]. He also introduced the terms *rank generating function*, for $R(G; x, y)$ [20, 18, 19], and *Tutte polynomial*, for $T(G; x, y)$ [20]. In the latter paper, Crapo generalised Tutte's work on internal and external activities, including (2), to matroids.

A separate stream of research on these polynomials began independently in the statistical mechanics literature with Potts in 1952 [57]. Potts introduced a statistical mechanical model, based on the set of mappings from the vertex set of a square lattice graph G to a set of q possible states, and investigated some properties of the *partition function* as the size of the lattice tends to infinity. This partition function depends on two variables: $q \in \mathbb{N}$ and a real-valued quantity K related to interaction energy along edges. (Ashkin and Teller [4] had earlier introduced a generalisation, in a different direction, of the case $q = 4$.) The function had not, at that stage, been put in a form that allowed the connection with Tutte-Whitney polynomials to be readily seen. It turned out to be essentially the coboundary polynomial in q and e^K.

The two streams seem to have joined with the work of Fortuin and Kasteleyn [25] (see also [35] for some of their relevant prior work). They introduced the *random* *cluster model partition function* Z_{RC} (see, e.g., [30, 74, 77]):

$$Z_{RC}(G; p, q) = \sum_{X \subseteq E} p^{|X|}(1-p)^{|E \setminus X|} q^{n-\rho(X)}. \tag{4}$$

They showed that: the q-state Potts model partition function ($q \in \mathbb{N}$) can be put in this form[4]; $Z_{RC}(G; p, q)$ is related to $R(G; x, y)$ by a simple algebraic transformation (with $xy = q$, $y = p/(1-p)$); and Z_{RC} contains a number of other polynomials from the literature. The terms *random cluster representation* and *Fortuin-Kasteleyn representation* have come to be used for any representation in terms of Z_{RC} (i.e., an algebraic transformation to the variables (p, q) that shows how some quantity or polynomial of interest can be obtained from Z_{RC}).

A great deal of work was done on these polynomials, in both the combinatorial and statistical mechanics literature, in the 1970s and since. In combinatorics, the work of Brylawski [13, 14] in the early 1970s was notable.[5] Welsh's book [73, Ch. 15] helped make the polynomials widely known in the combinatorial community. He attaches Whitney's name to the term 'rank generating function'.

4 Generalisations

The Tutte-Whitney polynomials may be extended in many ways. We find it convenient to group these extensions loosely into four types:

(i) generalisations of graphs that have *less structure* (in general) than graphs;

(ii) generalisations of graphs that have *more structure* than graphs;

(iii) objects different to graphs that are in some way *analogous* to graphs;

(iv) extensions of the *polynomials* (rather than the graphs), typically by introducing extra variables.

Clearly, there is the possibility of overlap between these, or of extending in more than one direction at the same time. It is an informal classification, not a precise taxonomy. Also note that, strictly speaking, the real starting point in all these generalisations is graphic matroids, rather than graphs themselves.

We mention several generalisations, grouped according to these broad types. Completeness is not intended.

Type (i)

Tutte-Whitney polynomials were first generalised to matroids by Tutte in his thesis [68], as recounted in [32], although until recently this was not widely known. The

[4] Once this is done, the restriction $q \in \mathbb{N}$ no longer matters, in that the expression makes sense for all real (or complex) q. In this form, and with q as an indeterminate, the Potts model partition function is equivalent, as a formal expression, to $Z_{RC}(G; p, q)$, and hence has all the expressive power of $R(G; x, y)$.

[5] Incidentally, Brylawski gives a tabular representation of part of the Tutte plane in [13, Table 4.12].

first published extension to matroids was by Crapo [20], who extended the definition of the Whitney rank generating function (1) to matroid rank functions in general. Independently, Smith [59] (see also [60]) extended the Tutte polynomial to representable matroids.

A further relaxation of the requirements on ρ in (1) may be found in Gordon and McMahon's generalisation to greedoids [28].

The generalisation of [22] is also of type (i). This is discussed in §6.

Welsh and Whittle [84] and Ardila [1, 2] independently introduced generalisations based on arrangements of linear spaces, which we now describe. These generalisations may be viewed as starting out from the well-known generalisation of the Tutte-Whitney polynomial to matroids representable over $\mathbb{R}$, viewed in terms of hyperplanes. From there, they generalise to arrangements of affine spaces, and (in [84]) beyond.

A graphic matroid is completely specified by the rowspace over $\mathbb{R}$ of its matrix representation A. Each column determines a hyperplane[6] in $\mathbb{R}^n$ consisting of vectors orthogonal to that column. Thus, the column $\mathbf{a}^{(e)} = (a_w^{(e)} : w \in V)^T$ corresponding to edge $e = uv$ determines the hyperplane $H_e = \{(x_w)_{w \in V} | x_u = x_v\}$. [7] It is well known that, for any $X \subseteq E$,

$$\rho(X) = n - \dim\left(\cap_{e \in X} H_e\right). \tag{5}$$

A *hyperplane arrangement* is a set of affine hyperplanes (i.e., $(n-1)$-dimensional affine subspaces of $\mathbb{R}^n$) (see, e.g., [31, Ch. 18]). A *central arrangement* is a hyperplane arrangement in which the intersection of all the hyperplanes is nonempty. The *rank* of an arrangement is the largest of the ranks (computed using the appropriate generalisation of (5)) of its central subarrangements. The rank generating function of an arrangement is then a sum, analogous to (1), but summing just over all central subarrangements, and the Tutte polynomial is obtained by the usual co-ordinate shift [2]. When the arrangement is just the set $\{H_e \mid e \in E\}$ for a graph, then we have the usual Tutte-Whitney polynomials of the graph. Central arrangements give the usual Tutte-Whitney polynomials of matroids representable over $\mathbb{R}$. Both papers [2, 84] define a coboundary polynomial for hyperplane arrangements, analogous to the right side of (3), and prove a result like (3) for counting points contained in various numbers of members of a collection of affine subspaces of $GF(q)^n$ for sufficiently large q.

Welsh and Whittle [84] extend the coboundary polynomial further, to any set of elements of a geometric lattice, which they call an *arrangement*. More generally still, they consider *configurations* consisting of a ground set together with a function that takes value 0 on the empty set and is monotone (i.e., satisfies rank axiom (ii) in §2). The configurations they are interested in mostly come from arrangements in some way.

More recently, Ardila [3] has extended the Tutte polynomial of an affine hyperplane arrangement to *semimatroids*.

Type (ii)

[6] Here, a *hyperplane* is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. This is not the same thing as the hyperplanes (maximal sets of rank $< \rho(E)$) of the corresponding matroid.

[7] A loop is regarded as determining the space $\mathbb{R}^n$, as a special "hyperplane".

There has been an abundance of extensions to objects obtained by adding extra structure to graphs or matroids. Examples include Tutte-Whitney polynomials of signed graphs [45], rooted or K-terminal graphs [65, 87], biased graphs [89], bimatrices and bimatroids [36, 40], matroid perspectives and relatives (see, e.g., [42]), and matroid pairs [83] (described in this volume in [50]).

Some extensions involve attaching one or more weights to each edge (or matroid element) [8, 21, 25, 35, 39, 62, 64, 65, 66, 88]. If the weights are treated as indeterminates then we have a multivariate polynomial of the original graph. We will discuss such extensions under type (iv) below, though they may be regarded as being in the overlap between the two types.

Type (iii)

The first step towards extending Tutte-Whitney polynomials to 2-polymatroids seems to have been taken by Borzacchini [9, 10], though he did not present it in such terms. He introduced two-variable generating polynomials for the numbers of subgraphs of a graph with a given number of vertices and edges. He gave a simple recursive rule [9], expressions in terms of such polynomials of vertex-deleted subgraphs [9, 10], and relations with some similar polynomials (one where the subgraphs in question can have any number of isolated vertices, and another where the subgraphs must be induced) [10].

Oxley and Whittle [54, 55] extended the Tutte-Whitney polynomial to 2-polymatroids in general, and developed a theory for them analogous to that of ordinary Tutte-Whitney polynomials. Oxley and Whittle's *2-polymatroid rank generating function* is

$$S(\nu; x, y) = \sum_{X \subseteq E} x^{\nu(E)-\nu(X)} y^{2|X|-\nu(X)}.$$

This polynomial is distinct from the Tutte-Whitney polynomial, even if both are restricted to graphs. For graphic 2-polymatroids, simple algebraic transformations relate $S(\nu; x, y)$ to the polynomials of Borzacchini described above. The computational complexity of $S(\nu; x, y)$ (for graphs) has been investigated by Noble [46], who gives a classification of points in its domain analogous to that given for the ordinary Tutte-Whitney polynomial in [33].

A different kind of analogue of graphs is the class of greedoids arising from rooted digraphs, whose members are given a Tutte-Whitney-like polynomial by Gordon and McMahon in [28].

Type (iv)

A number of authors have proposed more general polynomials that contain the Tutte-Whitney polynomial in some sense.

Borzacchini and Pulito [11] introduced a three-variable polynomial they called the *subgraph enumerating polynomial* (SEP).[8] It is essentially a generating function for the numbers of sets of edges X in a graph with a given size $|X|$, number $\nu(X)$ of incident vertices, and number of components. An easy change of variables gives a generating function for counting sets $X \subseteq E(G)$ according to their values of $|X|$, $\rho(X)$ and

[8]Note that Borzacchini also used this term for his two-variable polynomials, discussed above [9, 10].

$\nu(X)$. So this polynomial contains, as partial evaluations, both the Tutte-Whitney and Oxley-Whittle polynomials. Borzacchini and Pulito described the connection with Tutte-Whitney polynomials, gave recursive relations for the SEP, and showed that the SEP of a graph G can be reconstructed either from the SEPs of the vertex-deleted subgraphs of G or from the SEPs of the edge-deleted subgraphs of G.

The random cluster model partition function Z_{RC} has been generalised in many ways in the statistical mechanics literature. For example, several authors [58, 17, 56] consider sums over pairs of sets along the following lines:

$$Z_{RC}(G; p_1, p_2, p_3, q_1, q_2) = \sum_{X \subseteq E} \sum_{Y \subseteq E} p_1^{|X \setminus Y|} p_2^{|Y \setminus X|} p_3^{|X \cap Y|} (1 - p_1 - p_2 - p_3)^{|E \setminus (X \cup Y)|} q_1^{n-\rho(X)} q_2^{n-\rho(Y)}. \qquad (6)$$

This form was considered because it contains (with $q_1 = q_2 = 2$) the partition function of the Ashkin-Teller model [4] (a four-state model of which the four-state Potts model is a special case). We return to this model at the end of §7.

Noble and Welsh [48] defined a multivariate polynomial $U_G(\mathbf{x}, y)$, where $\mathbf{x} = (x_i)_{i=1}^n$ is an n-tuple of variables, by

$$U_G(\mathbf{x}, y) = \sum_{X \subseteq E(G)} \left(\prod_{i=1}^{n} x_i^{k_i} \right) (y-1)^{|X| - \rho(X)}.$$

Here k_i denotes the number of components of the graph $(V(G), X)$ with i vertices. Noble and Welsh show that appropriate variable substitutions give a variety of specialisations including the Tutte-Whitney polynomial and the Oxley-Whittle polynomial. They then show that U_G determines, and is determined by, another multivariate generalisation of the Tutte polynomial, involving symmetric functions, due to Stanley [63]. They also study a more general multivariate polynomial W_G for graphs with positive integer weights on vertices and investigate connections with knot theory. They conclude by considering the computational complexity of their polynomials.

Cameron has introduced polynomials that generalise both the Tutte-Whitney polynomial and the cycle index polynomial (which is used for counting up to symmetry) [16].

Several generalisations are based on assigning "local" variables to the edges of a graph. These may be regarded as belonging to both types (ii) and (iv).

Fortuin and Kasteleyn's random cluster model partition function [35, 25] was originally defined with a different variable p_e for each edge e, and such multivariate generalisations are often used in statistical mechanics. Traldi's dichromatic polynomial of weighted graphs [64] may be viewed as a multivariate Tutte-Whitney polynomial by using indeterminates for weights, and in that form is essentially equivalent (via a simple algebraic transformation) to the multivariate Z_{RC}. In recent years Sokal [61, 62] has further promoted the use of the multivariate Z_{RC} in combinatorics, using it as a vehicle to introduce insights and techniques from statistical mechanics. In [62], he calls it (after a minor change of variables) the *multivariate Tutte polynomial* of a graph $G = (V, E)$, with variables q and $\mathbf{v} = (v_e)_{e \in E}$:

$$Z_G(q, \mathbf{v}) = \sum_{X \subseteq E} \mathbf{v}^X q^{n-\rho(X)}, \tag{7}$$

where $\mathbf{v}^X := \prod_{e \in X} v_e$. He investigates the locations in the complex plane of the roots of these polynomials, generalisations to matroids, and connections with statistical mechanics. Kung introduced such polynomials independently in [39]. His *Tugger polynomial* of a matroid (E, ρ) may be written by replacing n by $\rho(E)$ in the right-hand side of (7).

These polynomials were extended further by Zaslavsky [88] and Bollobás and Riordan [8]. Their multivariate generalisations are similar in spirit but significantly different in details. Again, the edges have weights or colours which may be regarded as indeterminates. Part of their aim was to characterise the most general multivariate polynomials of Tutte-Whitney type (albeit in different settings) in which deletion-contraction relations have coefficients that depend on the weight of the edge. A synthesis and common generalisation of these two approaches has been given recently by Ellis-Monaghan and Traldi [21].

These multivariate polynomials can be huge. For U_G, Z_G and their relatives, the size (i.e., number of non-zero terms) can be exponential in the size of the graph. They are not as efficient as (for example) $R(G; x, y)$ in storing information about counting problems on graphs. On the other hand, the multivariate Z_G can be very useful in proofs, even if the real object of interest is the normal Tutte-Whitney polynomial. Sokal explains this point wth examples in [62].

Another multivariate generalisation, closely related to the multivariate Z_{RC} and its relatives mentioned above, is introduced by Traldi [66] and Britz [12] for matroids with a pair of values on each element. Britz shows that it contains some generalisations of weight enumerators of linear codes.

5 Three univariate polynomials

From now on, our level of generality will mostly be that of arbitrary collections of subsets of a ground set S (placing us in type (i)). Such an object can be represented as the support of a Boolean function $f : 2^S \to \{0, 1\}$, and at times we will work at the even more general level of *binary functions*, i.e., functions $f : 2^S \to \mathbb{R}$.

We now consider three univariate polynomials defined for such objects, in order to set the scene for later sections.

5.1 Weight enumerators

If $\mathcal{C}$ is a code over $GF(2)$, then the *weight enumerator* $A_{\mathcal{C}}(z)$ of $\mathcal{C}$ is the generating function for the number of codewords of each possible weight (see [43, 44]):

$$A_{\mathcal{C}}(z) = \sum_{\mathbf{x} \in \mathcal{C}} z^{|\mathbf{x}|}.$$

Greene showed in [29] that the weight enumerator of a *linear* code is a partial evaluation of the Tutte-Whitney polynomial of the associated matroid. See curve H_2 in Figure 1.

For some other invariants of codes related to weight enumerators, and connections (for linear codes) with Tutte-Whitney polynomials or generalisations thereof, see [5, 12].

5.2 Clutter reliability

Suppose each edge of a graph is *chosen* with probability $p \in [0,1]$, so not chosen with probability $1-p$, with the choices being independent and all using the same p. The *all-terminal reliability* $\Pi(G;p)$ of a graph G is the probability that the set of chosen edges contains a spanning tree of G (see, e.g., [51]). This reliability (called the *percolation probability* in [51]) is easily extended to matroids, by letting the bases play the role of spanning trees in the above definition, and is readily shown to be a partial evaluation of the Tutte-Whitney polynomial of the matroid [51, §2], as illustrated in Figure 1.

In [51, 52], Oxley and Welsh extended the all-terminal reliability to clutters. A *clutter* is a collection $\mathcal{C}$ of subsets of some set S such that no member of $\mathcal{C}$ is a subset of any other member of $\mathcal{C}$. The *dual clutter* $\mathcal{C}^*$ is the set of minimal $X \subseteq S$ such that $X \cap Y \neq \emptyset$ for all $Y \in \mathcal{C}$. Observe that $\mathcal{C}^{**} = \mathcal{C}$. If $\mathcal{C}$ is the set of spanning trees of a graph (or bases of a matroid), then $\mathcal{C}^*$ is its set of cutsets (or cocycles).

If the elements of the ground set S are chosen with probability p, with the choices being independent, then the *reliability* $\Pi(\mathcal{C};p)$ of $\mathcal{C}$ is the probability that some member of $\mathcal{C}$ has all its elements chosen. If $\mathcal{C}$ is the set of bases of a matroid on S, then $\Pi(\mathcal{C};p)$ is just the reliability of that matroid. Note that $\Pi(\mathcal{C};p)$ can actually be defined for any $\mathcal{C} \subseteq 2^S$ at all, not just for clutters, but that nothing is really gained by doing so, since $\Pi(\mathcal{C};p) = \Pi(\mathcal{C}^{\min};p)$, where the clutter $\mathcal{C}^{\min}$ is the set of minimal members of $\mathcal{C}$.

Oxley and Welsh [51, 52] describe a standard way of extending deletion and contraction from matroids to arbitrary clutters. This extension works by considering how deletion and contraction work for the bases of a matroid, and then just applying that definition to clutters that do not necessarily have all the extra structure required of matroids. Oxley and Welsh show in [51, §4] that, if a clutter invariant is to behave like a Tutte-Whitney polynomial evaluation with respect to these operations of deletion and contraction, then it must be essentially an evaluation of the (one-variable) clutter reliability, or possibly one of a handful of trivial cases. So a two-variable Tutte-Whitney polynomial defined for all clutters, compatible with these particular deletion and contraction operations, does not exist. Furthermore, the only specific clutters for which such a Tutte-Whitney polynomial exists are those that form the set of bases of a matroid [51, §5].

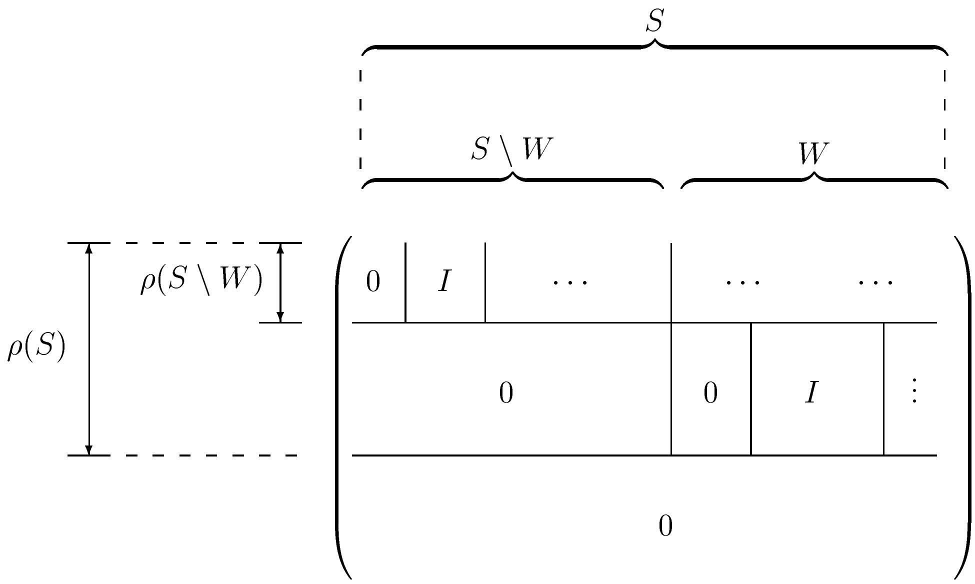

Figure 2: Relationship between rank function and rowspace

5.3 Kung's generalised chromatic polynomial

Let $\mathcal{C}$ be a collection of subsets of a set S. In [37], Kung defined $\zeta_{\mathcal{C}}(r)$ to be the number of r-tuples of members of $\mathcal{C}$ whose union is S. He showed that it extends the characteristic or chromatic polynomial of a representable matroid and satisfies a deletion-contraction relation. Kung's work prefigures the definition of rank-like functions for arbitrary binary functions given in [22] and to be described below. For further information and related work by Kung, see [38]. The relationship between his approach and that of [22] is discussed in [23, §2.2].

6 From binary matroids to binary functions

Let M be a matrix over $GF(2)$ with columns indexed by S and rank function ρ, so that (S, ρ) is a binary matroid. Of course, the rank function depends only on the rowspace $\mathcal{R}$ of M, not on the specific representation of $\mathcal{R}$ given by M: different matrices with the same rowspace will give the same rank function. The rank function can therefore be determined by the indicator function of the rowspace, which we will call f. To see the exact form of this relationship, let $W \subseteq S$ and suppose the columns of M are ordered so that the columns indexed by W are on the right, with the rest on the left. By performing elementary row operations on M (and perhaps some cosmetic rearrangement of columns) we can form a matrix of the form shown in Figure 2 with the same rowspace $\mathcal{R}$ as M.

Consider the three sets of rows here (Figure 2). The rows in the last of these sets are all 0, so do not concern us. The first two sets of rows together generate the rowspace of the entire matrix M, so these rows are $\rho(S)$ in number. The first set of rows alone, when restricted to positions indexed by $S \setminus W$, generates the rowspace of the submatrix $M[S \setminus W]$, so the number of these rows is $\rho(S \setminus W)$. The second set of rows generates a subspace $\mathcal{R}_W$ of $\mathcal{R}$ consisting of those rows in $\mathcal{R}$ that are 0 in positions outside those indexed by W, and we now know that there are $\rho(S)-\rho(S\backslash W)$ rows in this set. We therefore have

$$|\mathcal{R}_W| = 2^{\rho(S)-\rho(S\backslash W)}.$$

Another way to count the members of $\mathcal{R}_W$ is to use f, the indicator function for $\mathcal{R}$, to essentially restate the definition of $\mathcal{R}_W$, giving

$$|\mathcal{R}_W| = \sum_{\mathbf{x}\in\mathcal{R}_W} 1 = \sum_{\mathbf{x}\leq\mathbf{w}} f(\mathbf{x}) = \sum_{X\subseteq W} f(X),$$

moving freely between a set $X \in 2^S$ and its incidence vector $\mathbf{x} \in GF(2)^S$.

Combining these expressions for $\mathcal{R}_W$, putting $Y = S \setminus W$ and rearranging gives an expression for obtaining the rank function directly from f:

$$\rho(Y) = \log_2 \left(\frac{\sum_{X\subseteq S} f(X)}{\sum_{X\subseteq S\backslash Y} f(X)} \right). \tag{8}$$

A key step in [22] is to turn this into a definition and use it for functions f that are not necessarily indicator functions of linear spaces. We denote by Q the transform that takes any $f : 2^S \to \mathbb{R}$ and constructs a function $Qf = \rho$ according to (8). Although Qf will not in general be a rank function, it can still play the role of a rank function in some circumstances. We can still find its dual $(Qf)^*$. In [22], Qf is used to play the role of a rank function in order to extend the Tutte-Whitney polynomial to arbitrary binary functions:

$$\begin{aligned} R(f;x,y) &= \sum_{X\subseteq S} x^{(Qf)(S)-(Qf)(X)} y^{(Qf)^*(S)-(Qf)^*(S\backslash X)}, \\ R_1(f;x,y) &= y^{-(Qf)(S)} \sum_{X\subseteq S} (xy)^{(Qf)(S)-(Qf)(X)} y^{|X|}. \end{aligned} \tag{9}$$

These functions are not, in general, polynomials, since the exponents may be nonintegral, so we shall call them *Tutte-Whitney functions* of f. Usually $R = R_1$, but the effects of complex arithmetic can make them differ if both arguments are negative and some of the exponents are nonintegral. The representation R_1, after putting $xy = q$ and $y = p/(1-p)$, may be regarded as generalising the random cluster model partition function: see (4).

As we saw in §1, much of the interest in ordinary Tutte-Whitney polynomials arises from their interesting partial evaluations. It turns out that the Tutte-Whitney functions considered here also have interesting partial evaluations. In fact, the three polynomials described in §5, for arbitrary collections of sets, are all specialisations of $R(f;x,y)$. These specialisations occur along H_2, the y-axis, and the line $y = -1$, respectively. A further specialisation, along H_{2^r} for $r \in \mathbb{N} \cup \{0\}$, is obtained by generalising the Potts model partition function from statistical mechanics to arbitrary collections of sets [23]. This generalised 2^r-state Potts model partition function is

$$Z(\mathcal{C};r) = 2^r \sum_{(W_0,\ldots,W_{r-1})\in\mathcal{C}^r} e^{-K\left|\cup_{i=0}^{r-1} W_i\right|},$$

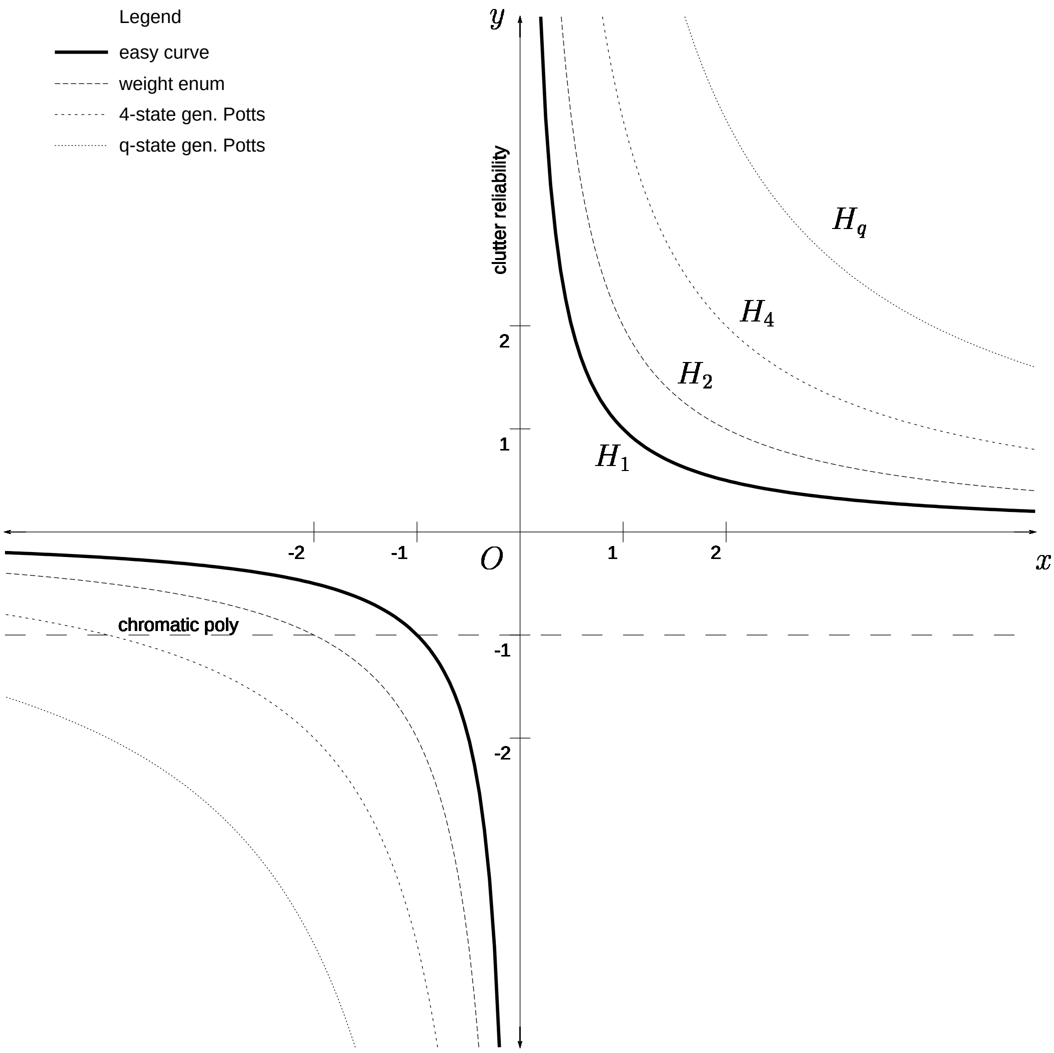

Figure 3: The generalised Tutte-Whitney plane.

where the sum is over all r-tuples of members of $\mathcal{C}$. This comes from the Tutte-Whitney function on H_{2^r}.

The next theorem summarises these partial evaluation results, with $I_{\mathcal{C}}$ denoting the indicator function of $\mathcal{C}$.

Theorem 1
(a) $A_{\mathcal{C}}(x) = (1-x)^{Qf(S)} x^{|S|-Qf(S)} R(I_{\mathcal{C}}; 2x/(1-x), (1-x)/x)$, *where* $x \in (0,1)$;
(b) $\Pi(\mathcal{C}; p) = p^{Qf(S)}(1-p)^{|S|-Qf(S)} R(I_{(\mathcal{C}^{\min})^*}; 0, p/(1-p))$;
(c) $\zeta_{\mathcal{C}}(r) = (-1)^{Qf(S)} R_1(I_{\mathcal{C}}; -2^r, -1)$;
(d) $Z(\mathcal{C}; r) = 2^r(e^K - 1)^{Qf(S)} e^{-K|S|} R(I_{\mathcal{C}}; 2^r/(e^K-1), e^K - 1)$. □

These results come from [22], [22], [37] (implicitly), and [23], respectively.

We can now draw a diagram analogous to Figure 1, showing the above four partial evaluations as particular curves in the domain $\mathbb{R}^2$ of $R(f; x, y)$ for a Boolean function f. See Figure 3.

It is also shown in [22] that $R(f; x, y)$ satisfies a deletion-contraction relation. This uses deletion and contraction operations based on the following observation. If f is the indicator function of the cutset space of a graph G and $e \in E(G)$, then the indicator functions $f\setminus\!\!\setminus e$ and $f/\!\!/e$ of the cutset spaces of the graphs $G\setminus e$ and G/e, respectively, are given by

$$\begin{aligned} f\setminus\!\!\setminus e(X) &= \frac{f(X)+f(X\cup\{e\})}{f(\emptyset)+f(\{e\})}, & (10)\\ f/\!\!/e(X) &= \frac{f(X)}{f(\emptyset)}, & (11)\end{aligned}$$

for $X\subseteq E(G)\setminus\{e\}$. If we make this our definition of deletion and contraction for arbitrary binary functions (with ground set S instead of $E(G)$), then we find that these operations are the right ones to use in this context, in the sense that under the transform Q they correspond to ordinary deletion and contraction for Qf (using the usual rank function deletion and contraction expressions[9], even though Qf may not be a rank function). A deletion-contraction expression for $R(f;x,y)$, using these operations, is given in [22, Theorem 5.1], with correction in [23, p. 256].

Recall from §5.2 that Oxley and Welsh showed that no generalised Tutte-Whitney polynomial for clutters can be compatible with the clutter deletion and contraction operations: the nearest you can come, by generalising in that direction, is their univariate clutter reliability [51]. Those deletion/contraction operations are straightforward extensions of deletion and contraction for matroids (specifically, of the form that deletion/contraction rules take when expressed in terms of the bases of a matroid). The deletion/contraction operations $\setminus\!\!\setminus$ and $/\!\!/$ described above amount to the same thing as matroid base deletion/contraction (via the usual links between bases, rank functions and cocycle spaces) for binary matroids, but diverge when one generalises further. For arbitrary collections of sets, we have seen that there is a generalised Tutte-Whitney function that respects the operations $\setminus\!\!\setminus$ and $/\!\!/$. The price we pay is that the Tutte-Whitney functions might not be polynomials, and that our deletion and contraction operations may take us out of the class of collections of sets (or Boolean functions): hence the need to use binary functions.

7 A space of Tutte-Whitney functions

One way of representing the deletion, or contraction, of an edge e of a graph G is to leave the graph itself unaltered and to assign a label $\lambda_e\in\{1,0\}$, respectively, to e. The effect of λ_e is to indicate how to determine whether a set of edges $X\subseteq E\setminus\{e\}$ of the modified graph ($G\setminus e$ or G/e) belongs to the cutset space of that graph: the label indicates which of (10) and (11) is to be used.

A more general operation can be obtained by choosing any $\lambda_e\neq -1$ and using the following expression (from [23]) to extend (and interpolate between) the expressions (10) and (11):

$$f\,\|_{\lambda_e} e(X) = \frac{f(X)+\lambda_e f(X\cup\{e\})}{f(\emptyset)+\lambda_e f(\{e\})}. \tag{12}$$

[9]If $\rho: 2^S\to\mathbb{R}$ and $e\in S$, then the operations of *deleting* and *contracting* e give the functions $\rho\setminus e, \rho/e: 2^{S\setminus\{e\}}\to\mathbb{R}$ defined for all $X\subseteq S\setminus\{e\}$ by $(\rho\setminus e)(X)=\rho(X)$ and $(\rho/e)(X)=\rho(X\cup\{e\})-\rho(\{e\})+\rho(\emptyset)$, respectively.

We call the function $f\,\|_{\lambda_e} e$ a λ_e-*minor* of f. It may be thought of as assigning, to any $X \subseteq E \setminus \{e\}$, a kind of "degree of membership" in a "cutset space" of the modified "graph". Note, though, that $f\,\|_{\lambda_e} e$ is not, in general, just $\{0,1\}$-valued, or even just $[0,1]$-valued, and the sets $\{X \subseteq E \setminus \{e\} \mid f(X) \neq 0\}$ and $\{X \subseteq E \setminus \{e\} \mid f(X) = 1\}$ are not necessarily linear spaces. The combinatorial object described by $f\,\|_{\lambda_e} e$ is thus not necessarily a graph, or even a binary matroid. We may still use a graphical representation for it, along the lines of the one described above for deletion and contraction: just label the edge e with λ_e. However, this must be understood only as a means of indicating how the expression (12) is to be used to assign values to edge sets $X \subseteq E \setminus \{e\}$. Essentially, the value of $f\,\|_{\lambda_e} e$ on X is a weighted sum of the values of f on X and $X \cup \{e\}$, with weights 1 and λ_e respectively, and scaled so that $f\,\|_{\lambda_e} e(\emptyset) = 1$ (by dividing by the constant $f(\emptyset) + \lambda_e f(\{e\})$) (see [23, §7]).

The definition (12), although stated for indicator functions of cutset spaces, can be used for any f (provided $f(\emptyset) + \lambda_e f(\{e\}) \neq 0$). It can therefore be applied again, to take minors of $f\,\|_{\lambda_e} e$, even though that function may not be the indicator function of a linear space.

Suppose we do a sequence of such minor operations, on edges $e \in T$ for some set $T \subseteq E$, with each $e \in T$ being used to form a λ_e-minor (where the values λ_e are not necessarily the same). The result is a function denoted by $f\,\|_{\boldsymbol{\lambda}} T$, where $\boldsymbol{\lambda} = (\lambda_e)_{e \in T}$, and which is given by the following expression (using the shorthand $\boldsymbol{\lambda}^U := \prod_{e \in U} \lambda_e$ for all $U \subseteq T$):

$$f\,\|_{\boldsymbol{\lambda}} T(X) = \frac{\sum_{U \subseteq T} \boldsymbol{\lambda}^U f(X \cup U)}{\sum_{U \subseteq T} \boldsymbol{\lambda}^U f(U)}, \tag{13}$$

where $X \subseteq E \setminus T$. The denominator here may be thought of as a scaling factor to ensure that $f\,\|_{\boldsymbol{\lambda}} T(\emptyset) = 1$. It is easily shown that repeated application of (12), for each $e \in T$, regardless of the order, yields the same final expression (13).

Again, it is possible to picture the result of this operation as a graph with appropriate labels on the edges of T. However, it must be borne in mind that it is only the values of $f\,\|_{\boldsymbol{\lambda}} T$ that are relevant, not the detailed graph structure of such a representation. This is analogous to the fact that an ordinary graph may contain more information than is needed to determine its cutset space.

This operation $\|_{\boldsymbol{\lambda}} T$ extends that considered in [23, §4], which just considered cases where all the λ_e are the same (and are denoted just by λ).

It is shown in [23, Theorem 5] that the λ-minor operations $\|_\lambda$ come in dual pairs, of which deletion and contraction are just one pair. If $\lambda^* := (1-\lambda)/(1+\lambda)$, then $\|_\lambda$ and $\|_{\lambda^*}$ are dual operations, in the sense that $\hat{f}\,\|_\lambda T = c \cdot \widehat{f\,\|_{\lambda^*} T}$, where c is a constant (depending on f, λ and T, but not on the function argument) that ensures that $\hat{f}\,\|_\lambda T(\emptyset) = 1$. It is straightforward to extend this result to operations $\|_{\boldsymbol{\lambda}}$ having potentially different values λ_e for each edge. Put $\boldsymbol{\lambda}^* = (\lambda_e^*)_{e \in E}$.

Theorem 2 *The operations $\|_{\boldsymbol{\lambda}}$ and $\|_{\boldsymbol{\lambda}^*}$ are dual, in the sense that* $\hat{f}\,\|_{\boldsymbol{\lambda}} T(X) = \widehat{f\,\|_{\boldsymbol{\lambda}^*} T}(X) \;/\; \widehat{f\,\|_{\boldsymbol{\lambda}^*} T}(\emptyset)$ *for all* $X \subseteq E \setminus T$. $\square$

For any f and any $\boldsymbol{\lambda}$, the following function can be used to play the role of a rank function, at least in extending the Tutte-Whitney polynomials:

$$(Q^{(\boldsymbol{\lambda})}f)(X) = \log_2\left(\frac{(\mathbf{1}+\boldsymbol{\lambda}^*)^X \sum_{W\subseteq E} \boldsymbol{\lambda}^W f(W)}{\sum_{W\subseteq E} \boldsymbol{\lambda}^{W\cap(E\setminus X)}(\boldsymbol{\lambda}^*)^{W\cap X} f(W)}\right).$$

(Here, $\mathbf{1} = (1)_{e\in E}$, a constant vector.)

The $\boldsymbol{\lambda}$-*Tutte-Whitney function* of f is defined by

$$R^{(\boldsymbol{\lambda})}(f;x,y) = \sum_{X\subseteq E} x^{Q^{(\boldsymbol{\lambda})}f(E)-Q^{(\boldsymbol{\lambda})}f(X)} y^{(Q^{(\boldsymbol{\lambda})}f)^*(E)-(Q^{(\boldsymbol{\lambda})}f)^*(E\setminus X)},$$

and $R_1^{(\boldsymbol{\lambda})}$ is defined similarly, by analogy with (9). This extends the definition of $R^{(\lambda)}(f;x,y)$ in [23] from $\lambda \in \mathbb{R}$ to vectors $\boldsymbol{\lambda} = (\lambda_e)_{e\in E} \in \mathbb{R}^E$. When f is the indicator function of the cutset space of a graph G, we speak of the $\boldsymbol{\lambda}$-Tutte-Whitney function of G.

In [23] it is shown that the λ-Tutte-Whitney function satisfies a deletion-contraction relation and contains the weight enumerator of an arbitrary code. More recently, in [24], it is shown that the partition function of the symmetric Ashkin-Teller model [4] from statistical mechanics can be obtained from $R_1^{(\lambda)}(G;x,y)$ for appropriate choice of λ. Other generalisations of Tutte-Whitney polynomials, or equivalently of Z_{RC}, have previously been shown to contain the Ashkin-Teller model partition function: see (6) and surrounding discussion. However, the generalisation of (6) is based on a double sum, whereas $R_1^{(\lambda)}(f;x,y)$ uses just a single sum. The symmetric Ashkin-Teller model partition function cannot be found from the normal Tutte-Whitney polynomial, so $R_1^{(\lambda)}(f;x,y)$ contains some information that cannot be found from $R_1(f;x,y)$.

8 Complexity

It is shown in [33] that the value of the Tutte-Whitney polynomial of a graph is #P-hard to compute at all points (x,y) except at four special points and on one special curve (see Figure 1).[10] It is natural to ask about the complexity of computing the Tutte-Whitney function $R(f;x,y)$ when f is an arbitrary Boolean function. In fact, it is polynomial time computable! However, this is only because the input size is now so large. There are 2^{2^m} Boolean functions on m variables (and half as many if we require that $f(\emptyset) = 1$). So the input size, in (e.g.) a list of bits giving all the values of f, will be 2^m (or $2^m - 1$) bits. More generally, if f is a binary function where each value can be stored in $\leq s$ bits, the input size of f is $\leq 2^m s$ bits.

Given such a binary function f, fast transform techniques can be used to calculate Qf using $O(2^m m)$ additions. In such an approach, each value of Qf results from $\leq m$ additions, so the numbers involved may grow to occupy about $s + \log m$ bits at most. So the time spent on each addition is $O(s+\log m)$, and the total time required to find

[10] Here, (x,y) is restricted to $\mathbb{F}^2$, where $\mathbb{F}$ is a finite-dimensional algebraic extension of $\mathbb{Q}$. See [33] for details.

Qf is $O(2^m m(s+\log m))$ or $O(2^m m \log m)$ if f is Boolean. It is then straightforward to compute $R(f;x,y)$, and the time bounds for the whole computation are of the same order as for finding Qf. If we let $N \simeq 2^m s$ stand for the input size, the time complexity of finding $R(f;x,y)$ is $O(N \log N \log\log N)$. Similar complexity remarks could be made about $R^{(\lambda)}(f;x,y)$.

The curve H_1, on which $xy = 1$, is still, in a sense, easier than most of the rest of the plane: on this curve, $R_1(f;x,y) = y^{-Qf(S)}(1+y)^{|S|}$, which depends only on the sum of all the values of f, so it can be computed even more quickly than other specialisations of $R_1(f;x,y)$.

Acknowledgements

I am most grateful to Dominic Welsh for introducing me to the Tutte polynomial over two decades ago, and for many stimulating conversations about it since. I also thank a number of colleagues for reading earlier versions of this article and offering helpful comments: Henry Crapo, Keith Edwards, Andrew Goodall, Joseph Kung, James Oxley, Gordon Royle, Carrie Rutherford, Alan Sokal, Lorenzo Traldi, Geoff Whittle and the referee.

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