Planarization and fragmentability of some classes of graphs

Keith Edwards
University of Dundee
Dundee, DD1 4HN
U.K.

Graham Farr*
Monash University (Clayton Campus)
Clayton, Victoria 3168
Australia

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Abstract

The coefficient of fragmentability of a class of graphs measures the proportion of vertices that need to be removed from the graphs in the class in order to leave behind bounded sized components. We have previously given bounds on this parameter for the class of graphs satisfying a given constant bound on maximum degree. In this paper, we give fragmentability bounds for some classes of graphs of bounded *average* degree, as well as classes of given thickness, the class of k-colourable graphs, and the class of n-dimensional cubes. In order to establish the fragmentability results for bounded average degree, we prove that the proportion of vertices that must be removed from a graph of average degree at most $\bar{d}$ in order to leave behind a planar subgraph is at most $(\bar{d}-2)/(\bar{d}+1)$, provided $\bar{d} \geq 4$ or the graph is connected and $\bar{d} \geq 2$. The proof yields an algorithm for finding large induced planar subgraphs and (under certain conditions) a lower bound on the size of the induced planar subgraph it finds. This bound is similar in form to the one we found for a previous algorithm we developed for that problem, but applies to a larger class of graphs.

1 Introduction and Definitions

The *coefficient of fragmentability* of a class of graphs was introduced by the authors in [5]. It measures how small a proportion of vertices need to be removed from graphs in a class in order to break them into components of bounded size. We begin by recalling the definition.

*Some of the work of this paper was done while Farr was visiting the University of Dundee.

Let $\varepsilon \in [0,1]$ and $C \in \mathbb{N}$. A graph G is *(C,ε)-fragmentable* if there exists $X \subseteq V(G)$ such that $|X| \le \varepsilon\,|V(G)|$ and every component of $G - X$ has at most C vertices. X is here called the *fragmenting set*. A class Γ of graphs is *ε-fragmentable* if there exists $C \in \mathbb{N}$ such that every $G \in \Gamma$ is (C,ε)-fragmentable. The *coefficient of fragmentability* of Γ is

$$c_f(\Gamma) = \inf\{\varepsilon \mid \Gamma \text{ is } \varepsilon\text{-fragmentable}\}.$$

In [5] we showed that, if Γ_d is the class of graphs with maximum degree at most d and $d \ge 2$,

$$\frac{d-2}{2d-2} \le c_f(\Gamma_d) \le \frac{d-2}{d+1}. \tag{1}$$

Work on the upper bound here also led to a new algorithm for finding large induced planar subgraphs of a graph [6]. In fact, the proof of the upper bound relies on removing an appropriate proportion of the vertices of graphs of bounded maximum degree so as to leave behind planar subgraphs, and then using the fact that planar graphs are easily fragmented.

The present paper also uses this kind of planarization to establish results on fragmentability, and the planarization results found are of independent interest. In §3 we establish an upper bound of $(d-2)/(d+1)$ on the proportion of vertices that need to be removed from a graph of *average* degree at most d in order to leave behind a planar subgraph, provided either $d \ge 4$ or the graph is connected and $d \ge 2$. Our method leads (§4) to another algorithm for finding large induced planar subgraphs. We prove a similar bound on performance to that of [6], but note that the bound for the present algorithm applies to a larger class of graphs. We also establish (§5) bounds on the coefficient of fragmentability of some classes of graphs of bounded average degree, extending our earlier results [5] for maximum degree. Specifically, we consider the class $\overline{\Gamma}_d$ of graphs of average degree at most d, for $d \ge 4$, and for the class $\overline{\Gamma}^c_d$ of connected graphs of average degree at most d, for $d \ge 2$. We then present results on the coefficient of fragmentability of some other classes of graphs: graphs of given thickness (§6), k-colourable graphs, and n-dimensional cubes (§7).

We use the following notation. Let G be a graph. Throughout the paper, $n = |V(G)|$ and $m = |E(G)|$. If $X \subseteq V(G)$ then $\langle X \rangle$ denotes the subgraph of G induced by X. If $X, Y \subseteq V(G)$ then $E(X,Y)$ is the set of edges with one endpoint in X and the other in Y. If $v \in V(G)$ then $d(v) = d_G(v)$ denotes the degree of v in G.

2 Related work

In earlier work by Edwards and McDiarmid [7], a class Γ is said to be *fragmentable* if (in our terminology) $c_f(\Gamma) = 0$. They show that the existence of a suitable separator theorem for Γ implies that $c_f(\Gamma) = 0$. For further information on the relationship between separator theorems and fragmentability, and some classes that are thus shown to have $c_f(\Gamma) = 0$, see [5, 7].

Caro and Yuster [3] define a graph G to be *k-slim* if, for every subgraph $H \le G$ of at least k vertices, there exists a set X of k vertices such that $H - X$ consists of at least two components, each of at most $\frac{2}{3}\,|V(H)|$ vertices. It follows from the abovementioned result of Edwards and McDiarmid that if there is some $k \in \mathbb{N}$ such that every graph in a class Γ is k-slim, then $c_f(\Gamma) = 0$.

Barefoot, Entringer and Swart [1] define the *integrity* $I(G)$ of a graph G by

$$I(G) = \min\{|X| + m(G - X) \mid X \subseteq V(G)\},$$

where $m(H)$ denotes the number of vertices in the largest component of H. It is straightforward to use the definitions to show that

$$c_f(\Gamma) \geq \inf_{n \in \mathbb{N}} \max_{\substack{G \in \Gamma \\ |V(G)| = n}} \frac{I(G)}{n} . \tag{2}$$

There can be a significant gap between the two sides. For example, if $\Gamma = \Gamma_k = \{K_{1,k-1}[K_{n/k}] \mid n \in \mathbb{N}, k|n\}$, where $K_{1,k-1}[K_{n/k}]$ is a composition (or lexicographic or wreath product), then the right-hand side of (2) is $2/k$, but $c_f(\Gamma_k) = 1$.

3 Planarization

Let G be a graph, and consider the following four operations on G:

1. Delete an isolated vertex of G.
2. Delete a vertex of degree 1 (and its incident edge).
3. Let v be a vertex of degree 2 with non-adjacent neighbours x and y, delete v (and edges vx, vy) and join x and y.
4. Let v be a vertex of degree 2 with adjacent neighbours, delete v (and incident edges).

Proposition 1 *Let G be a graph, and G' the result of applying one of the above operations to G. Let X be any subset of $V(G')$. Then if $G' - X$ is planar, $G - X$ is also planar.* □

For any graph G, let $p(G)$ be the size of the smallest set X of vertices of G such that $G - X$ is planar.

Corollary 2 *Let G be a graph, and G' the result of applying one of the above operations to G. Then $p(G) \leq p(G')$.* □

Let $r(G)$ be a graph obtained from G by applying operations $1, 2, 3, 4$ above repeatedly until none is possible (because the graph has minimum degree at least 3). This construction has appeared, for example, in [4]. (The resulting graph $r(G)$ is in fact unique, but we will not need that here.) Then $p(G) \leq p(r(G))$.

Lemma 3 *Let $G = (V, E)$ be a graph with n vertices and minimum degree at least 3. Then*

$$p(G) \leq \sum_{v \in V(G)} \frac{d(v) - 2}{d(v) + 1}$$

Proof. By induction on n. We allow G to be the empty graph with no vertices and no edges, and regard it as vacuously having minimum degree ≤ 3 (in that it has no vertices of degree ≤ 2). This gives us the case $n = 0$, when the inequality is true since the sum is empty.

If $1 < n < 4$, there is nothing to prove. So suppose G is a graph with $n \geq 4$ vertices and minimum degree at least 3, and let w be a vertex of maximum degree. Then clearly

$$p(G) \leq 1 + p(G - w) \leq 1 + p(r(G - w)).$$

Now in the reduced graph $r(G - w)$ (which may be empty), each vertex has degree no more than its degree in $G - w$. Also, at least $d(w)$ vertices have degree reduced by at least one, or have been deleted.

Let $V' = V(G - w)$, and for $v \in V'$, let $d'(v)$ be the degree of v in $G - w$. Similarly, let $V^* = V(r(G - w))$, and for $v \in V^*$, let $d^*(v)$ be the degree of v in $r(G - w)$.

By induction,

$$p(r(G - w)) \leq \sum_{v \in V^*} \frac{d^*(v) - 2}{d^*(v) + 1}.$$

Therefore

$$\begin{aligned} p(G) &\leq 1 + \sum_{v \in V^*} \frac{d^*(v) - 2}{d^*(v) + 1} \\ &\leq 1 + \sum_{v \in V'} \frac{d'(v) - 2}{d'(v) + 1} \end{aligned}$$

because $(x - 2)/(x + 1)$ is an increasing function of x, and $G - w$ has minimum degree ≥ 2, so any vertex in $V' - V^*$ has $d'(v) \geq 2$. So, recalling that w is of maximum degree in G,

$$\begin{aligned} p(G) &\leq 1 + \sum_{v \in V', vw \in E} \frac{(d(v) - 1) - 2}{(d(v) - 1) + 1} + \sum_{v \in V', vw \notin E} \frac{d(v) - 2}{d(v) + 1} \\ &= 1 + \sum_{v \in V'} \frac{d(v) - 2}{d(v) + 1} + \sum_{v \in V', vw \in E} \left(\frac{d(v) - 3}{d(v)} - \frac{d(v) - 2}{d(v) + 1} \right) \\ &= 1 + \sum_{v \in V'} \frac{d(v) - 2}{d(v) + 1} - \sum_{v \in V', vw \in E} \frac{3}{d(v)(d(v) + 1)} \\ &\leq 1 + \sum_{v \in V'} \frac{d(v) - 2}{d(v) + 1} - \sum_{v \in V', vw \in E} \frac{3}{d(w)(d(w) + 1)} \\ &= \sum_{v \in V'} \frac{d(v) - 2}{d(v) + 1} + \left(1 - \frac{3}{d(w) + 1} \right) \\ &= \sum_{v \in V} \frac{d(v) - 2}{d(v) + 1} \end{aligned}$$

as required. $\square$

Corollary 4 *Let G be a graph, and $r(G)$ a reduced graph of G. Then*

$$p(G) \le \sum_{v \in V(r(G))} \frac{d_{r(G)}(v) - 2}{d_{r(G)}(v) + 1}.$$

□

We now define two functions which will be useful in deriving the upper bound on $p(G)$ for graphs of average degree at most d. First, for any real number $i \ge 0$, define a function g by

$$g(i) = \frac{i-2}{i+1}$$

Now for any positive integer k, define f_k to be the straight line such that $f_k(k) = g(k)$ and $f_k(k+1) = g(k+1)$. It is easily verified that for any real number i,

$$f_k(i) = \frac{3i + k^2 - 3k - 4}{(k+1)(k+2)} = g(i) - \frac{3(i-k)(k+1-i)}{(i+1)(k+1)(k+2)}.$$

Observe that

$$f_k(i) < g(i) \quad \text{when } k < i < k+1$$

and that for any non-negative integer i,

$$f_k(i) \ge g(i).$$

Proposition 5 *Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be a graph obtained by applying one of the four operations above to G. For each $i \ge 0$, let n_i, n'_i be the number of vertices of degree i in G, G' respectively. Let k be a positive integer. Then if $n_0 = 0$ or $k \ge 4$, we have*

$$\sum_{i \ge 0} f_k(i)\, n'_i \le \sum_{i \ge 0} f_k(i)\, n_i.$$

Proof. We consider the four operations. If $n_0 = 0$ then operation 1 is impossible. Otherwise, after operation 1 we have $n'_0 = n_0 - 1$, and $n'_i = n_i$ for $i \ne 0$. Thus

$$\sum_{i \ge 0} f_k(i)\, n'_i = \sum_{i \ge 0} f_k(i)\, n_i \; - f_k(0) \le \sum_{i \ge 0} f_k(i)\, n_i$$

since $f_k(0) \ge 0$ for $k \ge 4$.

For operation 2, we delete a vertex of degree 1, adjacent to some other vertex v of degree j say, where $j \ge 1$. The degree of v will change to $j - 1$, hence we have

$$\begin{aligned}
\sum_{i \ge 0} f_k(i)\, n'_i &= \sum_{i \ge 0} f_k(i)\, n_i \; - f_k(1) - f_k(j) + f_k(j-1) \\
&= \sum_{i \ge 0} f_k(i)\, n_i - \frac{3 + 3j - 3(j-1) + (k^2 - 3k - 4)}{(k+1)(k+2)} \\
&= \sum_{i \ge 0} f_k(i)\, n_i - \frac{(k-1)(k-2)}{(k+1)(k+2)} \\
&\le \sum_{i \ge 0} f_k(i)\, n_i.
\end{aligned}$$

For operation 3, we have $n'_2 = n_2 - 1$, and $n'_i = n_i$ for $i \neq 2$. Thus

$$\sum_{i \geq 0} f_k(i)\, n'_i = \sum_{i \geq 0} f_k(i)\, n_i \;\; - f_k(2) \leq \sum_{i \geq 0} f_k(i)\, n_i$$

since $f_k(2) \geq 0$ for $k \geq 1$.

For operation 4, we delete a vertex of degree 2, with two neighbours of degrees j, j' say, which both lose one neighbour. Hence

$$\sum_{i \geq 0} f_k(i)\, n'_i = \sum_{i \geq 0} f_k(i)\, n_i \;\; - f_k(2) - (f_k(j) - f_k(j-1)) - (f_k(j') - f_k(j'-1)) \leq \sum_{i \geq 0} f_k(i)\, n_i$$

since $f_k(2) \geq 0$ and $f_k(i) - f_k(i-1) = 3/(k+1)(k+2) > 0$ for any i. $\square$

Proposition 6 *Let $G = (V, E)$ be a graph. Let $r(G)$ be a reduced graph of G, and suppose that $r(G)$ is non-empty. For each $i \geq 0$, let n_i, n'_i be the number of vertices of degree i in G, $r(G)$ respectively. Let k be a positive integer. Then if G is connected or $k \geq 4$, we have*

$$\sum_{i \geq 0} f_k(i)\, n'_i \leq \sum_{i \geq 0} f_k(i)\, n_i.$$

Proof. Let $G_0 = G, G_1, \ldots, G_k = r(G)$ be the sequence of graphs in the reduction process. If G is connected, then since $r(G)$ is non-empty, we use only operations 2,3,4 in forming $r(G)$, and no graph in the reduction sequence has an isolated vertex. The result then follows from Proposition 5. $\square$

Lemma 7 *Let G be a graph, and for each $i \geq 0$, let n_i be the number of vertices of degree i in G. Let k be a positive integer. Then if G is connected, or $k \geq 4$,*

$$p(G) \leq \max\{0, \sum_{i \geq 0} f_k(i)\, n_i\}.$$

Proof. Let $r(G)$ be a reduced graph of G. If $p(G) = 0$, the result follows. Otherwise, $r(G)$ is non-empty. For each $i \geq 0$, let n'_i be the number of vertices of degree i in $r(G)$. Note that $n'_0 = n'_1 = n'_2 = 0$. Then by Corollary 4,

$$p(G) \leq \sum_{i \geq 3} g(i)\, n'_i = \sum_{i \geq 0} g(i)\, n'_i$$

But as noted above, $g(i) \leq f_k(i)$ for each non-negative integer i, so by Proposition 6, we have

$$\sum_{i \geq 0} g(i)\, n'_i \leq \sum_{i \geq 0} f_k(i)\, n'_i \leq \sum_{i \geq 0} f_k(i)\, n_i$$

as required. $\square$

Lemma 8 *Let G be a graph with n vertices, of average degree at most d, where $d \geq 2$. Let k be a positive integer. Then if G is connected, or $k \geq 4$,*

$$p(G) \leq f_k(d)\, n.$$

Proof. If $p(G) = 0$, then since $f_k(d) \geq 0$ when $d \geq 2$, the result follows. Otherwise, by Lemma 7,

$$\begin{aligned} p(G) &\leq \sum_{i \geq 0} f_k(i)\, n_i \\ &= \frac{3}{(k+1)(k+2)} \sum_{i \geq 0} i\, n_i + \frac{k^2 - 3k - 4}{(k+1)(k+2)} \sum_{i \geq 0} n_i \\ &\leq \frac{3}{(k+1)(k+2)} dn + \frac{k^2 - 3k - 4}{(k+1)(k+2)} n \\ &= f_k(d)\, n. \end{aligned}$$

□

Theorem 9 *Let G be a graph with n vertices and average degree at most d, where $d \geq 2$. Then if G is connected, or $d \geq 4$,*

$$\frac{p(G)}{n} \leq \frac{d-2}{d+1} - \frac{3(d - \lfloor d \rfloor)(\lceil d \rceil - d)}{(d+1)(\lfloor d \rfloor + 1)(\lceil d \rceil + 1)}.$$

Proof. Set $k = \lfloor d \rfloor$, so that if $d \geq 4$, then $k \geq 4$. By Lemma 8, $p(G) \leq f_k(d)n$. But

$$f_k(d) = \frac{d-2}{d+1} - \frac{3(d-k)(k+1-d)}{(d+1)(k+1)(k+2)}.$$

If d is an integer, then the second term is zero. Otherwise $\lceil d \rceil = k+1$ and the result follows.

□

Note that the requirement in Theorem 9 that G is connected or $d \geq 4$ cannot in general be dropped completely, for example K_5 together with 5 isolated vertices has average degree 2 but is not planar.

4 Finding large induced planar subgraphs

The Maximum Induced Planar Subgraph (MIPS) problem (see, e.g., [9]) asks for the largest $P \subseteq V(G)$ in a graph G such that the induced subgraph $\langle P \rangle$ is planar. Sometimes it is convenient to work with the complementary and computationally equivalent problem of finding the smallest $R \subseteq V(G)$ such that $G - R$ is planar. MIPS is NP-hard, and difficult to approximate: for references and background, see [6].

In [6], we presented an algorithm for finding an induced planar subgraph of at least $3n/(d+1)$ vertices in a graph G of n vertices and maximum degree at most d. The algorithm was essentially extracted from the proof of [5, Theorem 3.2], which gives the claimed bound on its performance.

In this section we extract from the proof of Lemma 3 another algorithm for finding a large induced planar subgraph in a graph. This algorithm achieves the same performance ratio as that of [6] on graphs of maximum degree $d \geq 4$, or on connected graphs of maximum degree $d \geq 2$, but its performance is also guaranteed for the extensions of these classes to graphs for which the bound is just on average degree. It also works in the "opposite direction" to [6]. The older algorithm builds up the induced planar subgraph (starting with the empty subgraph) by iteratively adding a new vertex to it, or sometimes swapping a vertex in the subgraph with one outside it. The vertices to be added have low degree in the subgraph being built, and that subgraph is kept planar throughout. By contrast, the present algorithm starts with the subgraph set to be the entire graph (which will be nonplanar in general) and iteratively removes high degree vertices from it until enough vertices have been removed to ensure planarity of the subgraph remaining. No swapping is done.

Algorithm 1. Finding an induced planar subgraph of a graph G.
If G has average degree at most $\bar{d} \geq 4$, or is connected and has average degree at most $\bar{d} \geq 2$, then the induced planar subgraph found has at least $3n/(\bar{d}+1)$ vertices.

1. Input: Graph G.
2. $P := V(G)$
   $R := \emptyset$
   $$\rho := \left\lfloor \sum_{v \in V(r(G))} \frac{d_{r(G)}(v) - 2}{d_{r(G)}(v) + 1} \right\rfloor$$
3. while ($|R| < \rho$)
   {
   $w :=$ vertex in P with maximum degree in $r(\langle P \rangle)$
   $P := P \setminus \{w\}$
   $R := R \cup \{w\}$
   }
4. Output: $\langle P \rangle$.

The condition $|R| < \rho$ for loop iteration (step 3) could easily be replaced by the condition "$\langle P \rangle$ is nonplanar". This will in some cases give better results, but the test requires more effort.

Theorem 10 *Algorithm 1 finds an induced planar subgraph of at least*

$$\left(\frac{3}{\bar{d}+1} + \frac{3(\bar{d} - \lfloor \bar{d} \rfloor)(\lceil \bar{d} \rceil - \bar{d})}{(\bar{d}+1)(\lfloor \bar{d} \rfloor + 1)(\lceil \bar{d} \rceil + 1)} \right) n$$

vertices in a graph G of n vertices if either G has average degree at most $\bar{d} \geq 4$ or G is connected and has average degree $\bar{d} \geq 2$. The algorithm has time complexity $O(nm)$.

Proof. The lower bound on subgraph size follows from the proofs of Lemma 3 and Theorem 9. The main loop is executed $< n$ times, and each iteration takes time $O(m)$. $\square$

5 Fragmentability of graphs of bounded average degree

Let $\overline{\Gamma}_d$ be the class of graphs of average degree at most d, and let $\overline{\Gamma}_d^c$ be the class of connected graphs in $\overline{\Gamma}_d$. Theorem 9, together with [5, Lemma 3.1], gives upper bounds for the coefficients of fragmentability of these classes, with the usual restrictions on d. The same upper bound, in the case when d is an integer, was established in [5, Theorem 3.2] for the coefficient of fragmentability of the more restricted class Γ_d of graphs of maximum degree d.

We now prove a lower bound for the coefficient of fragmentability of $\overline{\Gamma}_d^c$. For any real number $d \geq 2$, let

$$\alpha_d = \frac{d-2}{2d-2} - \frac{(d-\lfloor d \rfloor)(\lceil d \rceil - d)}{2(d-1)(\lfloor d \rfloor - 1)(\lceil d \rceil - 1)}.$$

Note: When d is an integer, $\alpha_d = (d-2)/(2d-2)$; otherwise, the value of α_d is linearly interpolated between the values at the two adjacent integers.

Lemma 11 *Let $\overline{\Gamma}_d^c$ be the class of connected graphs of average degree at most d, where $d \geq 2$. Then $c_f(\overline{\Gamma}_d^c) \geq \alpha_d$.*

Proof. If d is an integer, then the result is already given in [5, Theorem 3.3]. So assume d is not an integer.

First consider the case when d is rational. By the definition of c_f, it suffices to show that for any $\varepsilon > 0$, $\overline{\Gamma}_d^c$ is not $(\alpha_d - \varepsilon)$-fragmentable, i.e. that for any positive integer C, there is a graph G in $\overline{\Gamma}_d^c$ which is not $(C, \alpha_d - \varepsilon)$-fragmentable. Note that it suffices to show this when $C > 1/\varepsilon$, so we will assume this.

Let d_1, d_2 be integers less than and greater than d respectively. As shown in [5, Theorem 3.3], we can find graphs H_i, $i = 1, 2$, such that H_i is regular of degree d_i, with n_i vertices and girth at least C, and for any $X \subseteq V(H_i)$, if $H_i - X$ has components of size at most C, then $|X| > n_i(d_i - 2)/(2d_i - 2)$.

Note that we can assume that H_1, H_2 are connected (otherwise we could use a suitable component instead) and that both have at least C vertices.

Now let $(d - d_1)/(d_2 - d) = p/q$, where p and q are positive integers. Form a graph G' consisting of the disjoint union of qn_2 copies of H_1 and pn_1 copies of H_2. Now let the components of G' be $K^{(0)}, \ldots, K^{(t-1)}$. In each $K^{(j)}$ select an edge $(v^{(j)}, w^{(j)})$. Delete these edges, and for each j join $v^{(j)}$ to $w^{(j+1)}$ (with superscript addition modulo t). This forms a connected graph G with $n = n_1 n_2 (p + q)$ vertices and average degree

$$\frac{qn_2n_1d_1 + pn_1n_2d_2}{n_1n_2(p+q)} = \frac{qd_1 + pd_2}{p+q} = d.$$

Note that $t \leq n/C$.

Now suppose that $X \subseteq V(G)$ and that $G - X$ has components of size at most C. It is easy to see that X contains at least $n_i(d_i - 2)/(2d_i - 2) - 1$ of the vertices from each copy of H_i. Hence

$$\begin{aligned}
|X| &> \left(\frac{d_1-2}{2d_1-2}\right) n_1n_2q + \left(\frac{d_2-2}{2d_2-2}\right) n_1n_2p - n/C \\
&= \left[\left(\frac{d_1-2}{2d_1-2}\right)\left(\frac{q}{p+q}\right) + \left(\frac{d_2-2}{2d_2-2}\right)\left(\frac{p}{p+q}\right)\right] n_1n_2(p+q) - n/C \\
&> \left[\left(\frac{d_1-2}{2d_1-2}\right)\left(\frac{q}{p+q}\right) + \left(\frac{d_2-2}{2d_2-2}\right)\left(\frac{p}{p+q}\right) - \varepsilon\right] n.
\end{aligned}$$

But a little calculation shows that if we take $d_1 = \lfloor d \rfloor$, and $d_2 = \lceil d \rceil$, then

$$\left(\frac{d_1 - 2}{2d_1 - 2}\right)\left(\frac{q}{p+q}\right) + \left(\frac{d_2 - 2}{2d_2 - 2}\right)\left(\frac{p}{p+q}\right) = \alpha_d.$$

Hence G is not $(C, \alpha_d - \varepsilon)$-fragmentable, as required.

Finally, if d is irrational, note that α_d is continuous in d, hence for any $\varepsilon > 0$ we can choose a rational d' with $2 < d' < d$ and $\alpha_{d'} > \alpha_d - \varepsilon/2$. Then the result follows. $\square$

Corollary 12 *If $d \geq 4$ then*

$$\frac{d-2}{2d-2} - \frac{(d - \lfloor d \rfloor)(\lceil d \rceil - d)}{2(d-1)(\lfloor d \rfloor - 1)(\lceil d \rceil - 1)} \leq c_f(\overline{\Gamma}_d) \leq \frac{d-2}{d+1} - \frac{3(d - \lfloor d \rfloor)(\lceil d \rceil - d)}{(d+1)(\lfloor d \rfloor + 1)(\lceil d \rceil + 1)}.$$

If $d \geq 2$ then

$$\frac{d-2}{2d-2} - \frac{(d - \lfloor d \rfloor)(\lceil d \rceil - d)}{2(d-1)(\lfloor d \rfloor - 1)(\lceil d \rceil - 1)} \leq c_f(\overline{\Gamma}^c_d) \leq \frac{d-2}{d+1} - \frac{3(d - \lfloor d \rfloor)(\lceil d \rceil - d)}{(d+1)(\lfloor d \rfloor + 1)(\lceil d \rceil + 1)}.$$

$\square$

In the case when $2 \leq d \leq 3$, both sides reduce to $(d-2)/4$, so we have

Corollary 13 *If $2 \leq d \leq 3$, then $c_f(\overline{\Gamma}^c_d) = (d-2)/4$.* $\square$

6 Thickness

Recall that the *thickness* $\theta(G)$ of a graph $G = (V, E)$ is the minimum t such that there exists a partition $E = E_1 \cup \cdots \cup E_t$ for which each $G_i = (V, E_i)$ is planar $(1 \leq i \leq t)$. See, e.g., [2, 10].

Let Θ_t be the class of graphs of thickness at most t. Halton [8] showed that if G has maximum degree $\leq d$ then $\theta(G) \leq \lceil d/2 \rceil$. This, together with the lower bound of (1), implies that $c_f(\Theta_t) \geq (t-1)/(2t-1)$.

Halton conjectured that $d \leq 6$ implies $\theta(G) \leq 2$. If true, this would imply that $c_f(\Theta_2) \geq 2/5$.

Now we turn to an upper bound for $c_f(\Theta_t)$. It is well known that if G is planar then its average degree is $\leq 6 - 12/n$. Hence if G has thickness $\leq t$ then its average degree is $\leq (6 - 12/n)t$, and then Corollary 12 gives $c_f(\Theta_t) \leq (6t-2)/(6t+1)$. Summarising:

Theorem 14

$$\frac{t-1}{2t-1} \leq c_f(\Theta_t) \leq \frac{6t-2}{6t+1}.$$

$\square$

When $t \leq 2$, the upper bound can be improved. Consider the case $t = 2$. Let $G = (V, E) \in \Theta_2$, with partition $E = E_1 \cup E_2$ and each $G_i = (V, E_i)$ planar. Find an independent set Y in G_1 of at least $n/4$ vertices. Put $X = V \setminus Y$. Observe that $|X| \leq 3n/4$ and $\langle Y \rangle$ is planar since all its edges are in E_2. Use [5, Lemma 3.1] to conclude

$$c_f(\Theta_2) \leq 3/4.$$

This improves on the upper bound of Theorem 14 in this case. The approach can in principle be extended, for $t > 2$, by repeatedly taking sufficiently large independent sets in the G_i, but the upper bound so obtained, $1 - 4^{1-t}$, is worse than Theorem 14 for all $t > 2$.

7 k-colourable graphs and n-cubes

Theorem 15 *Let* $\mathrm{Col}(k)$ *be the class of k-colourable graphs. Then, for all* $k \geq 2$,

$$c_f(\mathrm{Col}(k)) = \frac{k-1}{k}.$$

Proof. It is easy to see that $c_f(\mathrm{Col}(k)) \leq (k-1)/k$: for any k-coloured $G \in \mathrm{Col}(k)$, just remove all colour classes except the largest.

We now show that $c_f(\mathrm{Col}(k)) \geq (k-1)/k$.

Suppose $\mathrm{Col}(k)$ is α-fragmentable. Then there exists $C \in \mathbb{N}$ such that every $G \in \mathrm{Col}(k)$ is (C, α)-fragmentable.

Take $p > C$ and let G_C be the complete k-partite graph: $G_C = K_k(p) = K_{p,\ldots,p} \in \mathrm{Col}(k)$. Set $G_C = (V, E)$ and let $V_1, \ldots, V_k$ be the k parts of the k-partition.

Let $X \subseteq V$ and suppose every component of $G_C - X$ has $\leq C$ vertices.

Suppose $V \setminus X$ contains two vertices v, w in different parts of the k-partition of G_C: say $v \in V_i$, $w \in V_j$, $i \neq j$. Certainly v and w are adjacent. Furthermore, v is adjacent to all vertices in all $V_l \setminus X$, $l \neq i$, and w is adjacent to all vertices in all $V_l \setminus X$, $l \neq j$. It follows that $\langle V \setminus X \rangle$ is connected. Hence $|V \setminus X| \leq C$, so $|X| \geq |V(G_C)| - C \geq kp - p = (k-1)p$.

On the other hand, if $V \setminus X$ is contained entirely in one part, say V_i, then $|V \setminus X| \leq p$, so $|X| \geq (k-1)p$.

So, however X interacts with the parts V_i, we find that $|X| \geq (k-1)p$. So $\alpha \geq (k-1)/k$. Hence $c_f(\mathrm{Col}(k)) \geq (k-1)/k$, since it is the infimum of the possible values of α. □

Theorem 16 *If a class Γ includes regular bipartite graphs of arbitrarily high degree, then*

$$c_f(\Gamma) \geq \frac{1}{2}.$$

Proof. Suppose Γ is such a class, and $c_f(\Gamma) < 1/2$. Then there exists $C \in \mathbb{N}$ and $\alpha < 1/2$ such that every $G \in \Gamma$ is (C, α)-fragmentable.

Let $G = (V, E) \in \Gamma$ be d-regular and bipartite, with bipartition (V_1, V_2). Put $n = |V|$ and note $|V_1| = |V_2| = n/2$ by regularity. Let X be a fragmenting set: $X \subseteq V$, $|X| \leq \alpha n$, and every component of $G - X$ has $\leq C$ vertices. Suppose without loss of generality that $|X \cap V_2| \leq |X \cap V_1|$, so $|X \cap V_2| \leq \alpha n/2$.

Consider $|E(V_1 \setminus X, V_2 \cap X)|$. We count from each side in turn. Firstly, counting from $V_2 \cap X$,

$$|E(V_1 \setminus X, V_2 \cap X)| \leq |X \cap V_2|\ d\ .$$

Secondly, counting from $V_1 \setminus X$, observe that each $v \in V_1 \setminus X$ is part of a component of $\langle V \setminus X \rangle$ of $\leq C$ vertices, so has $\leq C-1$ neighbours in $\langle V \setminus X \rangle$, hence $\geq d-C+1$ neighbours in X, and these must all be in $V_2 \cap X$. Hence

$$|E(V_1 \setminus X, V_2 \cap X)| \geq |V_1 \setminus X|\ (d - C + 1)\ .$$

Combining these inequalities, we have

$$(|V_1| - |X \cap V_1|)\,(d - C + 1) \leq |X \cap V_2|\ d\ .$$

Therefore

$$\begin{aligned} \frac{n}{2}(d - C + 1) &\leq |X \cap V_2|\ d + |X \cap V_1|\,(d - C + 1) \\ &= |X|\,(d - C + 1) + |X \cap V_2|\,(C - 1) \\ &\leq \alpha n(d - C + 1) + |X \cap V_2|\,(C - 1) \\ &\leq \alpha n\left((d - C + 1) + \frac{C-1}{2}\right). \end{aligned}$$

Hence

$$\left(\frac{1}{2} - \alpha\right)(d - C + 1) \leq \alpha\,\frac{C-1}{2}.$$

Since $\alpha < 1/2$ and d can be chosen to be arbitrarily high, we have a contradiction. So in fact $c_f(\Gamma) \geq 1/2$. □

Corollary 17 *If* $\Gamma \subseteq \{\text{bipartite graphs}\}$ *and* Γ *contains regular graphs of arbitrarily high degree, then* $c_f(\Gamma) = 1/2$.

Proof. Theorem 16 shows that $c_f(\Gamma) \geq 1/2$. Choosing the smallest part of any bipartition as our fragmenting set shows that $c_f(\Gamma) \leq 1/2$. □

Corollary 18 *Let* Q_n *be the* n*-dimensional cube. Then*

$$c_f(\{Q_n \mid n \in \mathbb{N}\}) = \frac{1}{2}.$$

□

Note that, in Theorem 15 and Corollaries 17 and 18, the coefficient of fragmentability is attained (i.e., Γ is $c_f(\Gamma)$-fragmentable). This contrasts with classes in which maximum degree is bounded above, where the coefficient of fragmentability is not attained [5, Lemma 3.5].

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