How to Draw a Graph … Revisited

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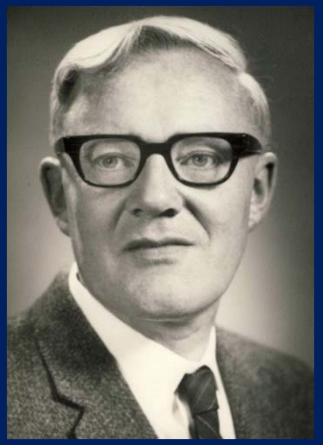
W. T. Tutte 1917 - 2002

- Code breaker at Bletchley park
- Pioneer of matroid theory
- Pioneer of graph theory
- Inventor of the first *graph drawing* algorithm
  - His 1963 paper "*How to Draw a Graph*"* has inspired much research

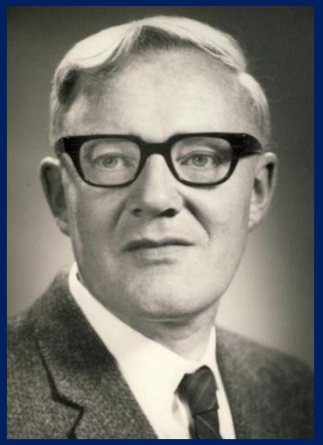


This talk is about two strands of Graph Drawing research that followed Tutte's work

*William T. Tutte. "How to draw a graph." Proc. London Math. Society, 13(52):743 -768, 1963.

This talk

1. Background
   a) Graphs
   b) Graph drawings
   c) Connectivity
   d) Planar graphs
   e) Topological embedding
   f) Good graph drawings



2. How to draw a graph?
   a) Before Tutte: 1920s – 1950s
   b) Tutte: 1960s
   c) After Tutte: 1970s – 1990s
   d) Recent work

(a) Graphs

A *graph* consists of

- Nodes, and
- Binary relationships called “edges” between the nodes

Example: a “Linked-In” style social network

Nodes:

- Alice, Andrea, Annie, Amelia, Bob, Brian, Bernard, Boyle

Edges

- Bob is connected to Alice
- Bob is connected to Andrea
- Bob is connected to Amelia
- Brian is connected to Alice
- Brian is connected to Andrea
- Brian is connected to Amelia
- Boyle is connected to Alice
- Boyle is connected to Andrea
- Boyle is connected to Annie
- Bernard is connected to Alice
- Bernard is connected to Andrea
- Bernard is connected to Annie

(b) Graph drawing

Graph

A graph consists of
- Nodes, and
- Edges

Graph Drawing

A *graph drawing* is a picture of a graph.

That is, a graph drawing is a mapping that assigns
- a location for each node, and
- a curve to each edge.

A social network

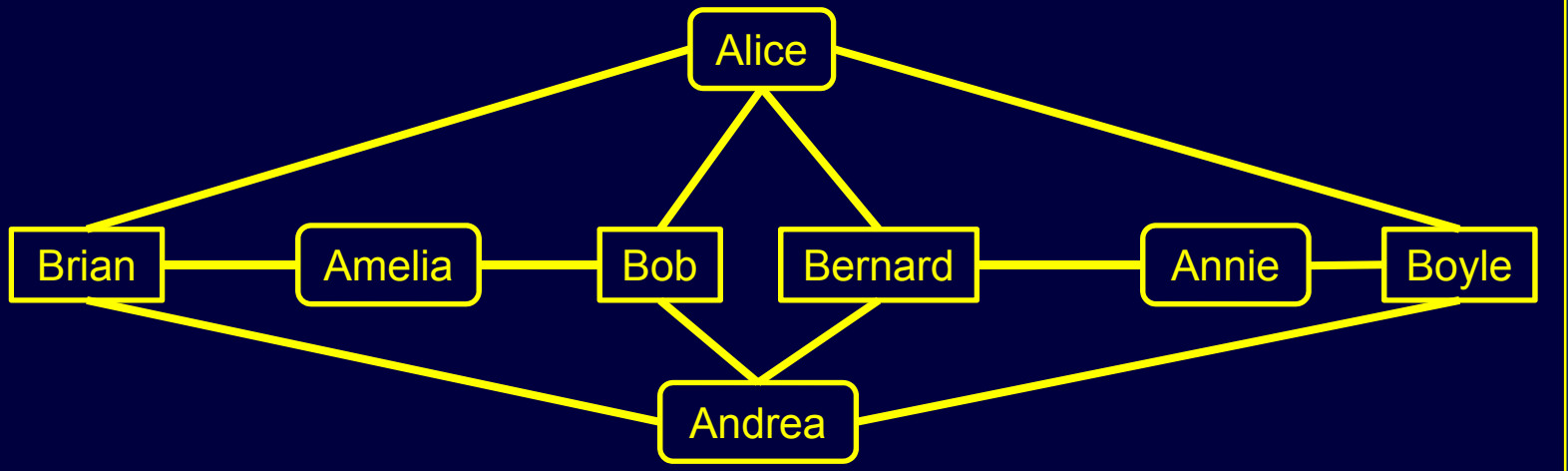
Nodes:

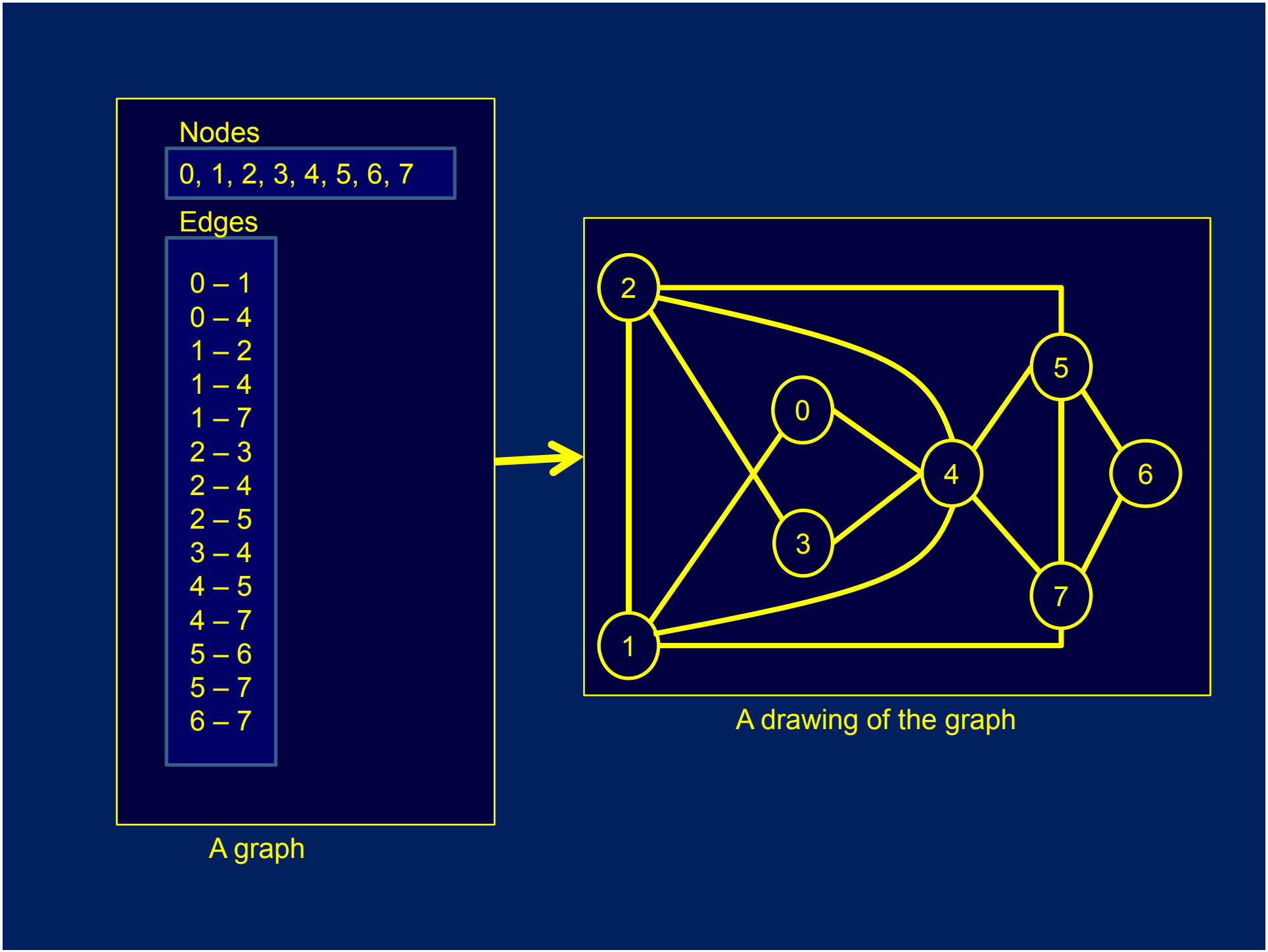
- Bob, Brian, Bernard, Boyle, Alice, Andrea, Annie, Amelia

Edges

- Bob is connected to Alice
- Bob is connected to Andrea
- Bob is connected to Amelia
- Brian is connected to Alice
- Brian is connected to Andrea
- Brian is connected to Amelia
- Boyle is connected to Alice
- Boyle is connected to Andrea
- Boyle is connected to Annie
- Bernard is connected to Alice
- Bernard is connected to Andrea
- Bernard is connected to Annie

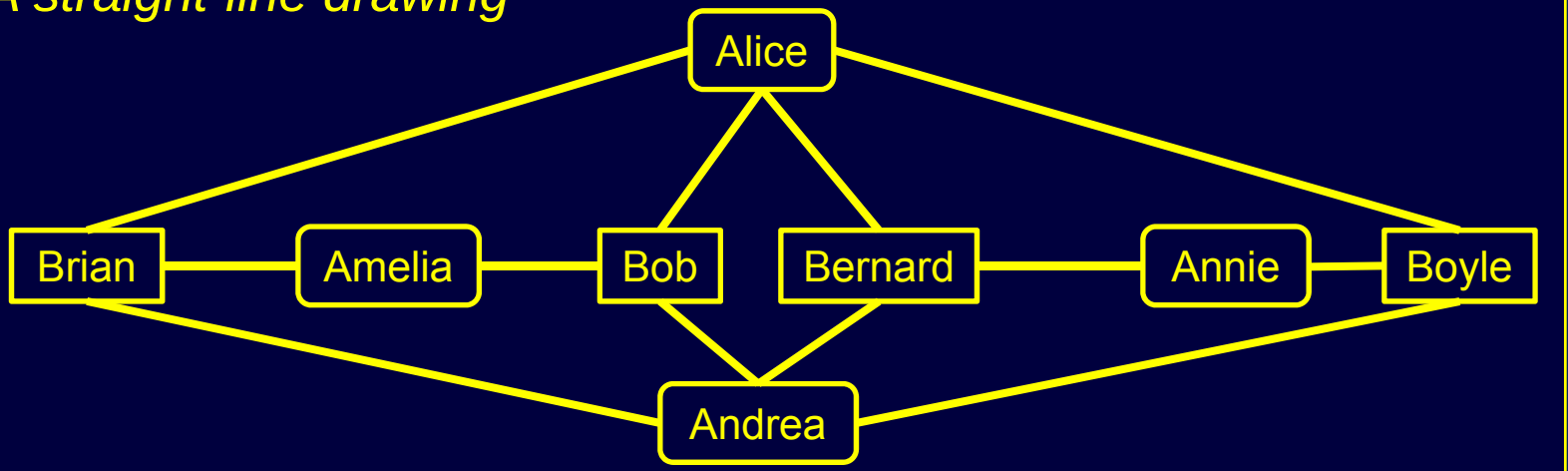
A drawing of the social network



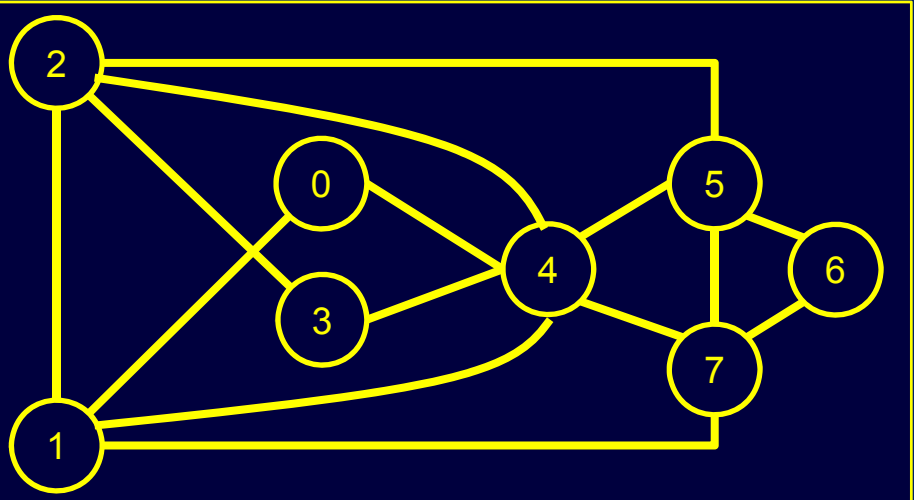

Nodes
0, 1, 2, 3, 4, 5, 6, 7
Edges
0 – 1
0 – 4
1 – 2
1 – 4
1 – 7
2 – 3
2 – 4
2 – 5
3 – 4
4 – 5
4 – 7
5 – 6
5 – 7
6 – 7
A graph
2
0
5
4
6
3
7
1
A drawing of the graph

A graph drawing is a *straight-line drawing* if every edge is a straight line segment.

A straight-line drawing



NOT a straight-line drawing



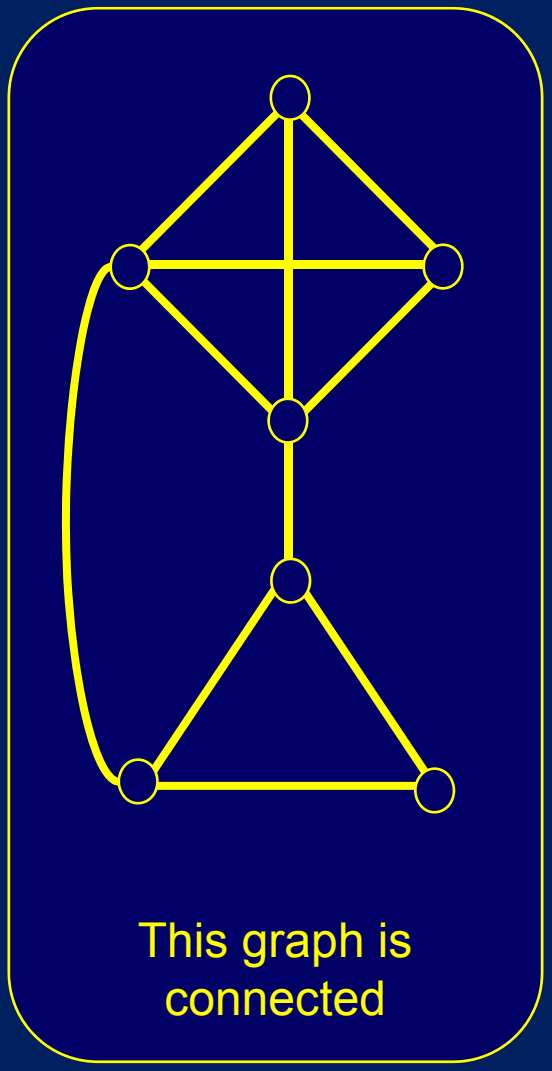
(c) Connectivity

Connectivity notions are fundamental in any study of graphs or networks

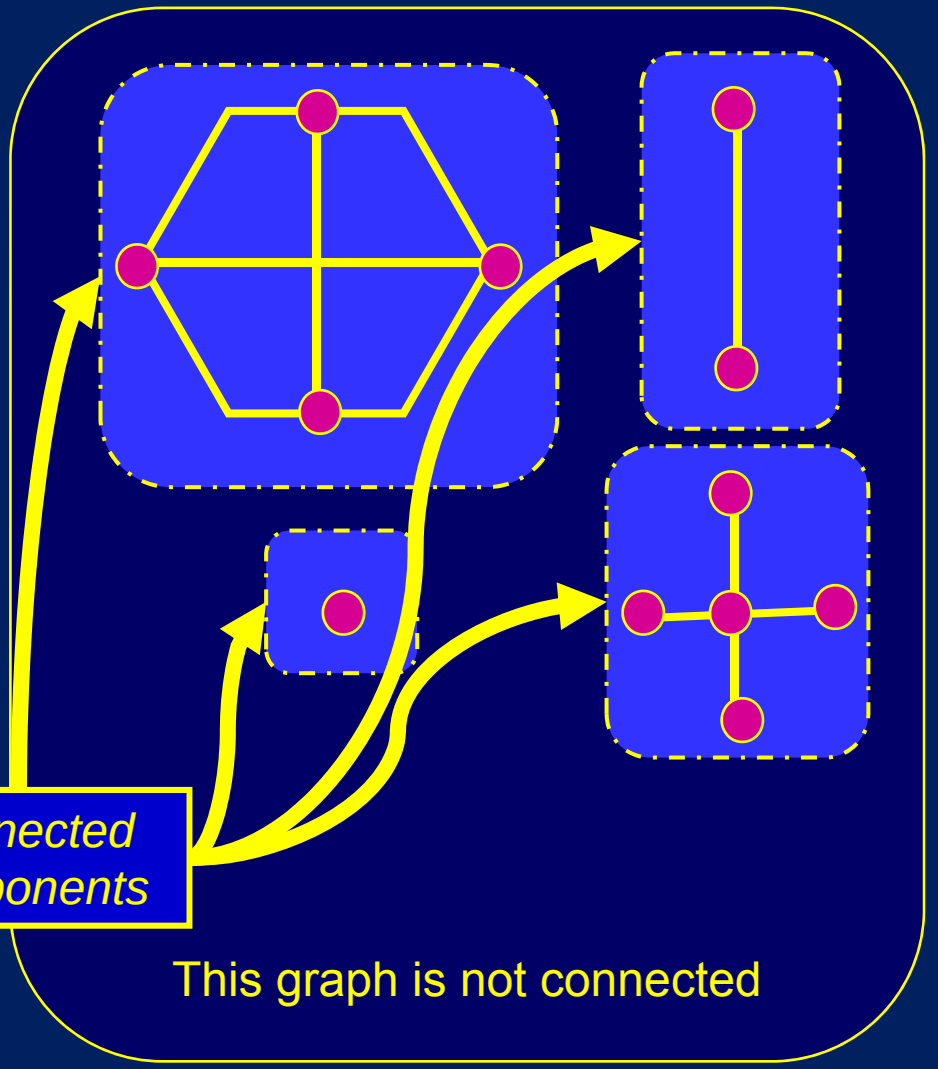
- A graph is *connected* if for every pair u, v of vertices, there is a path between u and v.

- A graph is *k-connected* if there is no set of (k-1) vertices whose deletion disconnects the graph.

  - k = 1: “1-connected” ≡ “connected”
  - k = 2: “2-connected” ≡ “*biconnected*”
  - k = 3: “3-connected” ≡ “*triconnected*”



This graph is connected



This graph is not connected

Connectivity notions are fundamental in any study of networks

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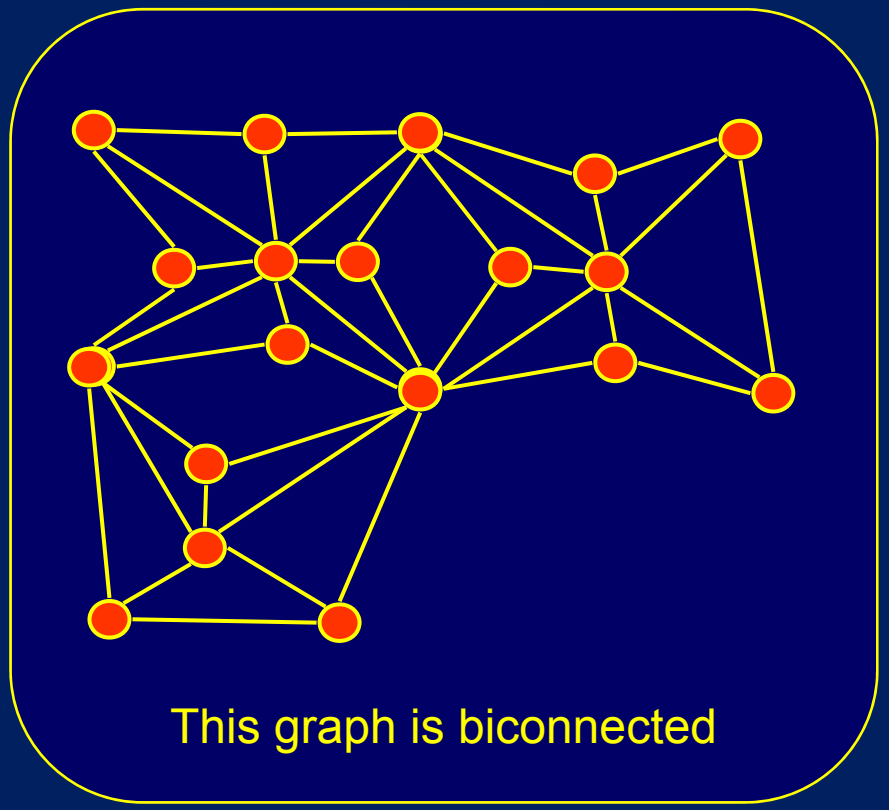
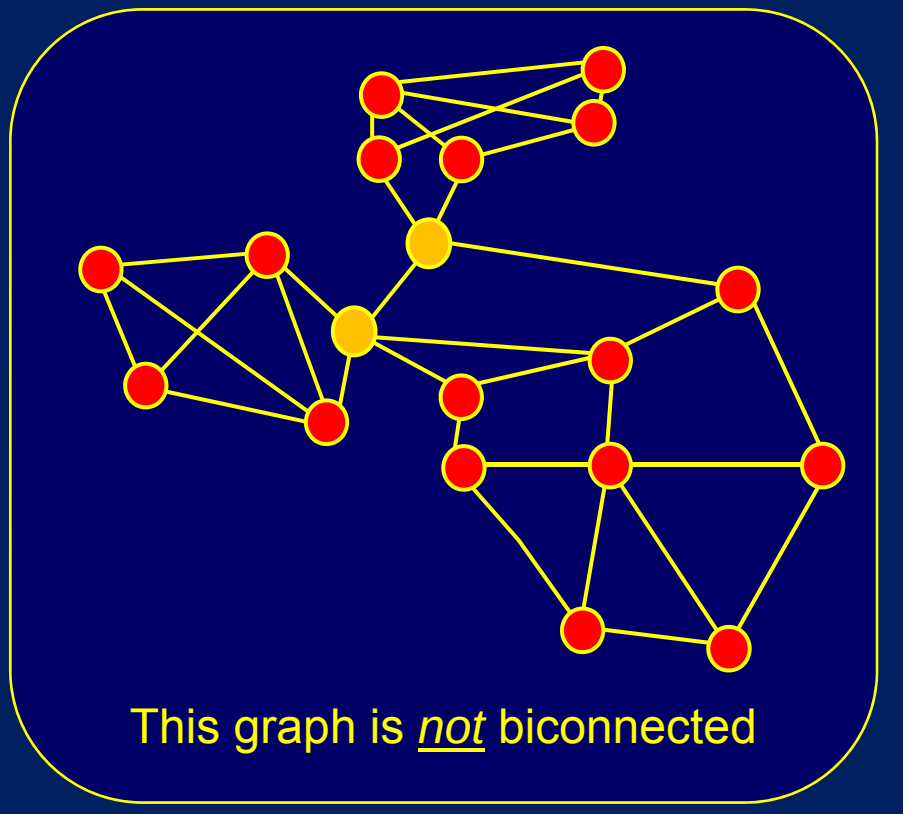
  - k = 3: “3-connected” ≡ “*triconnected*”

"2-connected" ≡ "*biconnected*"

- A *cutvertex* is a vertex whose removal would disconnect the graph.
- A graph without cutvertices is *biconnected*.



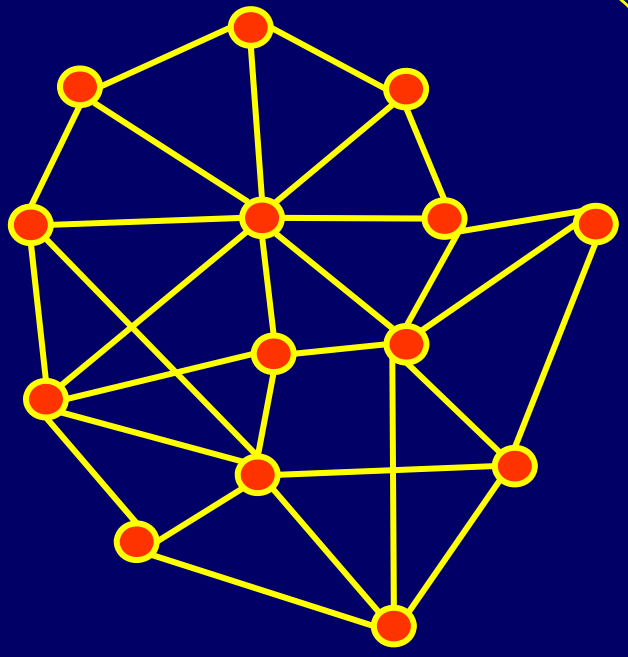
This graph is biconnected



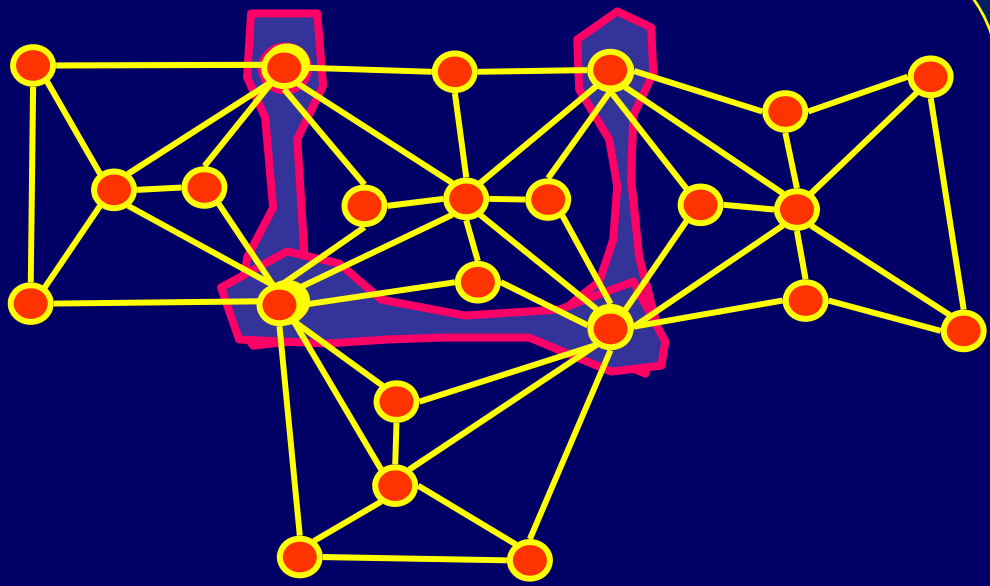
This graph is *not* biconnected

"3-connected" ≡ "*triconnected*"

- A *separation pair* is a pair of vertices whose removal would disconnect the graph.
- A graph without separation pairs is *triconnected*.



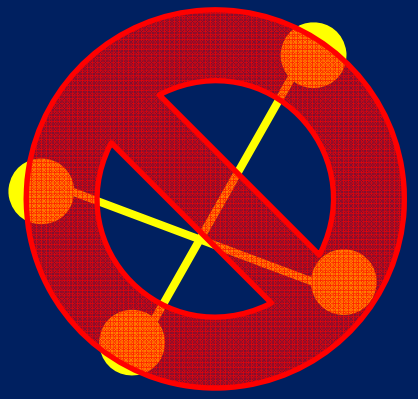
This graph is triconnected



This graph is *not* triconnected

(d) Planar graphs

A graph is *planar* if it can be drawn without edge crossings.



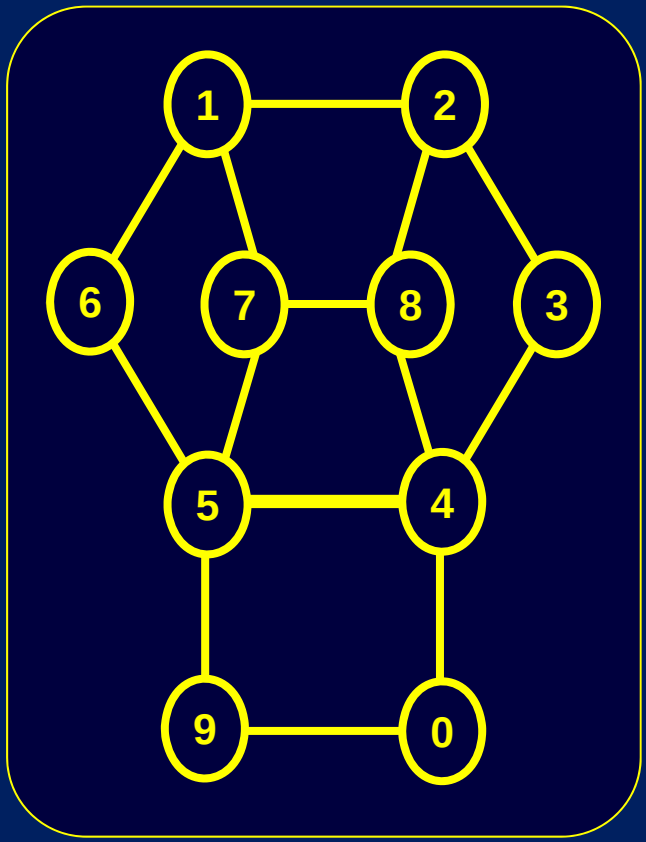
A graph is *planar* if it can be drawn without edge crossings.

Nodes:

- 0,1,2,3,4,5,6,7,8,9

Edges

- 0 – 4
- 0 – 9
- 1 – 2
- 1 – 6
- 1 – 7
- 2 – 3
- 2 – 8
- 3 – 4
- 4 – 5
- 4 – 8
- 5 – 6
- 5 – 7
- 7 – 8



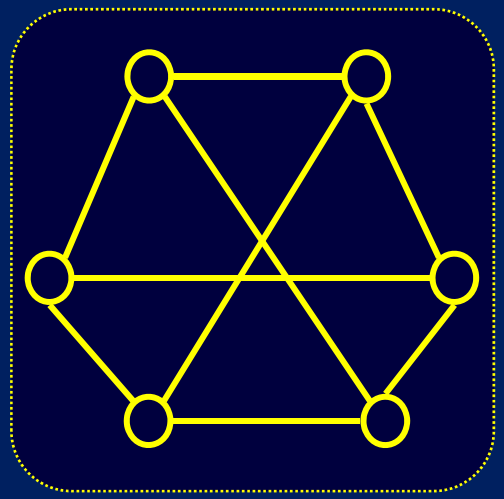
Non-planar

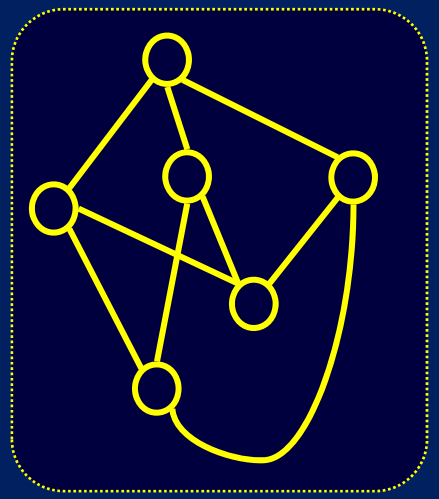
Nodes:

- 0,1,2,3,4,5

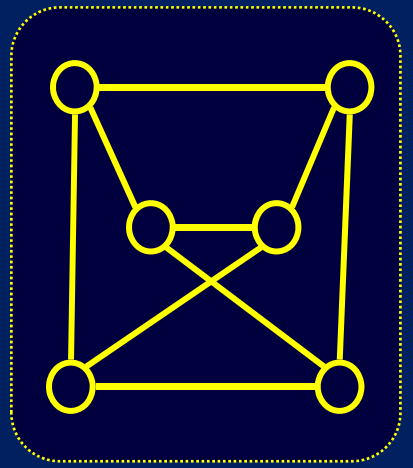
Edges

- 0 – 1
- 0 – 3
- 0 – 5
- 1 – 2
- 1 – 4
- 2 – 3
- 2 – 5
- 3 – 4
- 4 – 5



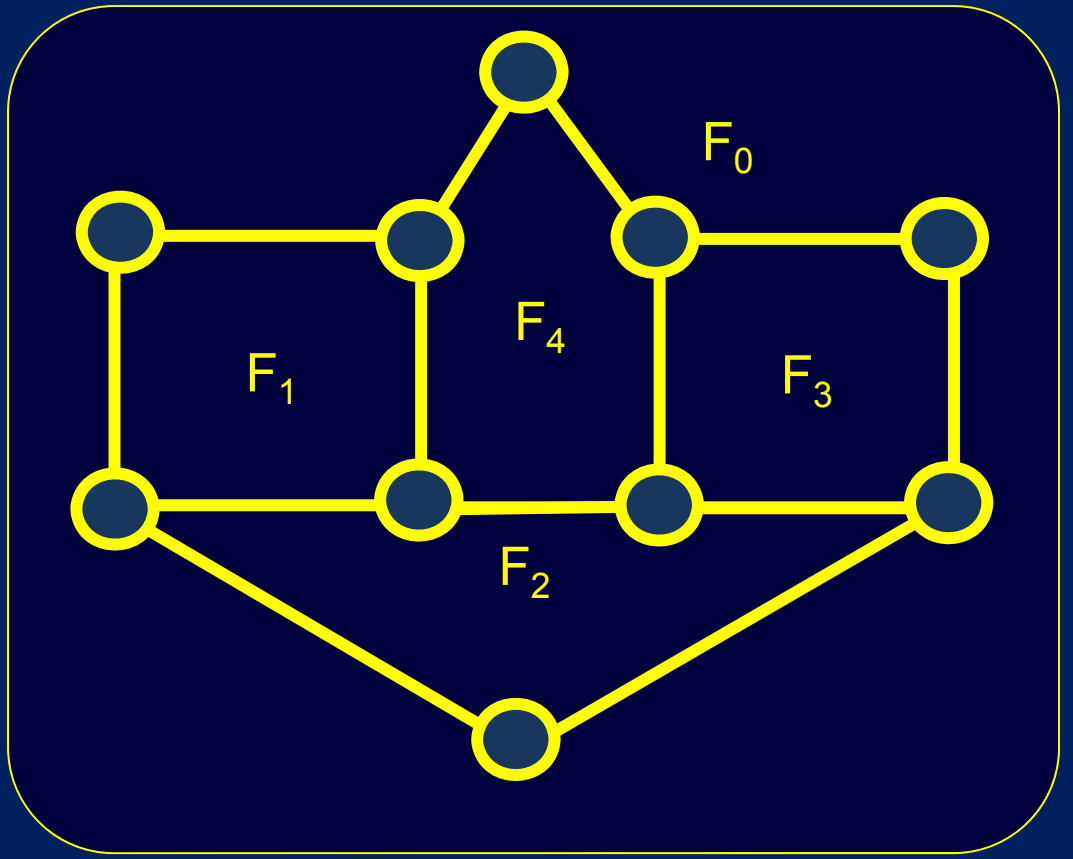


A graph is *non-planar* if *every* drawing has at least one edge crossing.



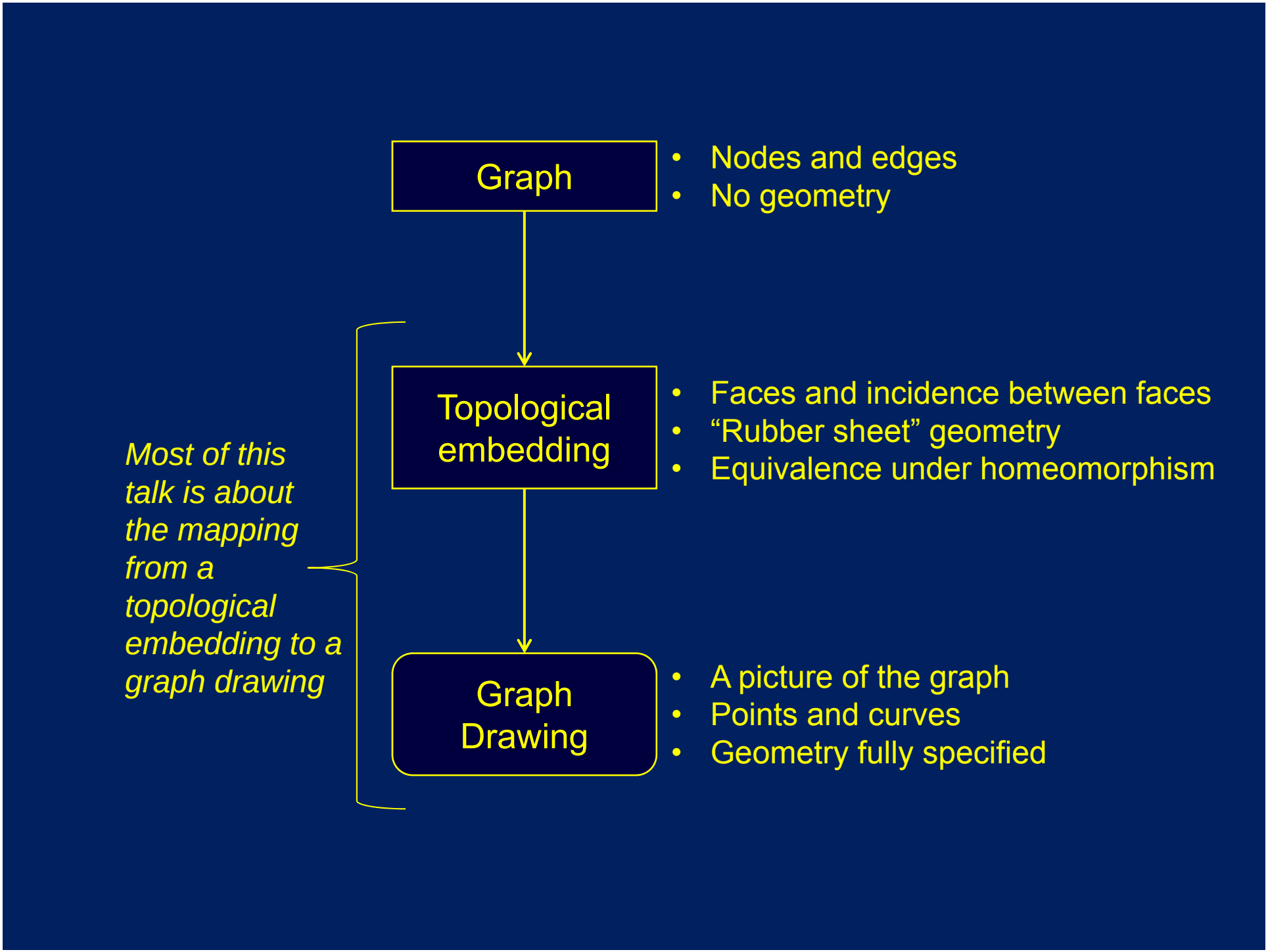
(e) Topological embedding

A planar drawing divides the plane into *faces*.



F_0 shares a boundary with F_1
F_0 shares a boundary with F_2
F_0 shares a boundary with F_3
F_0 shares a boundary with F_4
F_1 shares a boundary with F_2
F_1 shares a boundary with F_4
F_2 shares a boundary with F_1
F_2 shares a boundary with F_3
F_2 shares a boundary with F_4
F_3 shares a boundary with F_4

The boundary-sharing relationships of the faces defines a *topological embedding* of the graph drawing


Graph
Nodes and edges
No geometry
Topological embedding
Faces and incidence between faces
“Rubber sheet” geometry
Equivalence under homeomorphism
Most of this talk is about the mapping from a topological embedding to a graph drawing
Graph Drawing
A picture of the graph
Points and curves
Geometry fully specified

Euler formula

If

n = #vertices

f = #faces

m = #edges

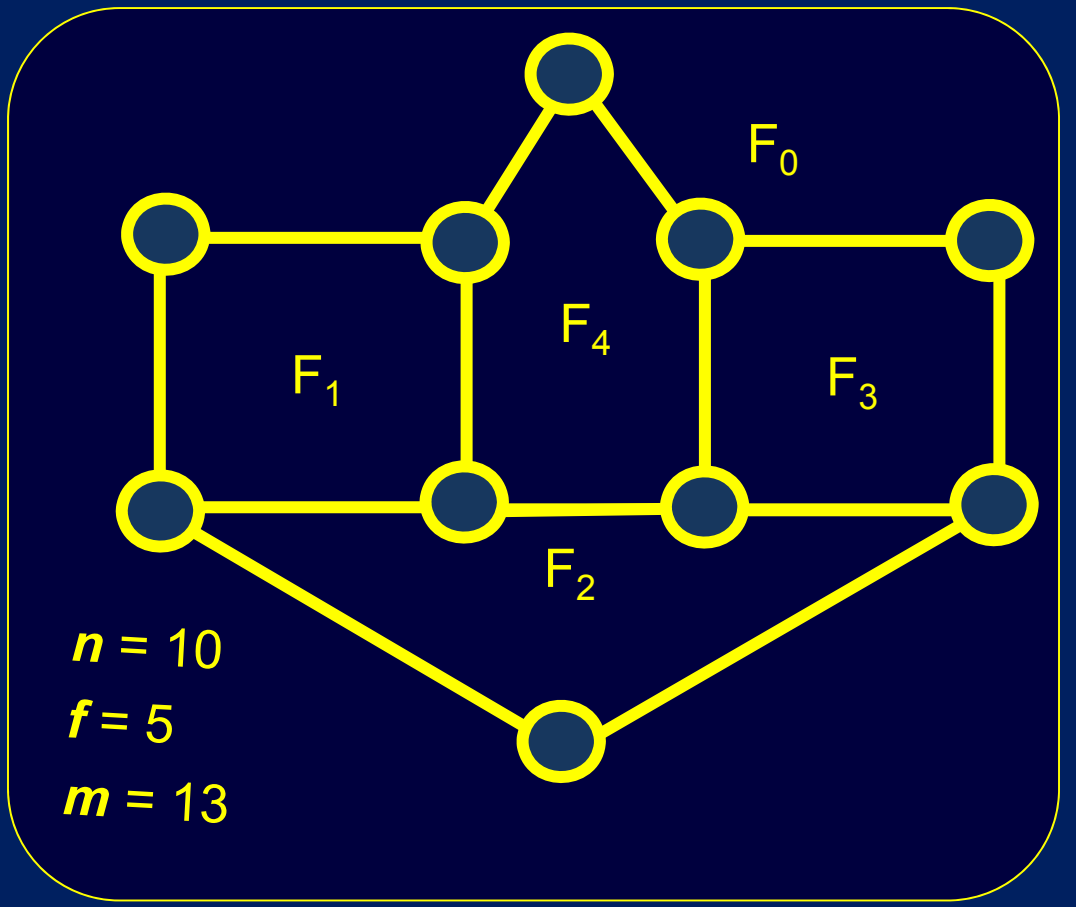
then

n+f = m+2

Corollary ***m ≤ 3n-6***

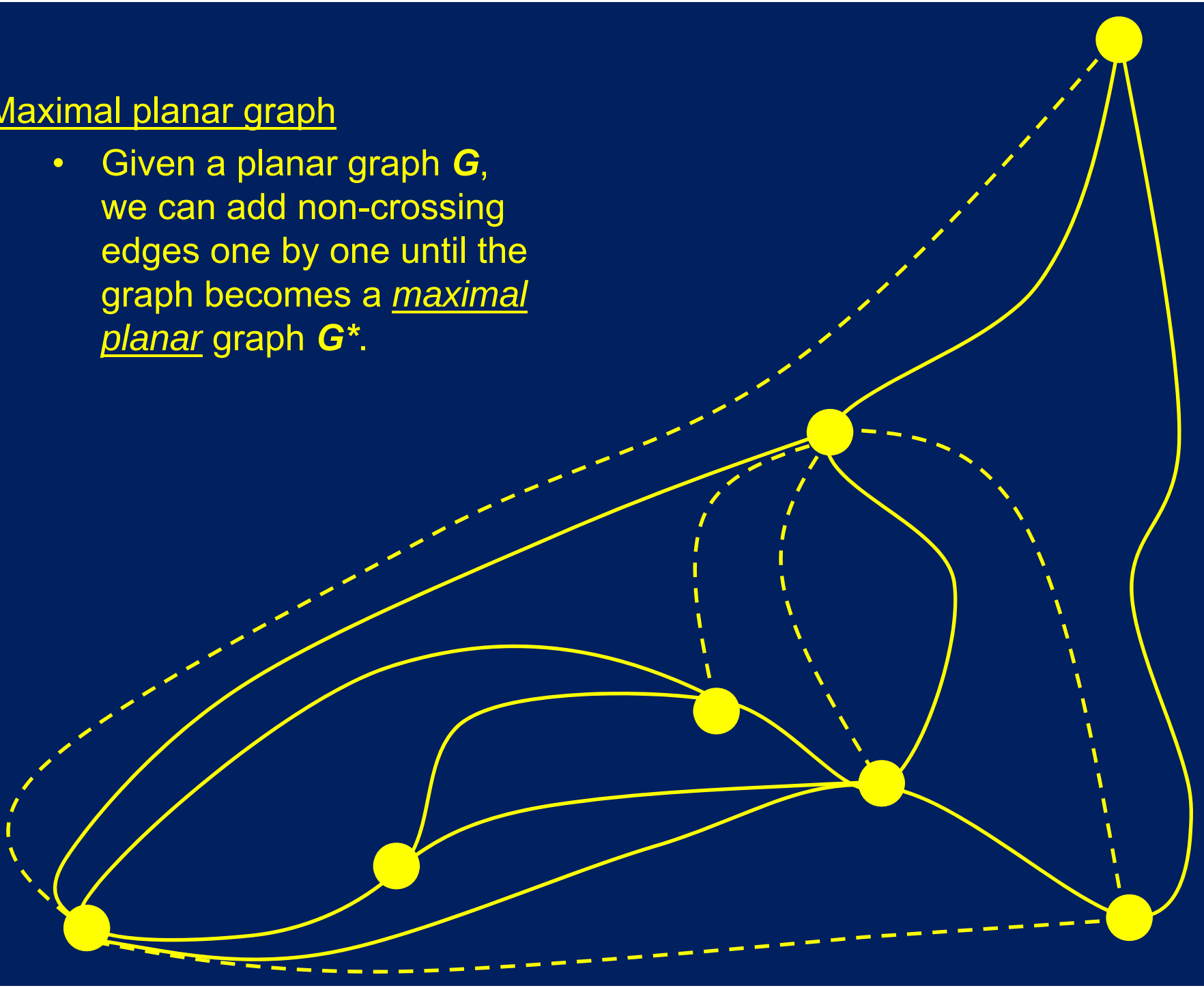
Corollary

If ***m = 3n-6*** then every face is a triangle



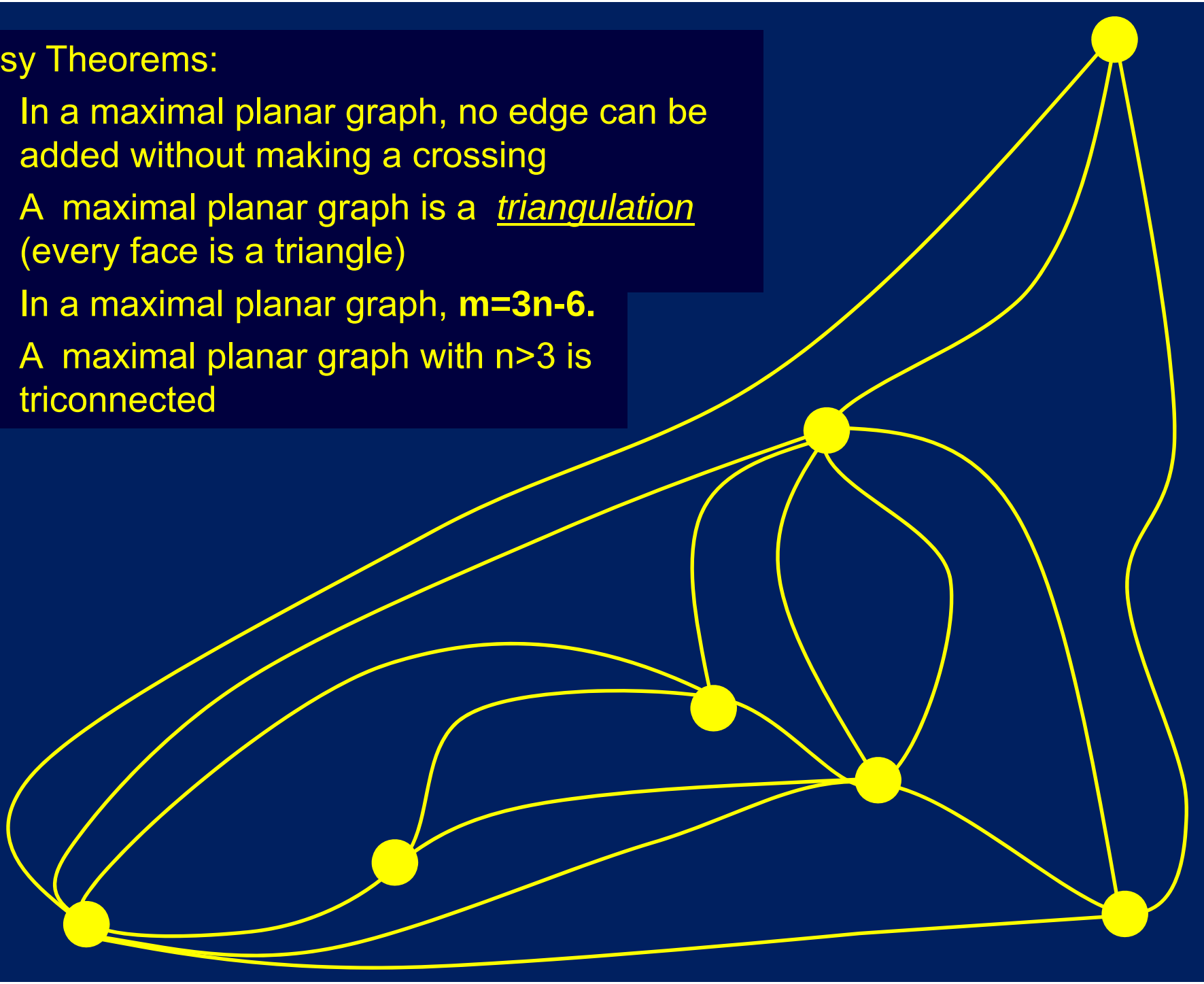
Maximal planar graph

- Given a planar graph ***G***, we can add non-crossing edges one by one until the graph becomes a *maximal planar* graph ***G****.



Easy Theorems:

- In a maximal planar graph, no edge can be added without making a crossing
- A maximal planar graph is a *triangulation* (every face is a triangle)
- In a maximal planar graph, **m=3n-6.**
- A maximal planar graph with n>3 is triconnected



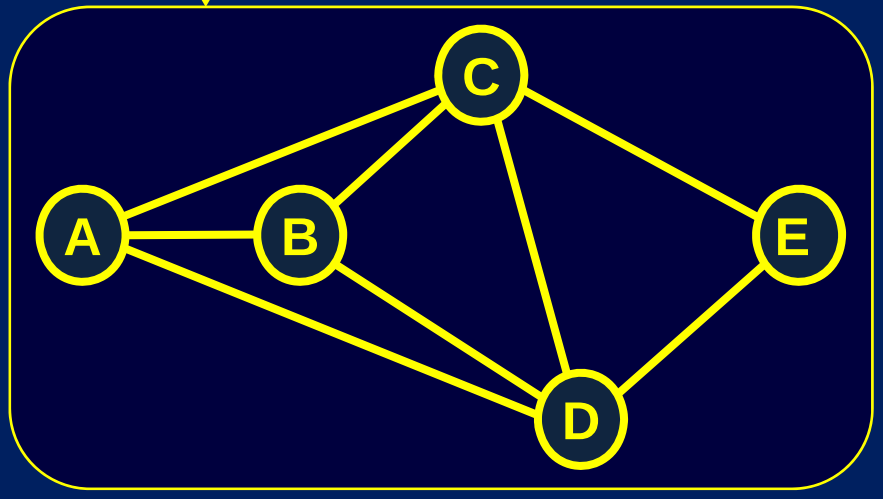
(e) Good graph drawing

Graph

A - B, C, D
B - A, C, D
C - A, B, D, E
D - A, B, C, E
E - C, D

The input is a graph with no geometry

Graph Drawing



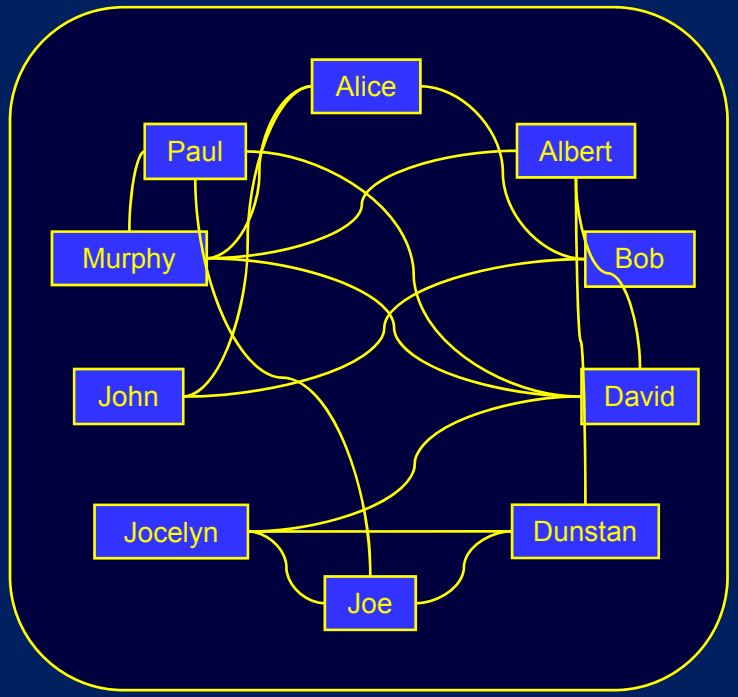
The output should be a <u>good</u> graph drawing:

- *<u>easy to understand</u>,*
- *<u>easy to remember</u>,*
- *<u>beautiful</u>.*

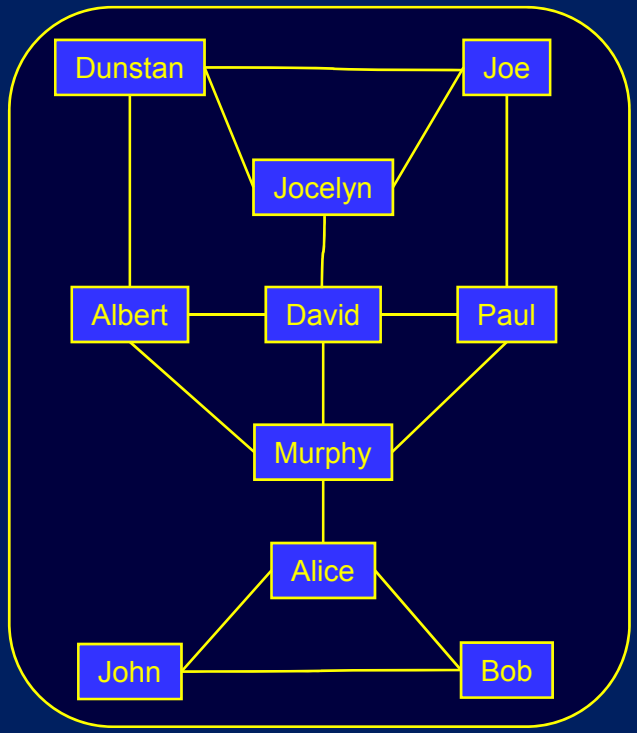
~1979 Intuition (Sugiyama et al. 1979; Batini et al 1982; etc):

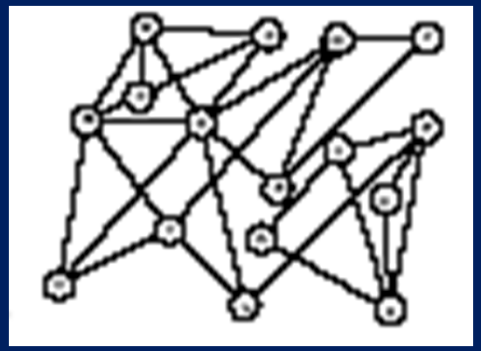
– Planar straight-line drawings make good pictures

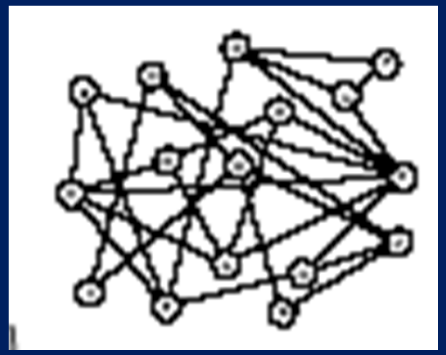
Bad drawing

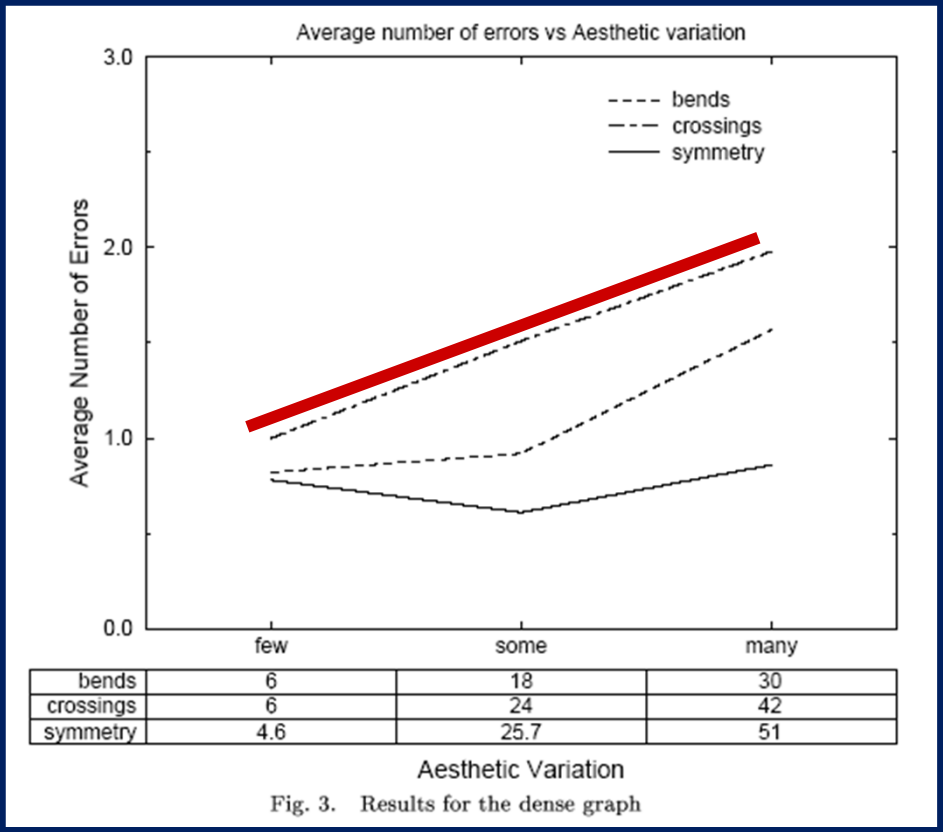


Good drawing









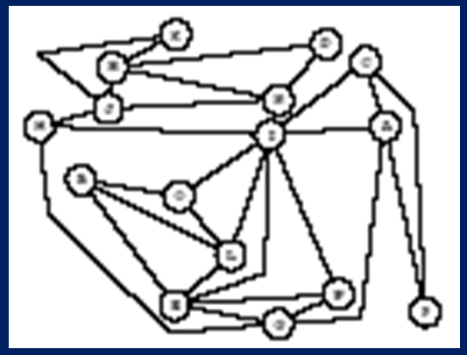
	few	some	many
bends	6	18	30
crossings	6	24	42
symmetry	4.6	25.7	51

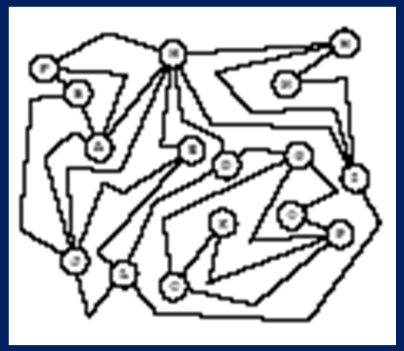
Fig. 3. Results for the dense graph

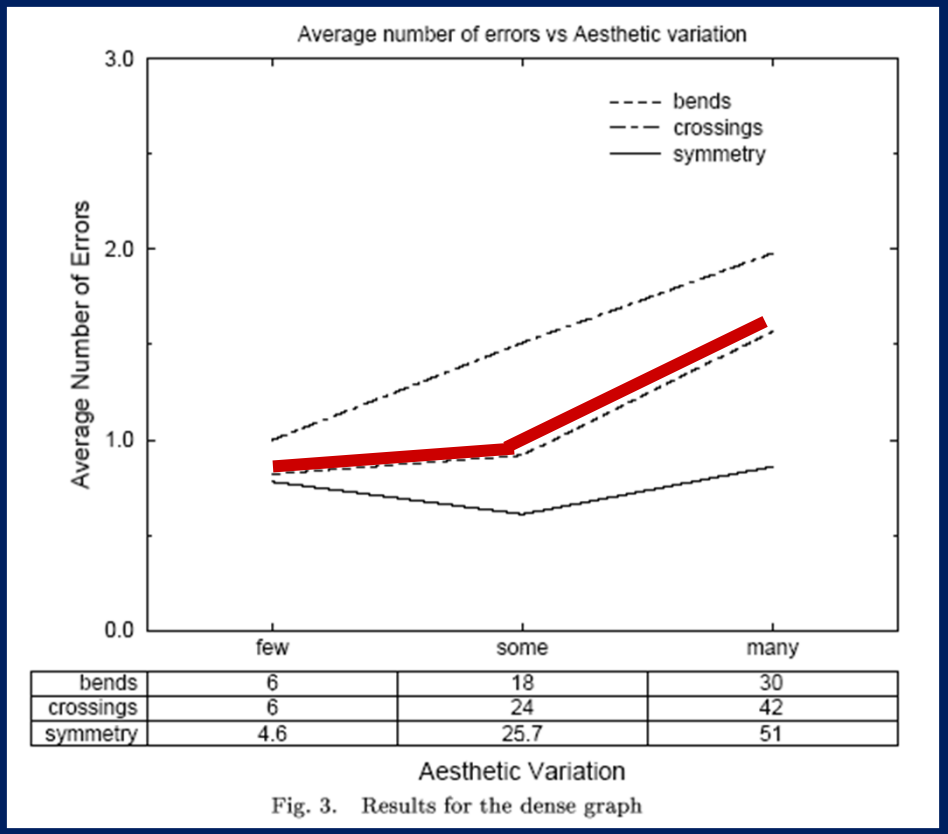
Purchase et al., 1997:

Significant correlation between *edge crossings* and human understanding

- *More edge crossings means more human errors in understanding*







bends	6	18	30
crossings	6	24	42
symmetry	4.6	25.7	51

Fig. 3. Results for the dense graph

Purchase et al., 1997:

Significant correlation between *straightness of edges* and human understanding

- *More bends mean more human errors in understanding*

What makes a *good* drawing of a graph?

- lack of edge crossings (planar drawings are good!)
- straightness of edges (straight-line drawings are good!)

(plus some other things)

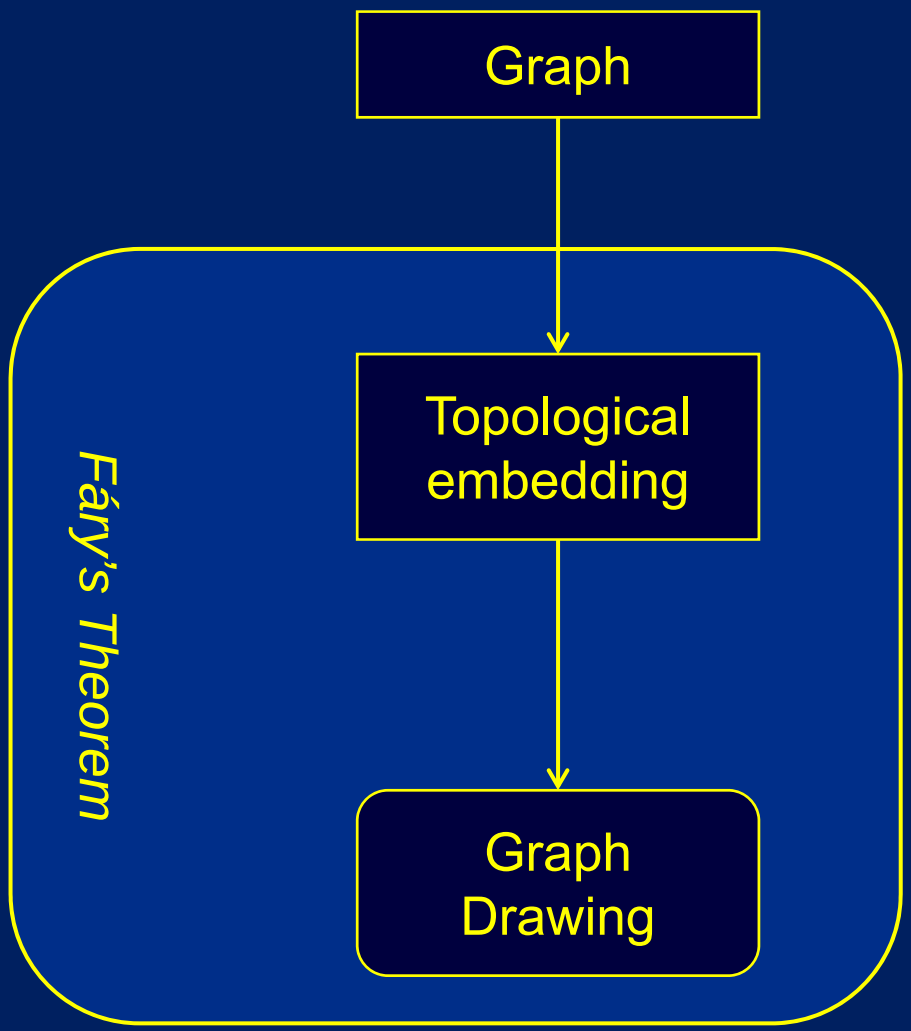
2. How to draw a graph

a) Before Tutte: 1920s – 1950s

Fáry's Theorem

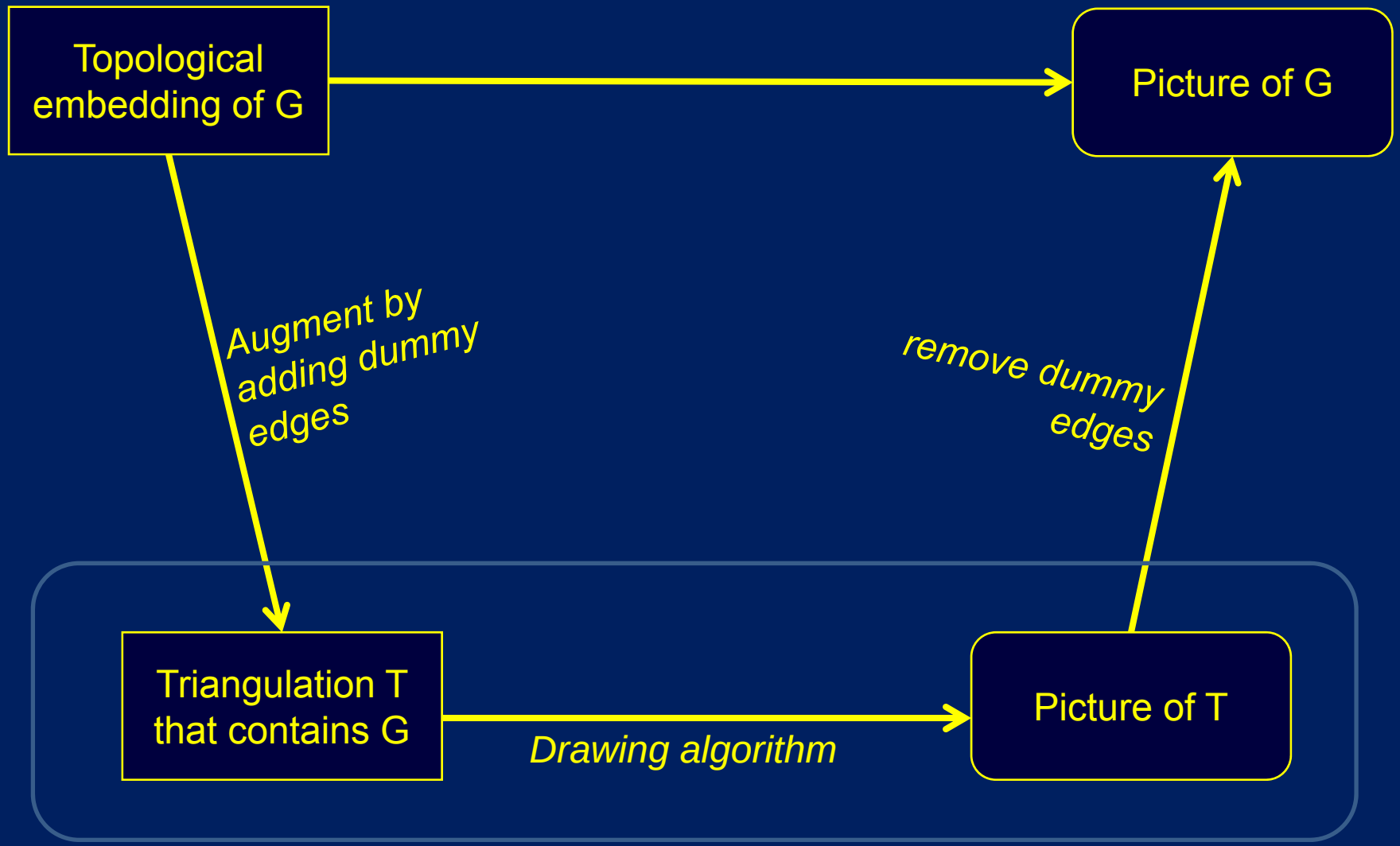
Every topological embedding of a planar graph has a straight-line planar drawing.

Proved independently by Wagner (1936), Fary (1948) and Stein (1951)


Graph
Topological embedding
Graph Drawing
Fáry's Theorem

Wikipedia proof of Fáry's Theorem:

- First note that it is enough to prove it for triangulations.



We prove Fáry's theorem by induction on the number of vertices.

If G has only three vertices, then it is easy.

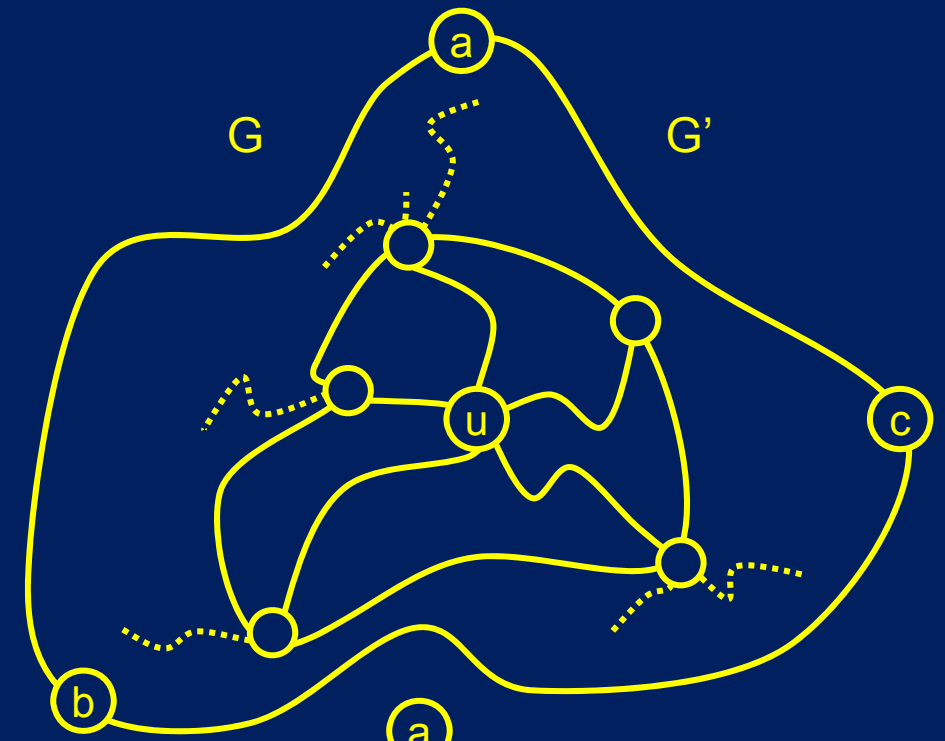
Suppose G has n>3 vertices and 3n-6 edges, and that the outer face of G is the triangle <abc>.

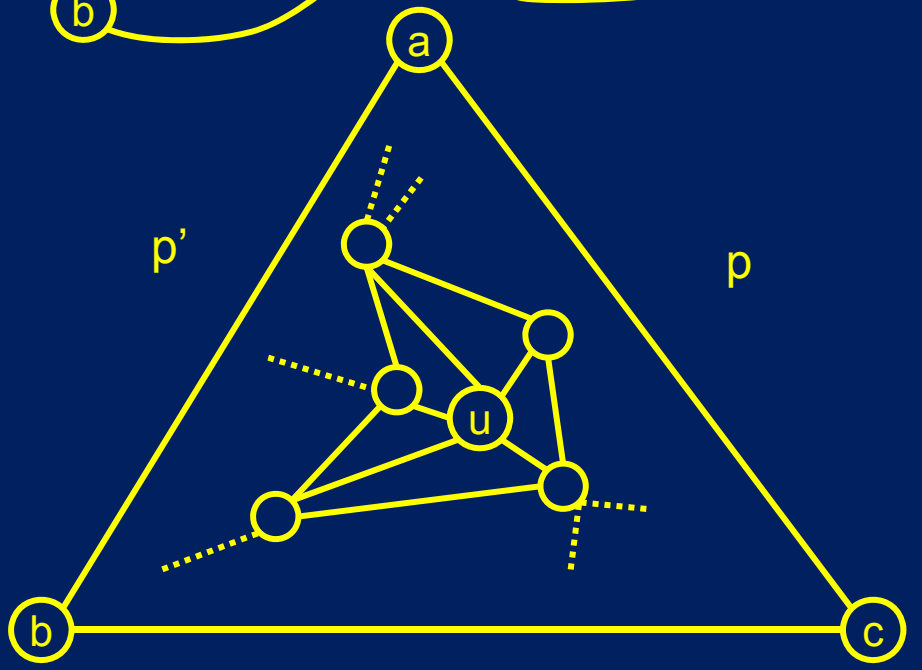
Since every vertex has degree at least 3, Euler implies that there is a vertex ***u*** not on the outside face with degree at most 5.

Delete ***u*** from G to form G'; this gives a face F of G of size at most 5.

Since G' has n-1 vertices, by induction it has a planar straight-line drawing p'.

Since F has at most 5 vertices, it is star-shaped, and we can place the vertex u in the kernel of F to give a planar straight-line drawing p of G





2. How to draw a graph?
   a) Before Tutte: 1920s – 1950s
   b) Tutte

W. T. Tutte,
"How to Draw a Graph",
Proceedings of the London Mathematical Society
13, pp743 – 767, 1960

Tutte's barycentre algorithm

Input:

- A graph G = (V,E)

Output

- A straight-line drawing p

Step 1. Choose a subset A of V

Step 2. Choose a location p(a) = (x_a, y_a) for each vertex a ∈ A

Step 3. For all u ∈ V-A, choose p(u) by

$$p(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} p(v)$$

where the sum is over all neighbors v of u

Vertex u is placed at the "*barycenter*" of its neighbors

Step 1. Choose a subset A of V

Step 2. Choose a location p(a) = (x_a, y_a) for each vertex a ∈ A

Step 3. For all u ∈ V-A,

$$p(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} p(v)$$

where the sum is over all neighbors v of u

$$x(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x(v)$$

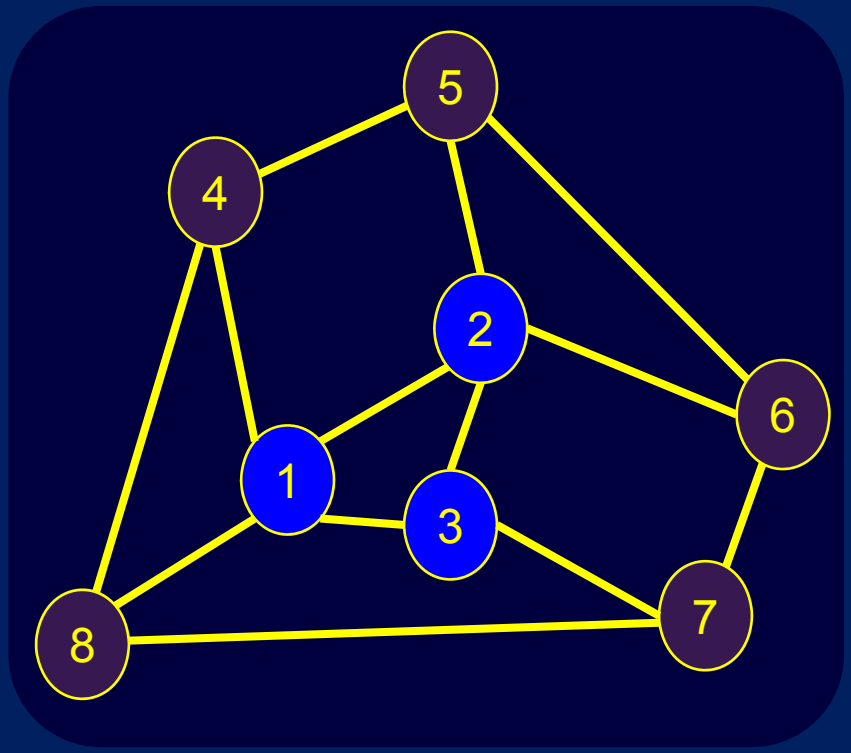
and

$$y(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} y(v)$$

2n equations

Tutte's barycenter algorithm

1. Choose a set A of vertices.
2. Choose a location p(a) for each a∈A
3. For each vertex ***u*** ∈V-A, place ***u*** at the barycentre of its graph-theoretic neighbors.



Example

Step 1. A = {4, 5, 6, 7, 8}

Step 2. For all i = 4, 5, 6, 7, 8,
choose x_i and y_i in some way.

Step 3. Find x_1, y_1, x_2, y_2, x_3, and y_3 such that:

$$x_1 = \frac{1}{4}(x_2 + x_3 + x'_4 + x'_8)$$

$$x_2 = \frac{1}{4}(x_1 + x_3 + x'_5 + x'_6)$$

$$x_2 = \frac{1}{3}(x_1 + x_2 + x'_7)$$

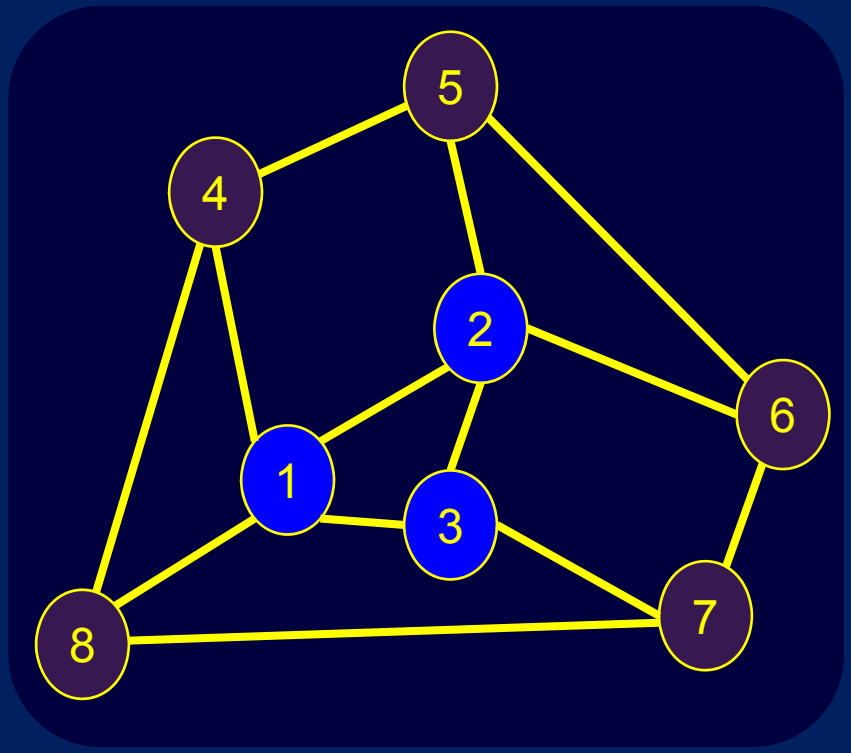
and

$$y_1 = \frac{1}{4}(y_2 + y_3 + y'_4 + y'_8)$$

$$y_2 = \frac{1}{4}(y_1 + y_3 + y'_5 + y'_6)$$

$$y_2 = \frac{1}{3}(y_1 + y_2 + y'_7)$$

where x'_i and y'_i are the values chosen at step 2.



Step 1. A = {4, 5, 6, 7, 8}

Step 2. For all i = 4, 5, 6, 7, 8, choose x_i and y_i in some way.

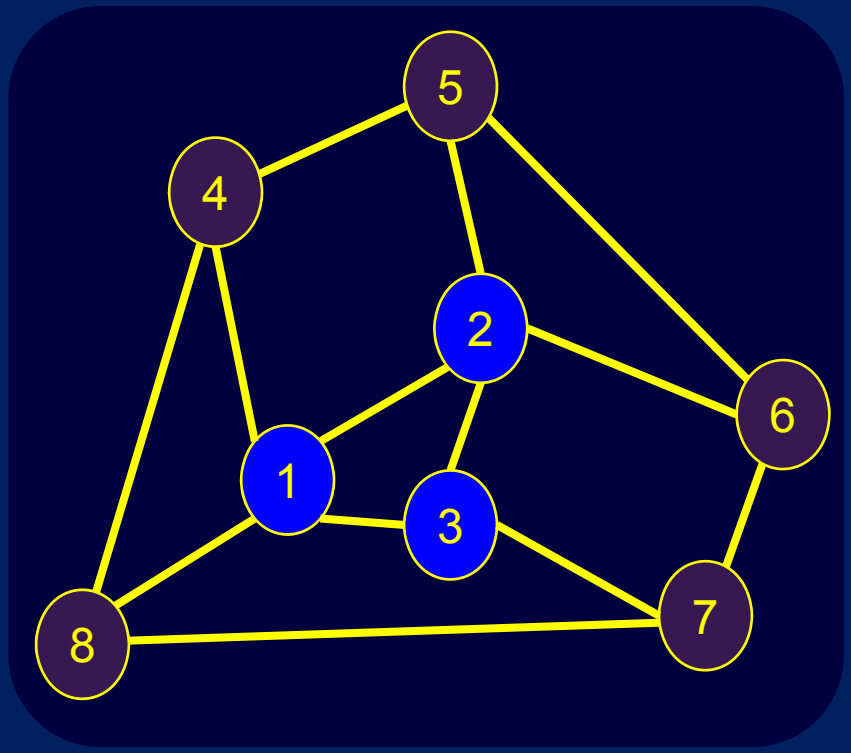
Step 3. Find x_1, y_1, x_2, y_2, x_3, and y_3 such that:

$$4x_1 - x_2 - x_3 = x'_4 + x'_8 = c_1$$
$$-x_1 + 4x_2 - x_3 = x'_5 + x'_6 = c_2$$
$$-x_1 - x_2 + 3x_3 = x'_5 = c_3$$

and

$$4y_1 - y_2 - y_3 = y'_4 + y'_8 = d_1$$
$$-y_1 + 4y_2 - y_3 = y'_5 + y'_6 = d_2$$
$$-y_1 - y_2 + 3y_3 = y'_5 = d_3$$

where c_1, c_2, c_3, d_1, d_2, d_3 are constants



Step 1. A = {4, 5, 6, 7, 8}

Step 2. For all i = 4, 5, 6, 7, 8, choose x_i and y_i in some way.

Step 3. Find vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ such that

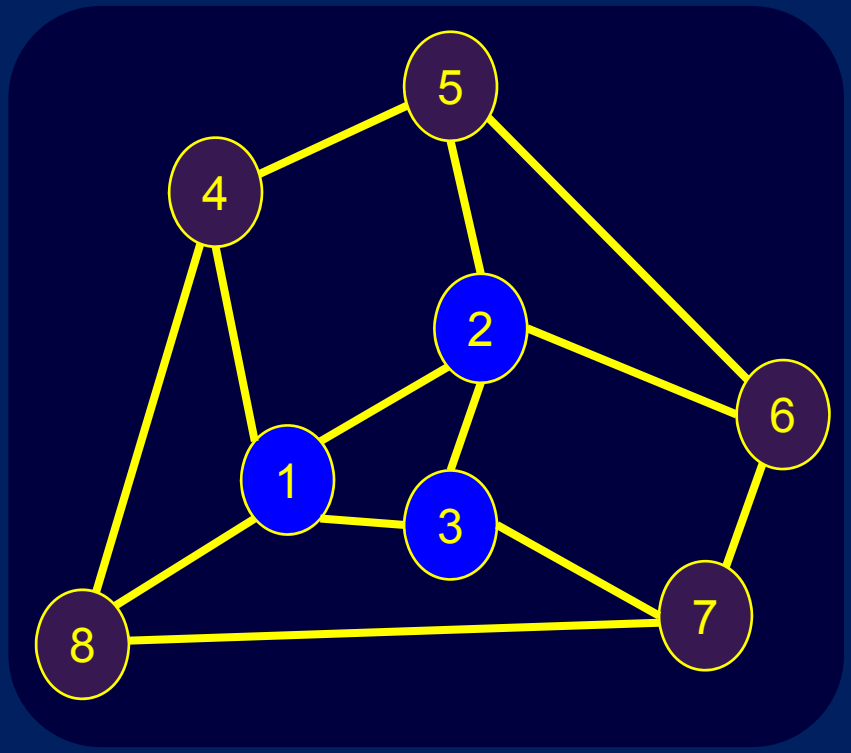
$$Mx = c$$

and

$$My = d$$

where

$$M = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$



The essence of Tutte's barycentre algorithm is inverting a matrix

Tutte's barycentre algorithm

Input: A graph G = (V,E)

Output: A straight-line drawing p

Step 1. Choose a subset A of V

Step 2. Choose a location p(a) = (x_a, y_a) for each vertex a ∈ A

Step 3. Let M be the matrix, indexed by V-A, defined by

$$M_{uv} = \begin{cases} \deg(u) \text{ if } u = v \\ -1 \text{ if } (u,v) \epsilon E \\ 0 \text{ otherwise} \end{cases}$$

Let c and d be the vectors, indexed by V-A, de

$$c_u = \sum_{w \in A} x_w, \quad \text{and} \quad d_u = \sum_{w \in A} y_w.$$

Then choose x and y to be the vectors, indexed over V-A, defined by

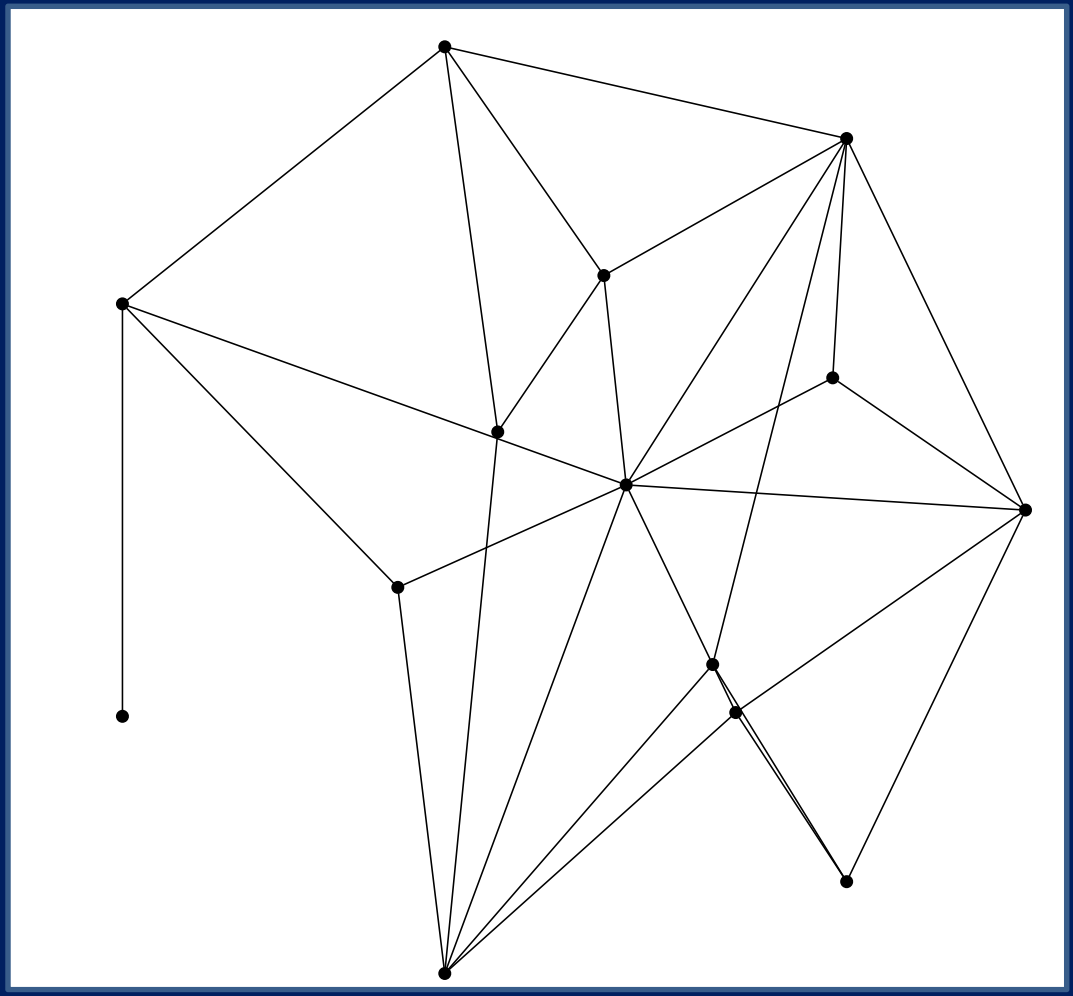
$$x = M^{-1}c \quad \text{and} \quad y = M^{-1}d.$$

Then choose $p(u) = (x_y, y_u)$ for all $u \in V - A$.

- This is a special matrix: *Laplacian submatrix*.
- Many software packages can invert such a matrix efficiently

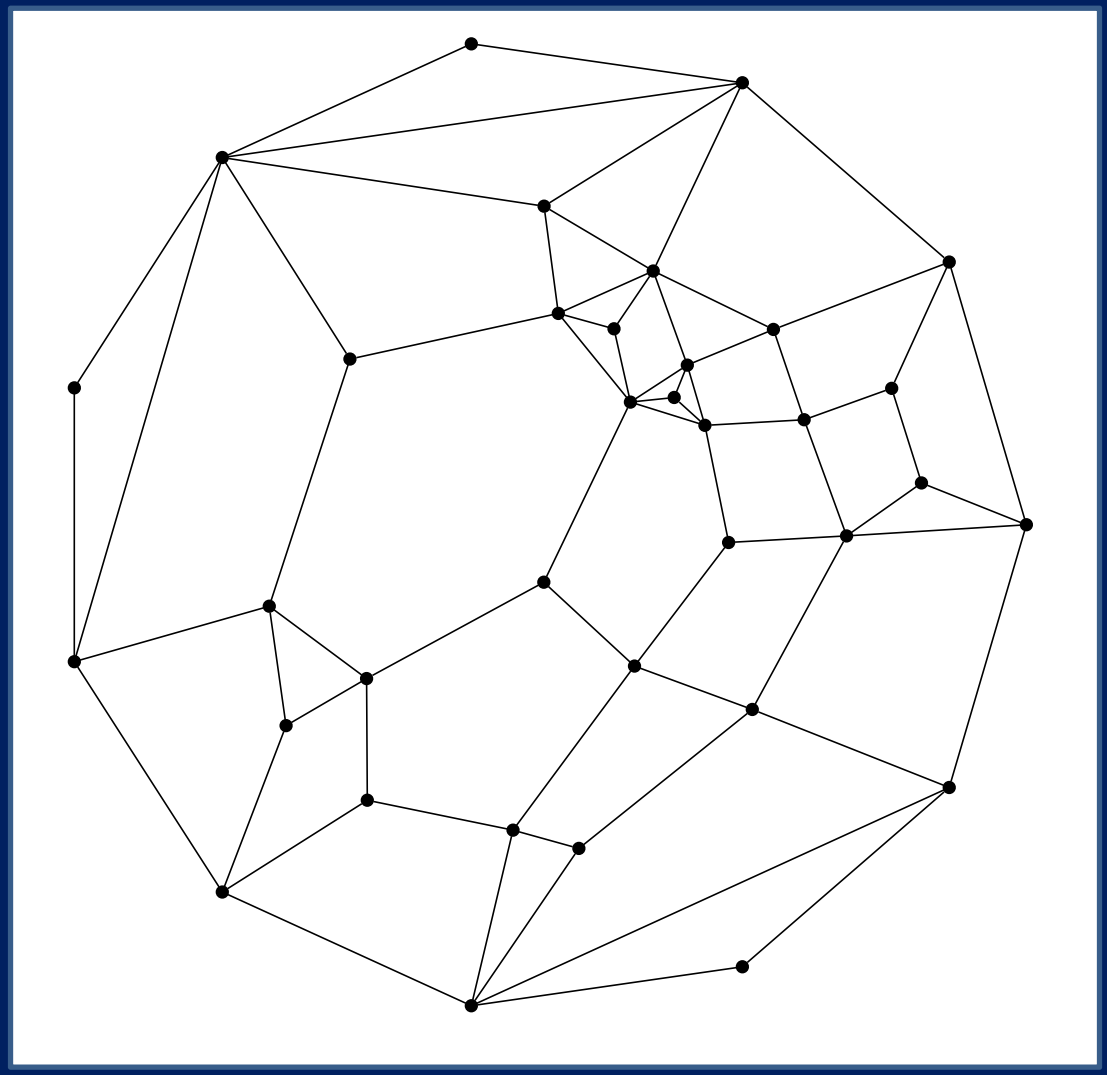
Tutte's barycentre algorithm

- Example output on a non-planar graph



Tutte's barycentre algorithm

- Example output on a *planar* graph



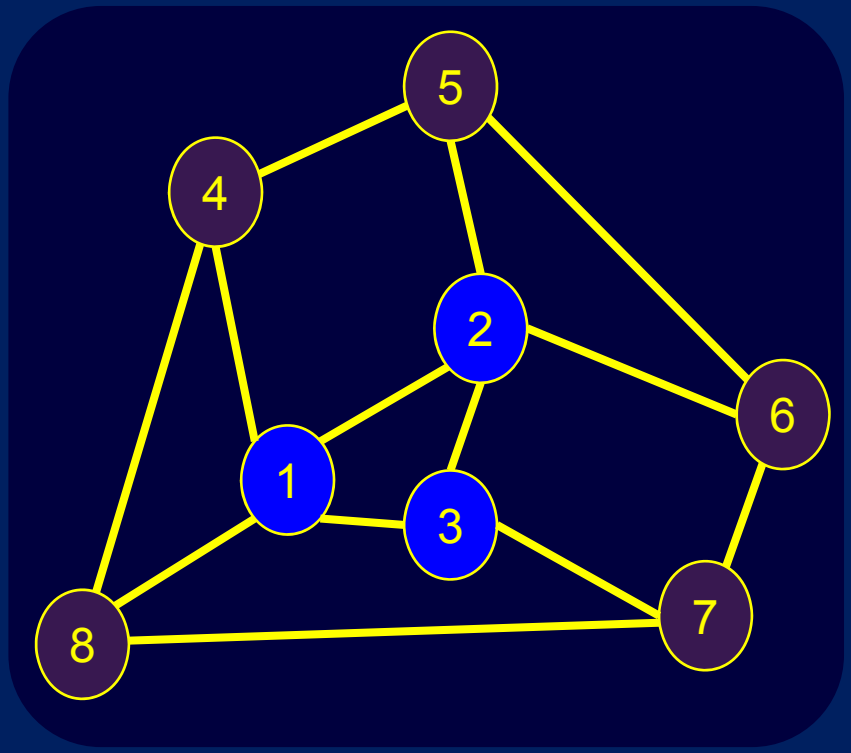
Tutte's barycenter algorithm for
triconnected planar graphs **…..**

Tutte's barycenter algorithm **for triconnected planar graphs**

1. Choose A to be *the outside face of the topological embedding*.

2. Choose the location p(a) for each a∈A to be *at the vertices of a convex polygon*.

3. For each vertex ***u*** ∈V-A, place ***u*** at the barycentre of its graph-theoretic neighbors.



Note: For planar graphs, the Laplacian matrix is sparse, and can be inverted fast.

Tutte's amazing theorem (1960)

If the input graph is planar and triconnected, then

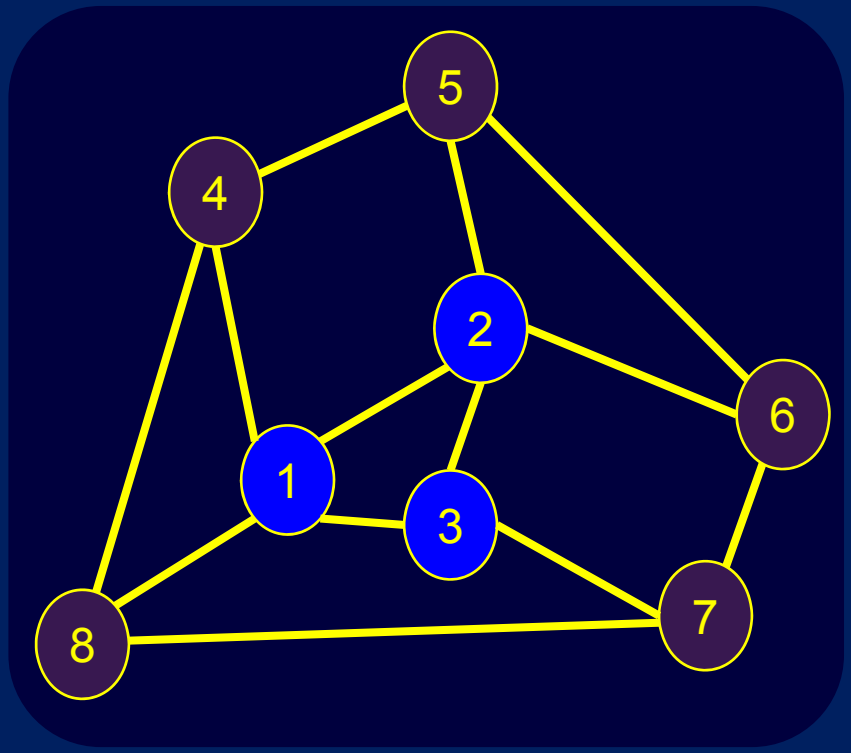
- the drawing output by the barycentre algorithm is planar, and
- every face is convex.

The *energy* view of Tutte's barycentre algorithm

Tutte's barycenter algorithm:

The energy view

1. Choose a set A of vertices.
2. Choose a location p(a) for each a∈A
3. *Place all the other vertices to minimize energy.*



What is the *energy* of a drawing p?

- The Euclidean distance between u and v in the drawing p is:
$$d(u,v) = \sqrt{(x_u - x_v) + (y_u - y_v)^2}$$
- The *energy in the edge (u,v)* is $d(u,v)^2 = (x_u - x_v)^2 + (y_u - y_v)^2$
- The *energy in the drawing p* is the sum of the energy in its edges, ie,

$$energy(p) = \sum_{(u,v)\in E} d(u,v)^2 = \sum_{(u,v)\in E} (x_u - x_v)^2 + (y_u - y_v)^2$$

Tutte's barycenter algorithm:

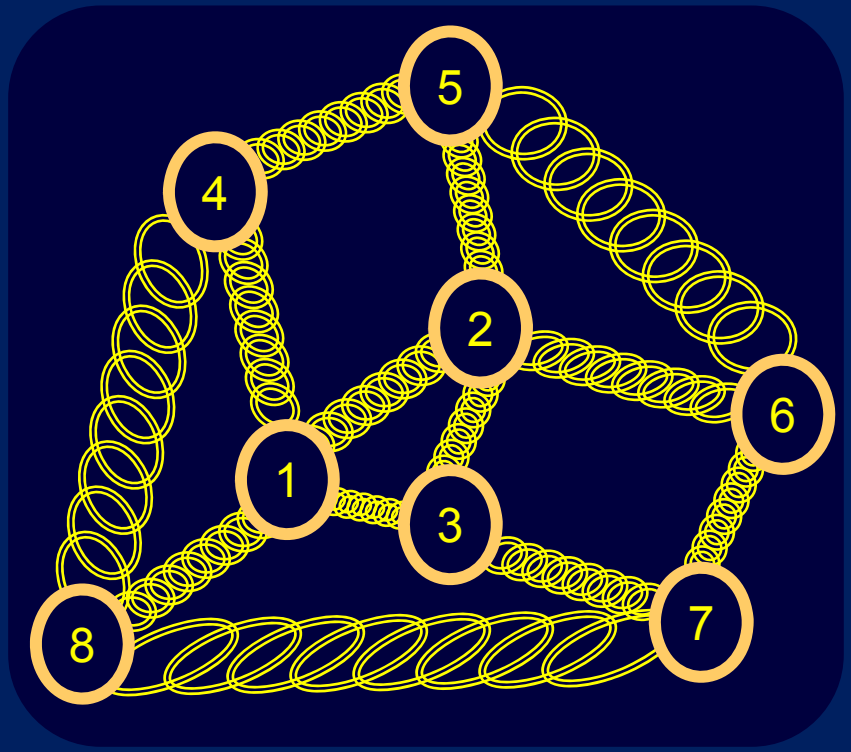
The energy view

We represent each vertex by a steel ring, and represent each edge by a spring of natural length zero connecting the rings at its endpoints.

1. Choose a set A of vertices.

2. For each a∈A, nail the ring representing a to the floor at some position.

3. The vertices in V-A will move around a bit; when the movement stops, take a photo of the layout; this is the drawing.



How to minimize energy:

- Choose a location $p(u) = (x_u, y_u)$ for each $u \in V - A$ to minimize

$$energy(p) = \sum_{(u,v)\in E} d(u,v)^2 \quad = \sum_{(u,v)\in E} (x_u - x_v)^2 + (y_u - y_v)^2$$

- Note that the minimum is unique, and occurs when

$$\frac{\partial\,(energy(p))}{\partial x_u} = 0 \quad \text{and} \quad \frac{\partial\,(energy(p))}{\partial y_u} = 0$$

for each $u \in V - A$.

For x_u, the minimum occurs when:

$$\frac{\partial\,(energy(p))}{\partial x_u} = 0$$

$$\frac{\partial}{\partial x_u}\left(\sum_{(u,v)\in E} (x_u - x_v)^2 + (y_u - y_v)^2\right) = 0$$

$$\sum_{v\in V\ s.t.(u,v)\in E} 2(x_u - x_v) = 0$$

$$x_u = \frac{1}{\deg(u)} \sum_{v\in V\ s.t.(u,v)\in E} x_v$$

Barycentre equations

How good is Tutte's barycentre algorithm?

Efficiency:

- In theory it is not bad: $O(n^{1.5})$ for planar graphs
- In practice it is fast, using numerical methods

Elegance:

- Very simple algorithm
- Easy to implement
- Numerical software available for the hard parts

Effectiveness:

- Planar graphs drawn planar
- Convex faces
- Straight-line edges

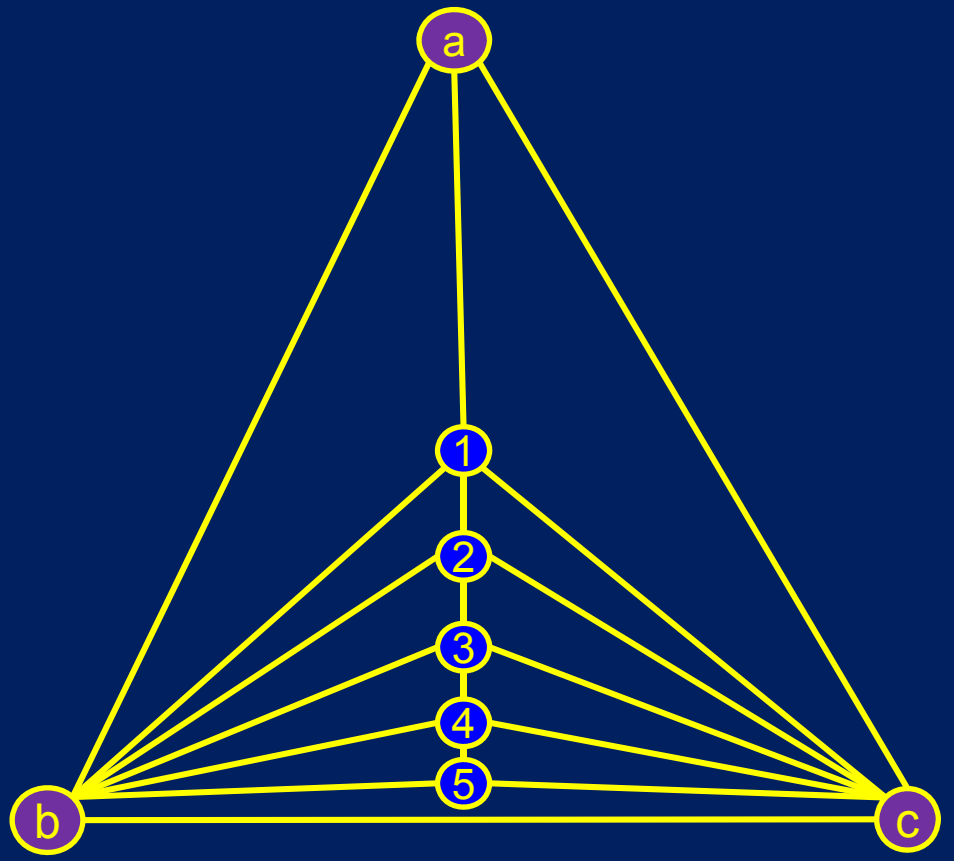
But unfortunately →

But unfortunately:

- Tutte's algorithm gives *poor vertex resolution* in many cases

Example:

- Vertex a is at (0.5, 0)
- b is at (0,0)
- c is at (1,0).
- Suppose that vertex j is at (x_j, y_j) for $0<j<n$.



From the barycentre equations:

$$\begin{cases} x_j = 0.5 \\ y_j = \frac{1}{4}(y_{j-1} + y_{j+1}) \end{cases}$$

One can deduce that

$$|y_{j-1} - y_j| < \frac{1}{2^j}$$

That is, some vertices are exponentially close together.

Aside:

- Commercial graph drawing software needs good resolution.

How good is Tutte's barycentre algorithm?

Efficiency:

- OK

Elegance:

- Excellent

Effectiveness:

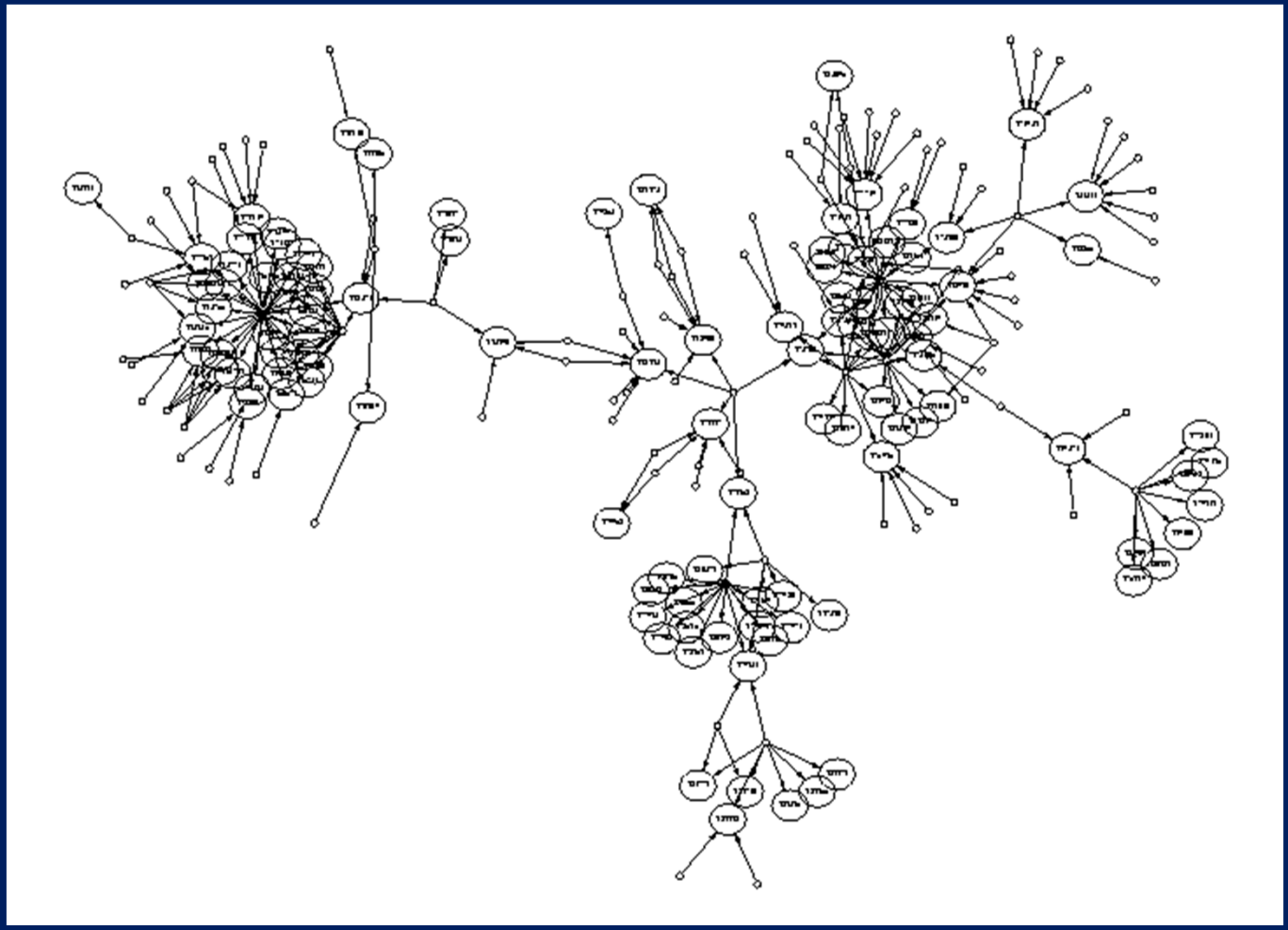
- So-so

2. How to draw a planar graph?
   a) Before Tutte: 1920s – 1950s
   b) Tutte: 1960s
   c) After Tutte: 1970s – 1990s

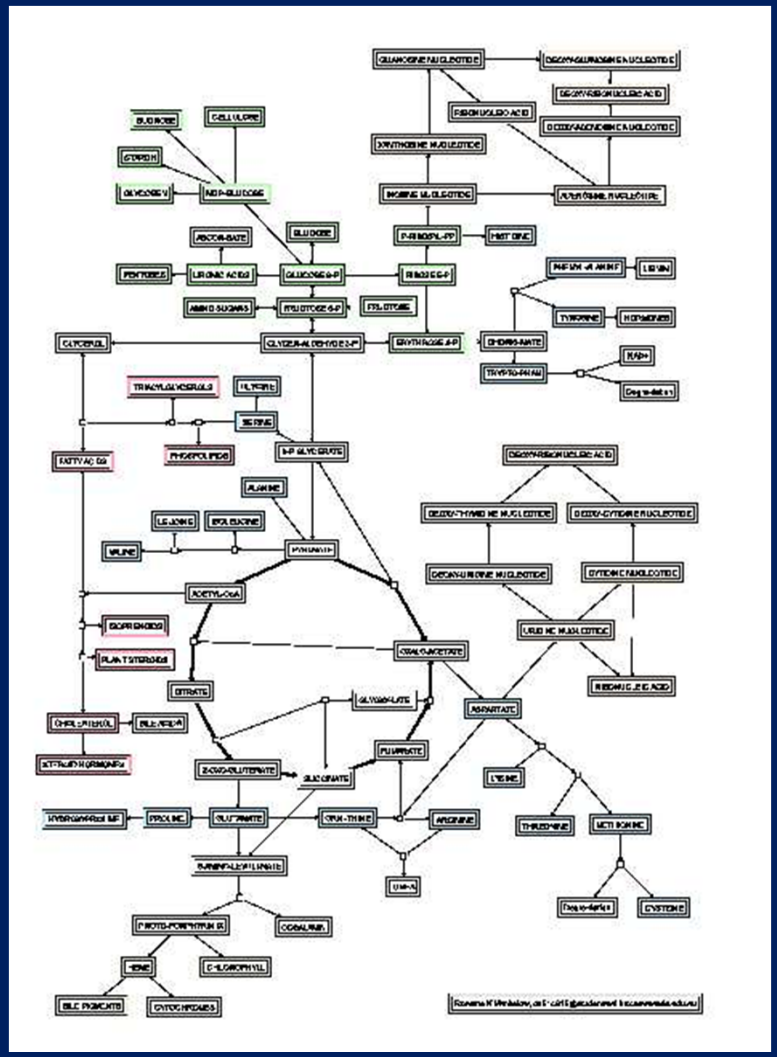
After Tutte: 1970s – 1990s

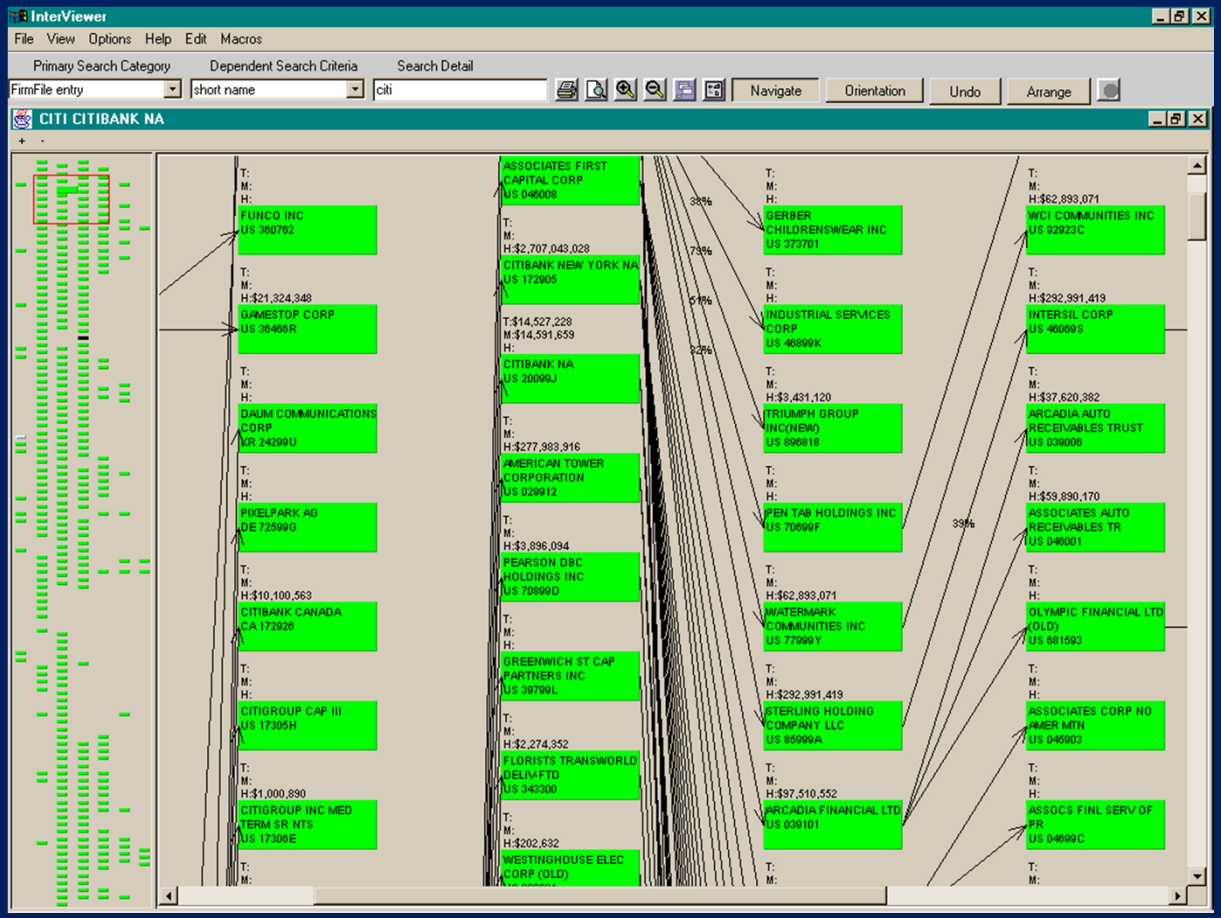
Sometime in the 1980s, the motivation for graph drawing changed from *Mathematical curiosity* to *visual data mining*.

Software



Biology





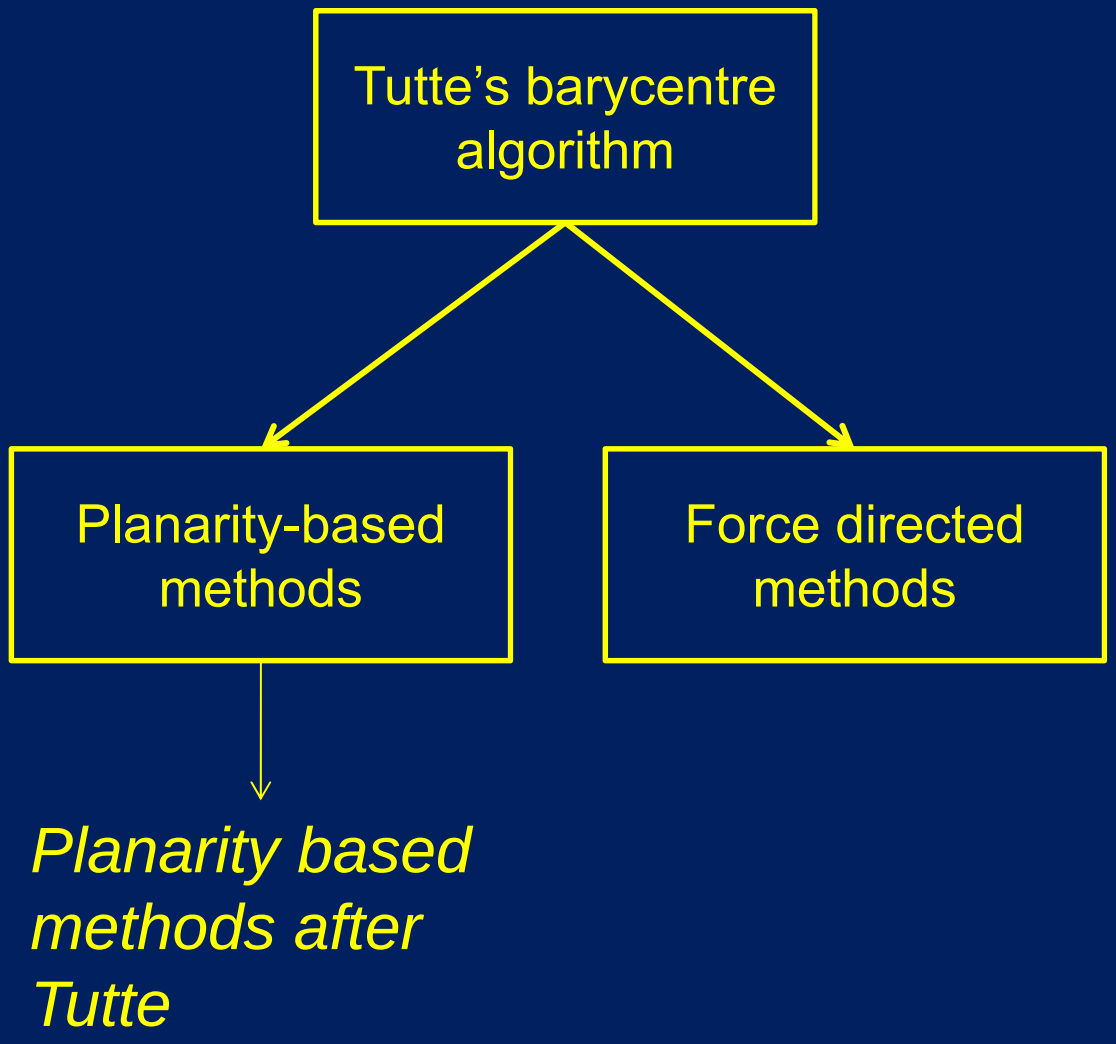
Risk Exposure

From the 1980s, industrial demand for graph drawing algorithms has grown

- Software engineering: CASE systems, reverse engineering
- Biology: PPI networks, gene regulatory networks
- Physical networks: network management tools
- Security: risk management, money movements
- Social network analysis
- Customer relationship management: value identification

Many companies buy graph drawing algorithms, many code them.

Currently the international market for graph drawing algorithms is in the hundreds of millions of dollars per year.


Tutte's barycentre algorithm
Planarity-based methods
Force directed methods
Planarity based methods after Tutte

Planarity based methods after Tutte

R.C. Read (1979, 1980)

1. Efficient?
   - Yes, linear time algorithm

2. Elegant?
   - Yes: follows proof of Fáry's theorem, with some tricks to make it into an efficient algorithm

3. Effective?
   - Maybe ...
     - Straight-line planar drawings of planar graphs
     - But unfortunately, output has poor vertex resolution

Planarity based methods after Tutte

Chiba-Nishizeki-Yamanouchi (1984)

1. Efficienct?
   - Yes, linear time algorithm

2. Elegant?
   - Yes, a simple divide&conquer approach

3. Effective?
   - Maybe ...
     - Straight-line planar drawings of planar graphs
     - Convex faces for well connected input

     - But unfortunately, output has poor vertex resolution

Planarity based methods after Tutte

Breakthrough in 1989:

de Fraysseix-Pach-Pollack Theorem (1989)
Every planar graph has a planar straight-line drawing with vertices located on a *2n X 4n* integer grid.

Notes:

- Good resolution: the minimum distance between vertices is at least *screensize/4n*.
- Linear time (Chrobak, 1990)

The deFraysseix-Pach-Pollack Theorem gave much hope for planarity-based methods, and many refinements appeared 1990 – 2000.

de Fraysseix-Pach-Pollack-Chrobak Algorithm

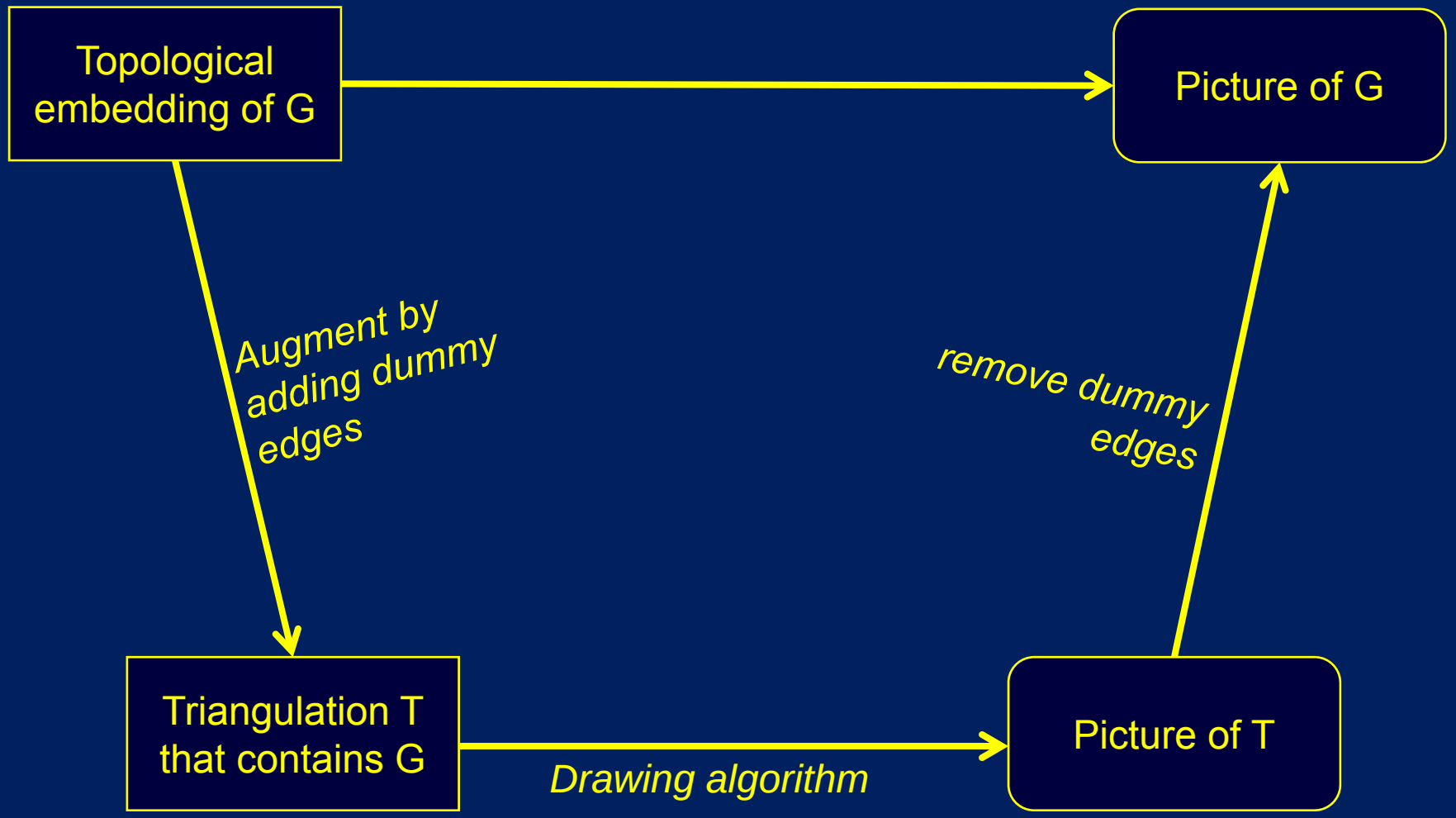
1. Add dummy edges to make the graph into a triangulation

2. Construct an ordering $u_1, u_2, \ldots, u_n$ of the vertices, called the *canonical ordering*.

3. Draw the graph, adding one vertex at a time, in order $u_1, u_2, \ldots, u_n$

Wikipedia proof of Fáry's Theorem

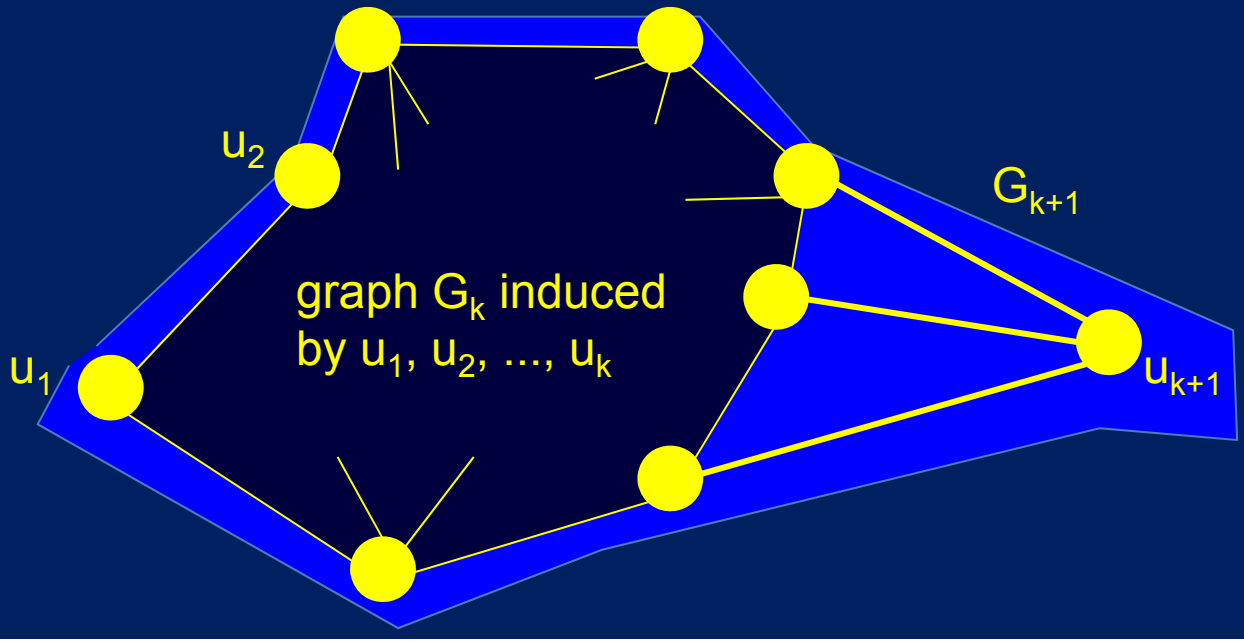
Step 1: Add dummy edges to make the graph into a triangulation



Step 2: Construct an ordering $u_1, u_2, \ldots, u_n$ of the vertices , called the "canonical ordering".

A *canonical ordering* is an ordering $u_1, u_2, ..., u_n$ of the vertices of a triangulation having the property that, for each $3 \leq k < n$, the graph G_k induced by $u_1, u_2, ..., u_k$ has the following properties

- G_k is biconnected
- G_k contains the edge (u_1, u_2) on its outer face,
- Any vertices in G_k adjacent to u_{k+1} are on the outer face of G_k
- The vertices in G_k adjacent to u_{k+1} form a path along the outer face of G_k

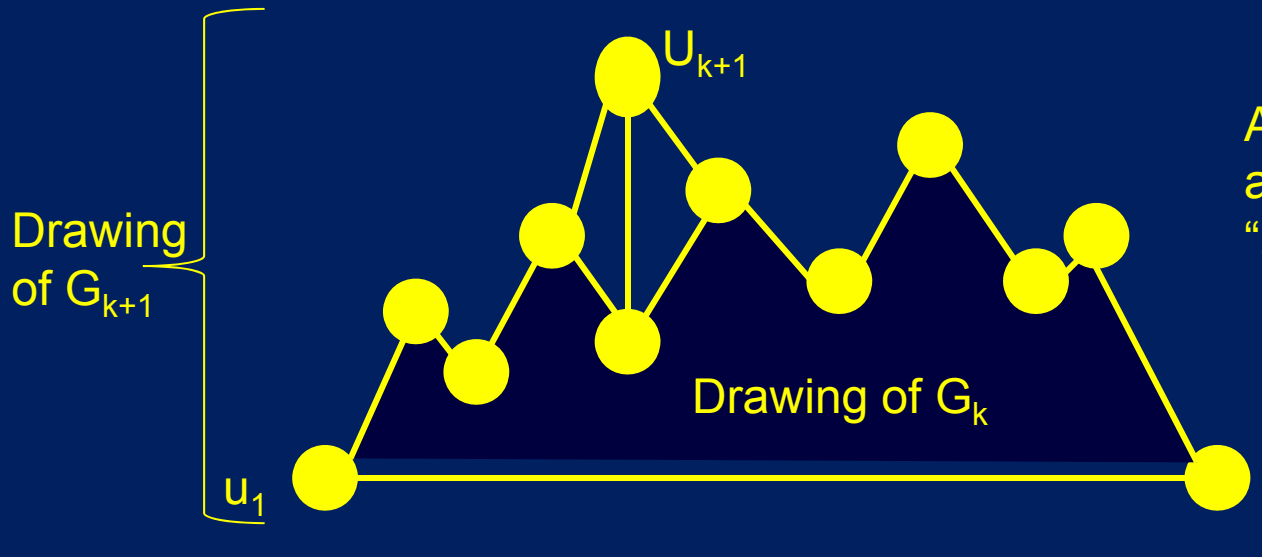


Step 3: Draw the graph, adding one vertex at a time in order $u_1, u_2, \ldots, u_n$

a) Start with the edge (u_1, u_2) at y=0

b) For each k>1:

- add u_{k+1} on y=k
- Choose x coordinate of u_{k+1} so that there are no edge crossings.



At each stage, there is a drawing of G_k as a *"terrain"*.

Some details of deFraysseix-Pach-Pollack-Chrobak algorithm are needed to show

- It runs in linear time
- It is possible to avoid edge crossings
- Each vertex lies on an integer grid of size at most 4nX2n

Also:

For restricted classes of planar graphs, there are algorithms with better resolution, for example:

- Outerplanar graphs: algorithms with output with area $\min(k_1 n^{1.48}, k_2 dn \log n)$ (Frati 2009)

And some lower bounds:

- Planar graphs: area $\Omega(n^2)$ is necessary (Leighton(?), about 1970)
- Series-parallel graphs:$\Omega\left(n2^{\sqrt{\log n}}\right)$ is necessary (Frati, 2010)

The deFraysseix-Pach-Pollack-Chrobak algorithm

Efficiency:

- Yes, linear time

Elegance:

- Not bad; can be coded by a student in a week or so.

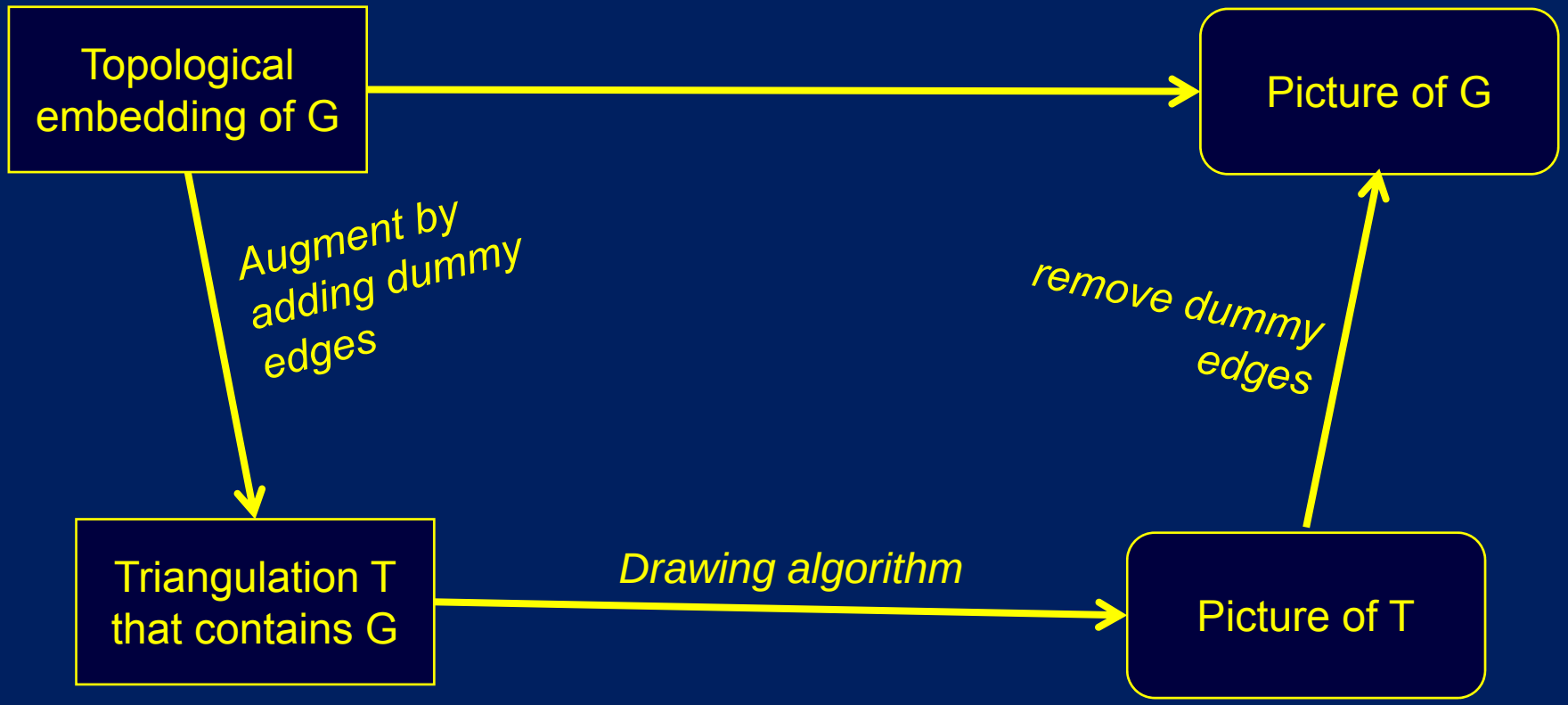
Effectiveness:

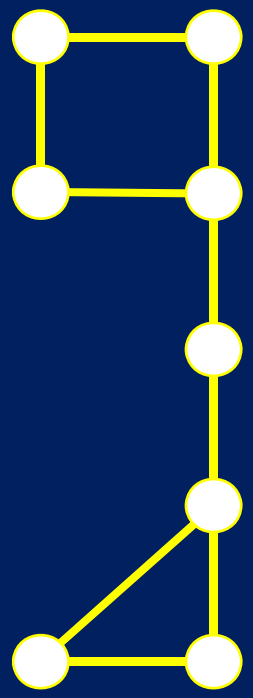
- Maybe pretty good:
  - Straight-line edges
  - No edge crossings
  - Good vertex resolution

  - But unfortunately -->

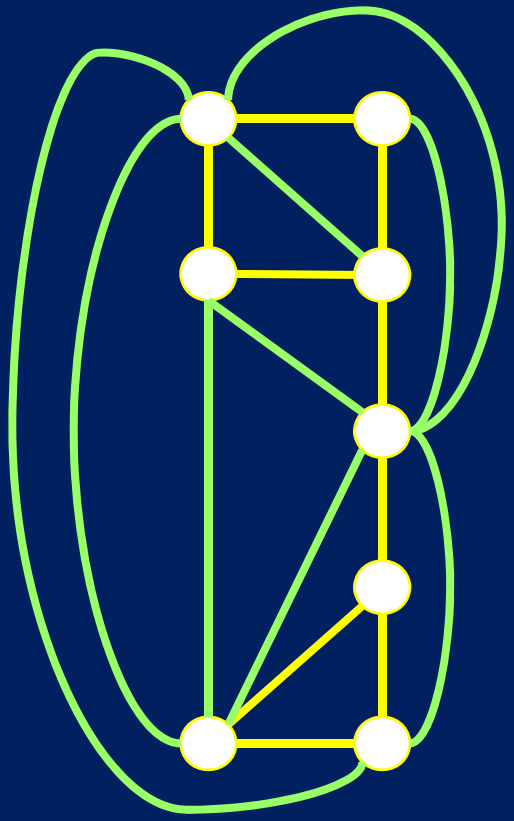
The deFraysseix-Pach-Pollack-Chrobak algorithm gave much hope for planarity-based methods, and many refinements appeared 1990 – 2000.

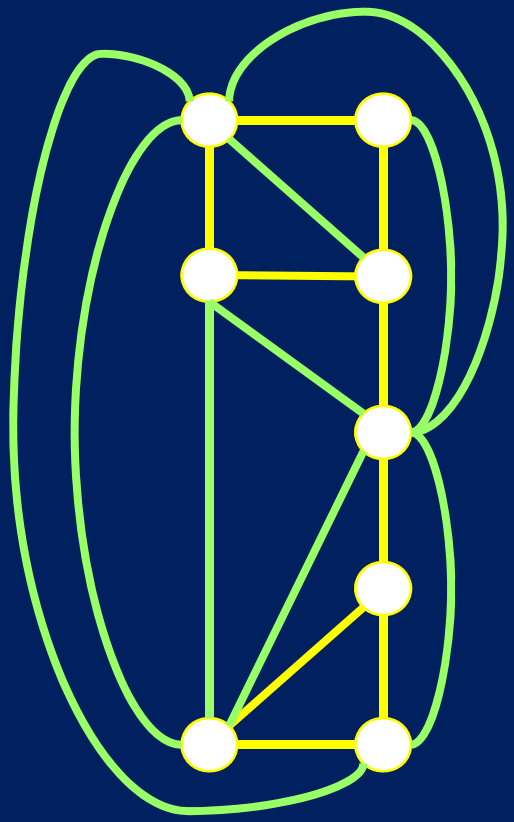
But unfortunately, we found that the first step (increasing connectivity by triangulation) gives some problems.



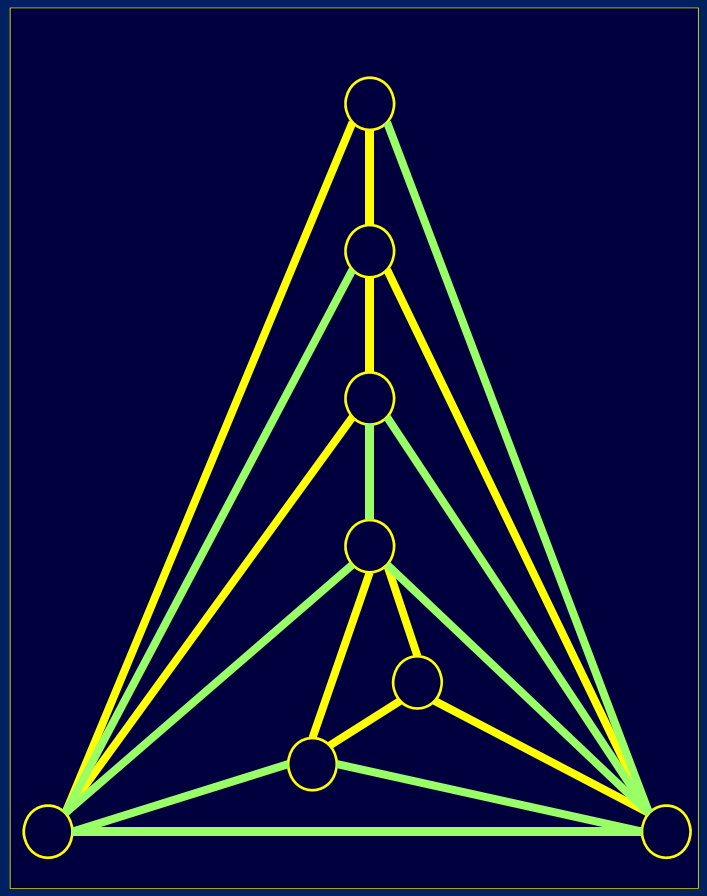


1. Add dummy edges to triangulate

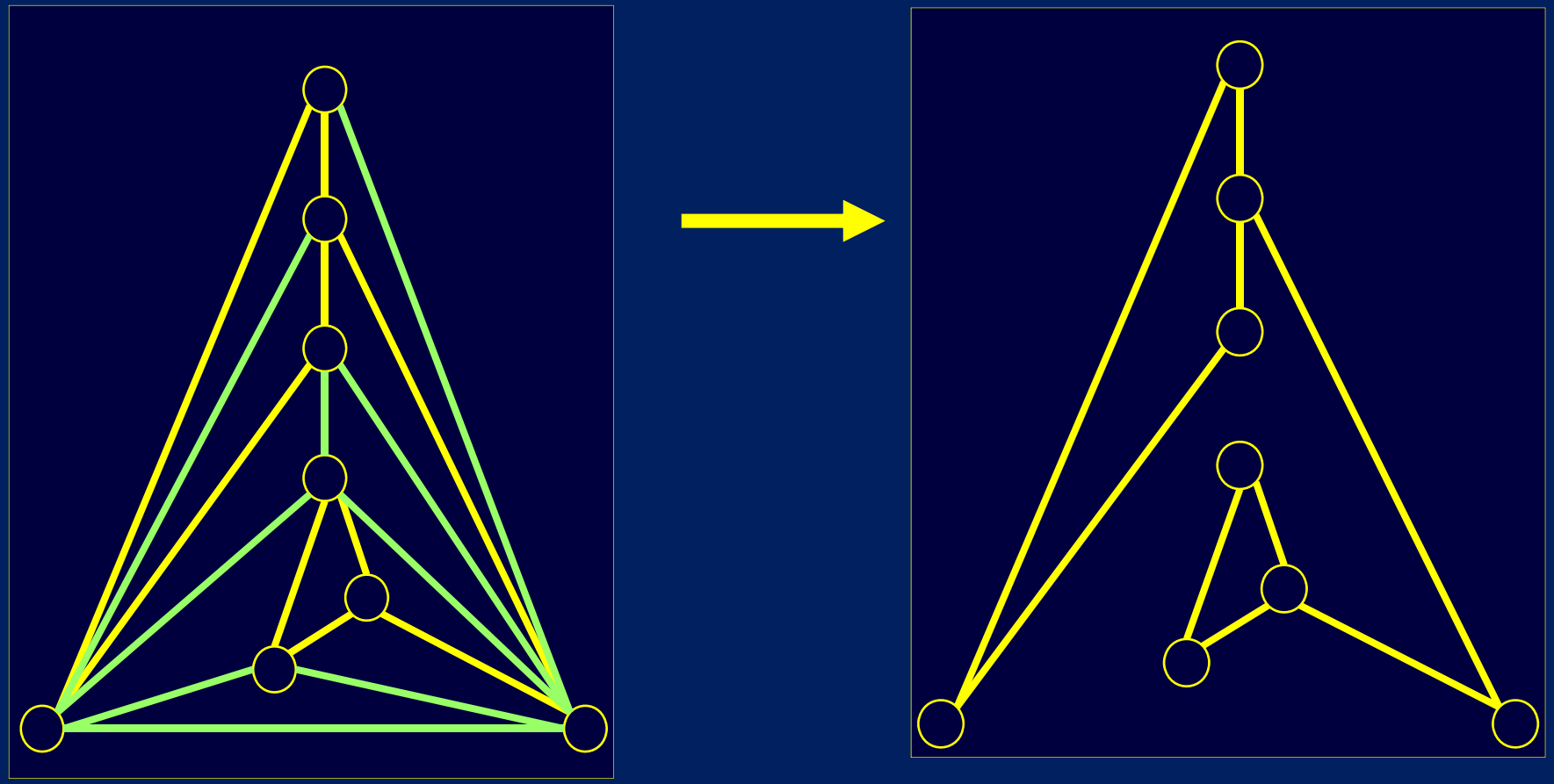




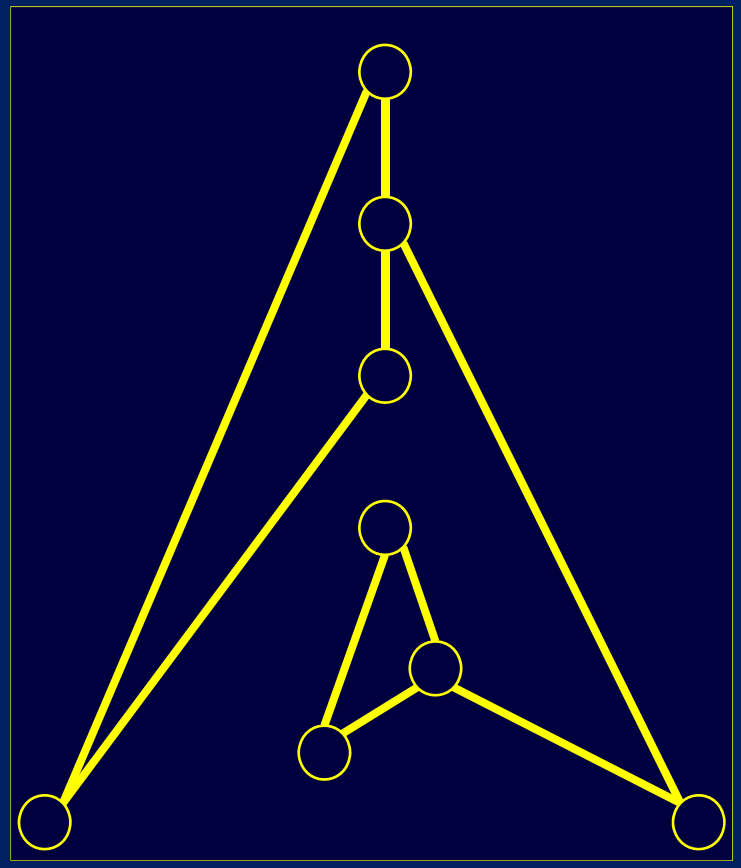
2.Draw the augmented graph.



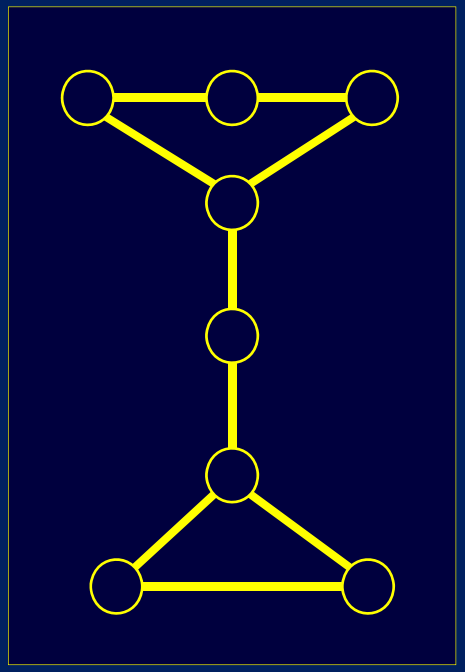
3. Delete the dummy edges



Note: the resulting drawing is ugly.



A better drawing



This kind of ugly drawing is a typical output from methods that use augmentation to increase connectivity

The deFraysseix-Pach-Pollack-Chrobak algorithm

Efficiency:

- Yes, linear time

Elegance:

- Not bad; can be coded by a student in a week or so.

Effectiveness:

- So-so
  - Straight-line edges
  - No edge crossings
  - Good vertex resolution
  - Augmentation step gives poor angular resolution and weird shapes

Aside

What if the input graph is non-planar?

Classical approach: *planarization*

Input: a graph $G = (V, E)$

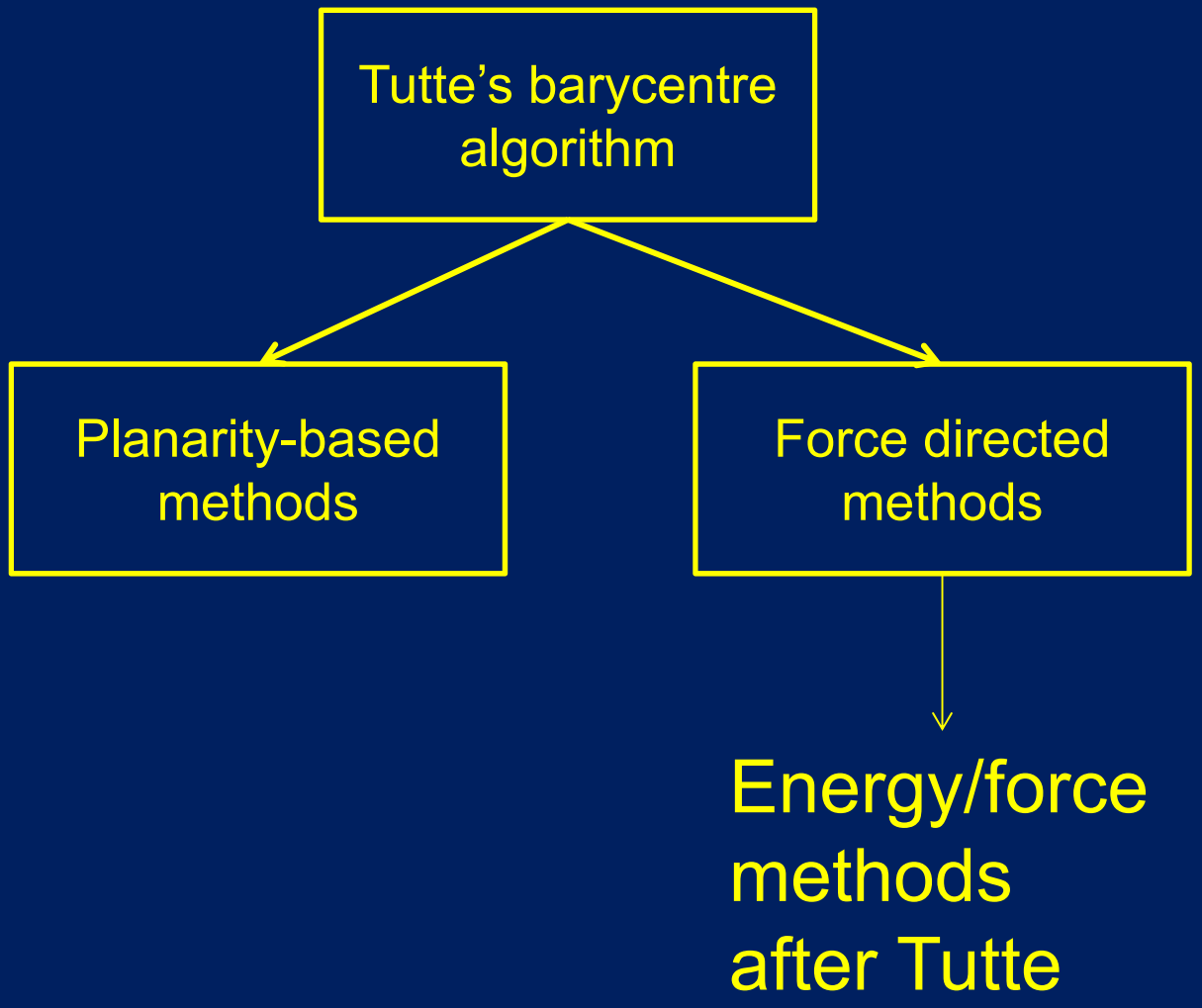
1. Find an approximately maximal planar subgraph $G' = (V, E')$ with $E' \subseteq E$.
2. Find a topological embedding of G'.
3. Route each edge $e \in E - E'$ in such a way to locally minimise crossings.
4. Put dummy vertices at each crossing point, to give a vertex set V' with $V \subseteq V'$.
5. The graph $G'' = (V', E'')$ is then planar. Use a planarity-based drawing algorithm to draw G''.

Current state-of-the-art for planarity based methods:

- There are many small improvements to the deFraysseix-Pach-Pollack-Chrobak algorithm.
- But none have overcome all the connectivity augmentation problem.

Current state-of-the-art for planarity based methods:

- *Almost no planarity based methods* have been adopted in commercial software … despite the fact that planarity is the single most important aesthetic criterion.


Tutte's barycentre algorithm
Planarity-based methods
Force directed methods
Energy/force methods after Tutte

Tutte's algorithm gives poor vertex resolution: vertices can be exponentially close to each other.

To solve this problem, we need to prevent vertices from becoming very close together.

This can be done with forces:-

1. Use springs of nonzero natural length

2. Use an inverse square law repulsive force between nonadjacent vertices.

Force exerted by a vertex v on a vertex u:

If u and v are adjacent:

$$f_{spring}(u,v) = k_{uv}(d(u,v) - q_{uv})i_{uv}$$

where

- k_{uv} is the *strength* of the spring between u and v
- $d(u,v)$ is the Euclidean distance between u and v
- q_u is the *natural length* of the u-v spring
- i_{uv} is a unit vector in the direction from u to v.

If u and v are not adjacent:

$$f_{nonadjac}(u,v) = \frac{r_{uv}}{d(u,v)^2} i_{uv}$$

where

- r_{uv} is the *strength* of the repulsive force

Total force on a vertex u:

$$F(u) = \sum_{(u,v)\in E} f_{spring}(u,v) + \sum_{(u,w)\notin E} f_{nonadjac}(u,w).$$

A (locally) minimum energy configuration satisfies

$$F(u) = 0$$

for each vertex u.

This is a system of *nonlinear* equations.

Note

1. In general, the solution to this system of equations is *not unique*, that is, there are local minima that may not be global.
2. Many methods to solve systems of equations like this are available. Some methods are fast, some are slow, depending on the specific equations.

Force-based techniques can be constrained in various ways.

The constants in the force definitions

$$f_{spring}(u,v) = k_{uv}(d(u,v) - q_{uv})i_{uv}$$

$$f_{nonadjac}(u,v) = \frac{r_{uv}}{d(u,v)^2} i_{uv}$$

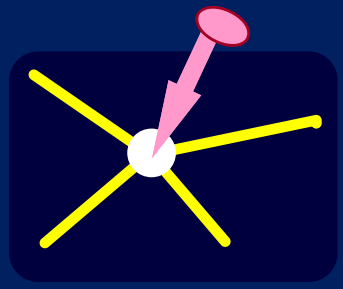
can be chosen to reflect the relationships in the domain.

For example

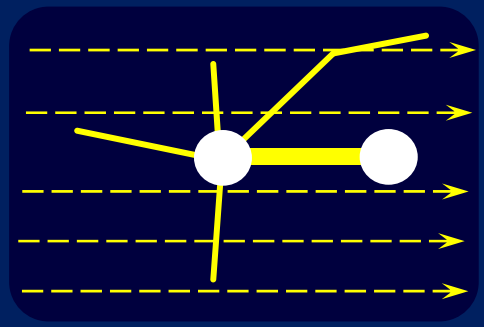
- If the edge between *u* and *v* is important in the domain, then we can choose k_{uv} to be large and q_{uv} to be small.

Force-based techniques can be constrained in various ways.

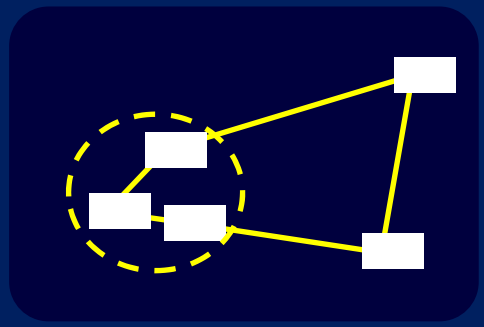
Nails can be used to hold a node in place.



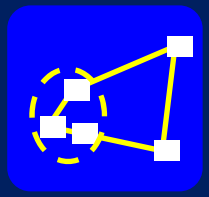
Magnetic fields and *magnetized springs* can be used to align nodes in various ways.

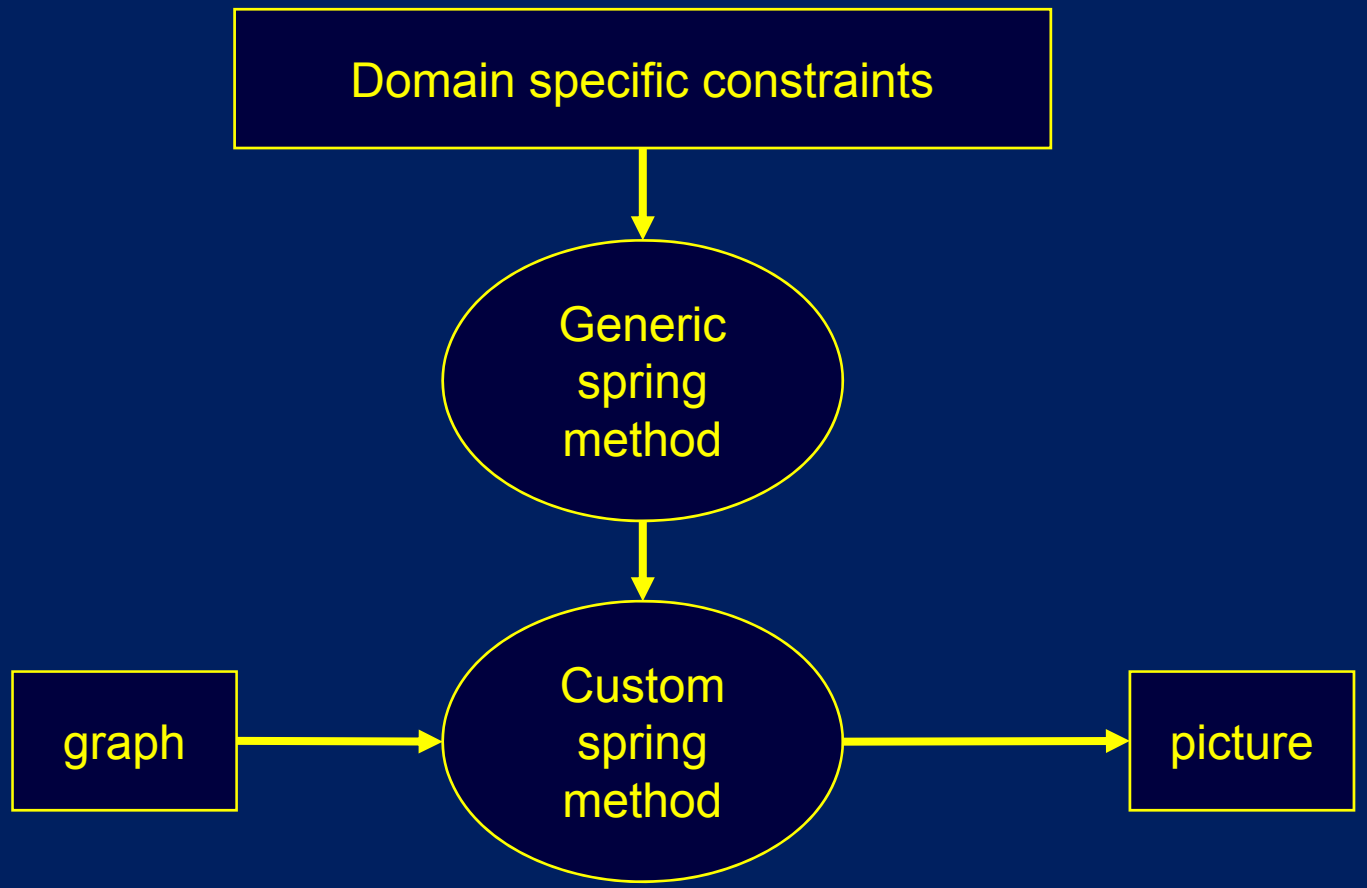


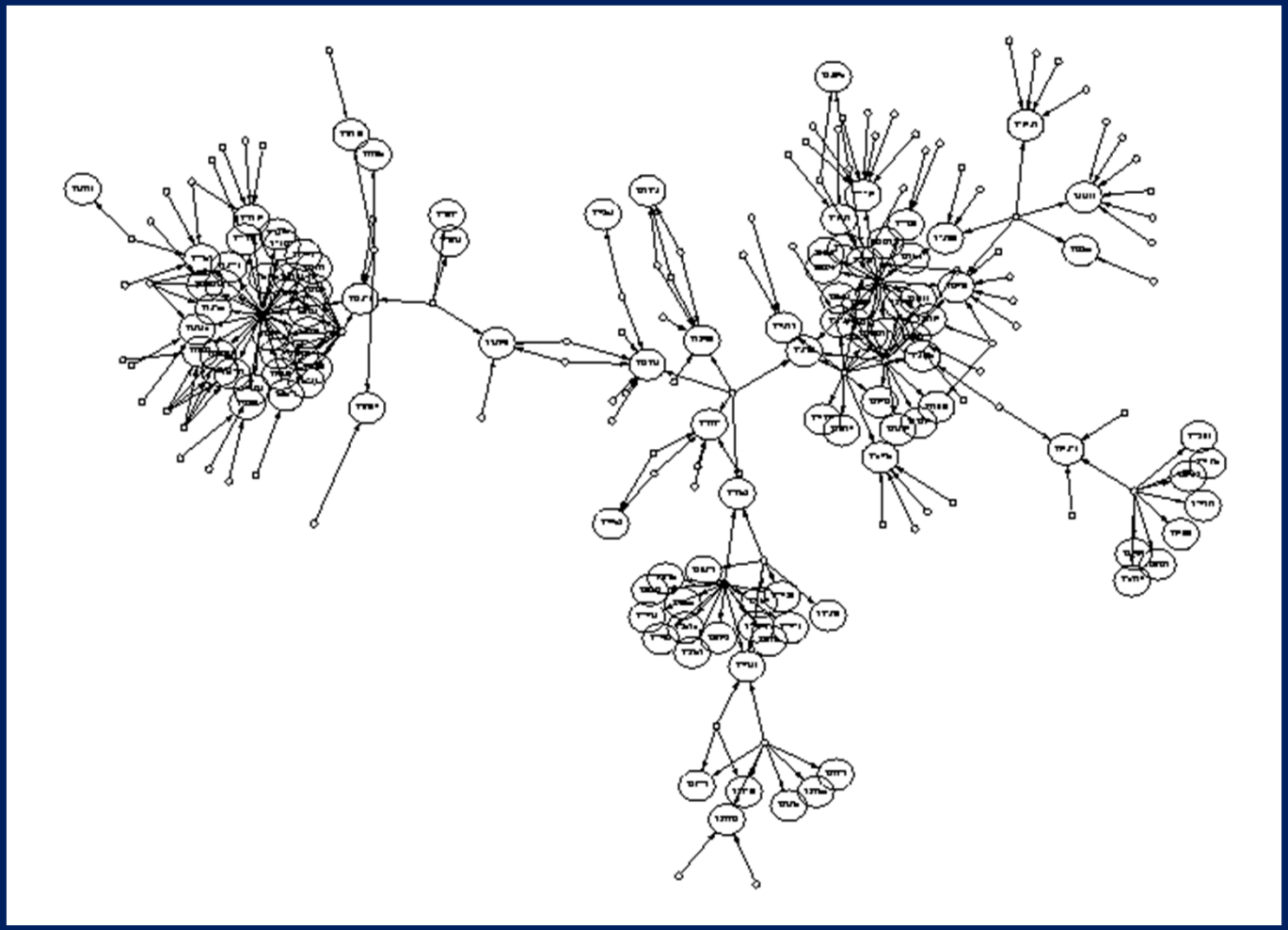
Attractive forces can be used to keep clusters together.

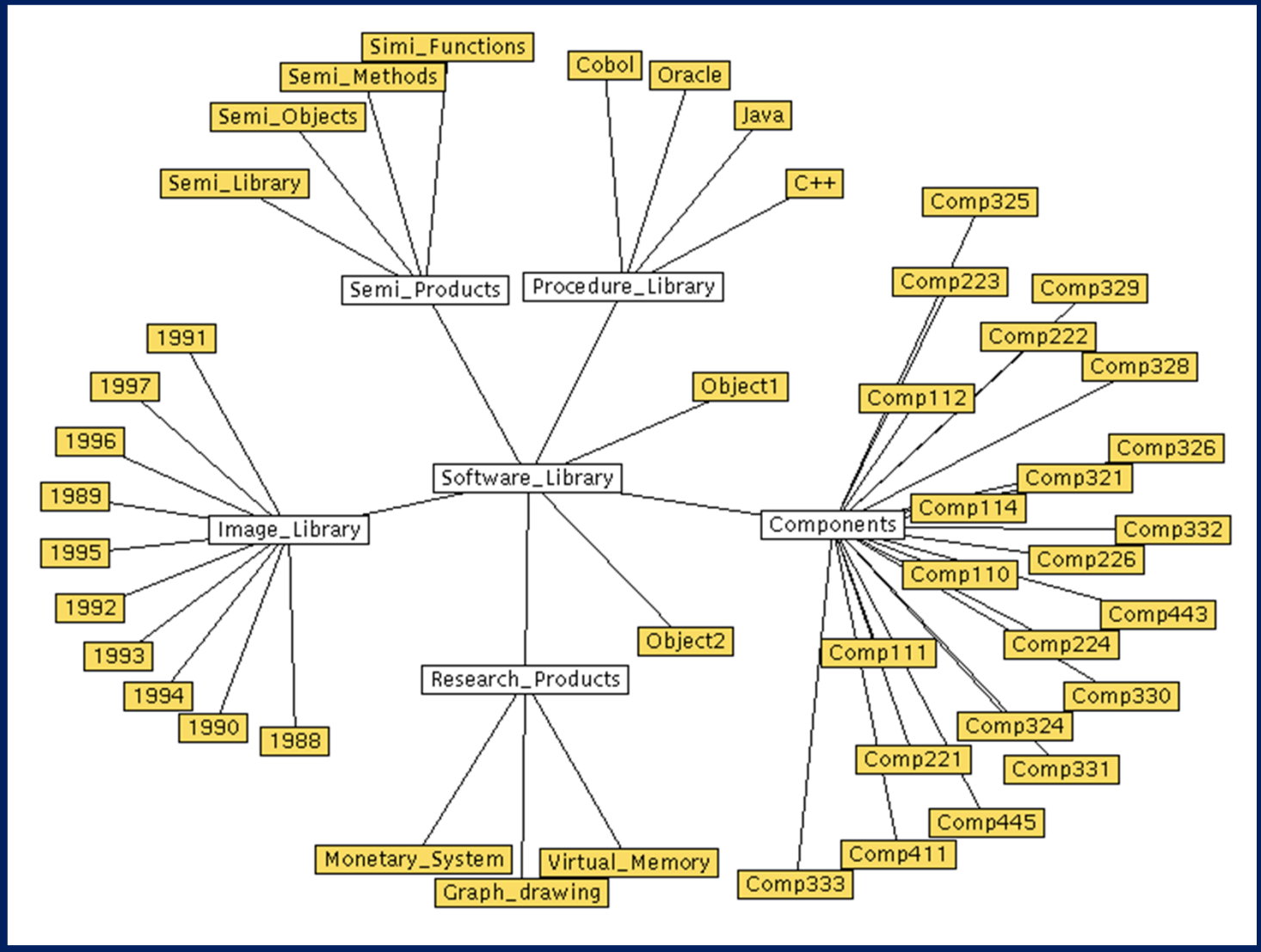


These constraints are very useful in customizing the general spring method to a specific domain.





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Comp332
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Comp221
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Comp445
Comp411
Comp333

Example:

Metro Maps

- *Damian Merrick*
- *SeokHee Hong*
- *Hugo do Nascimento*

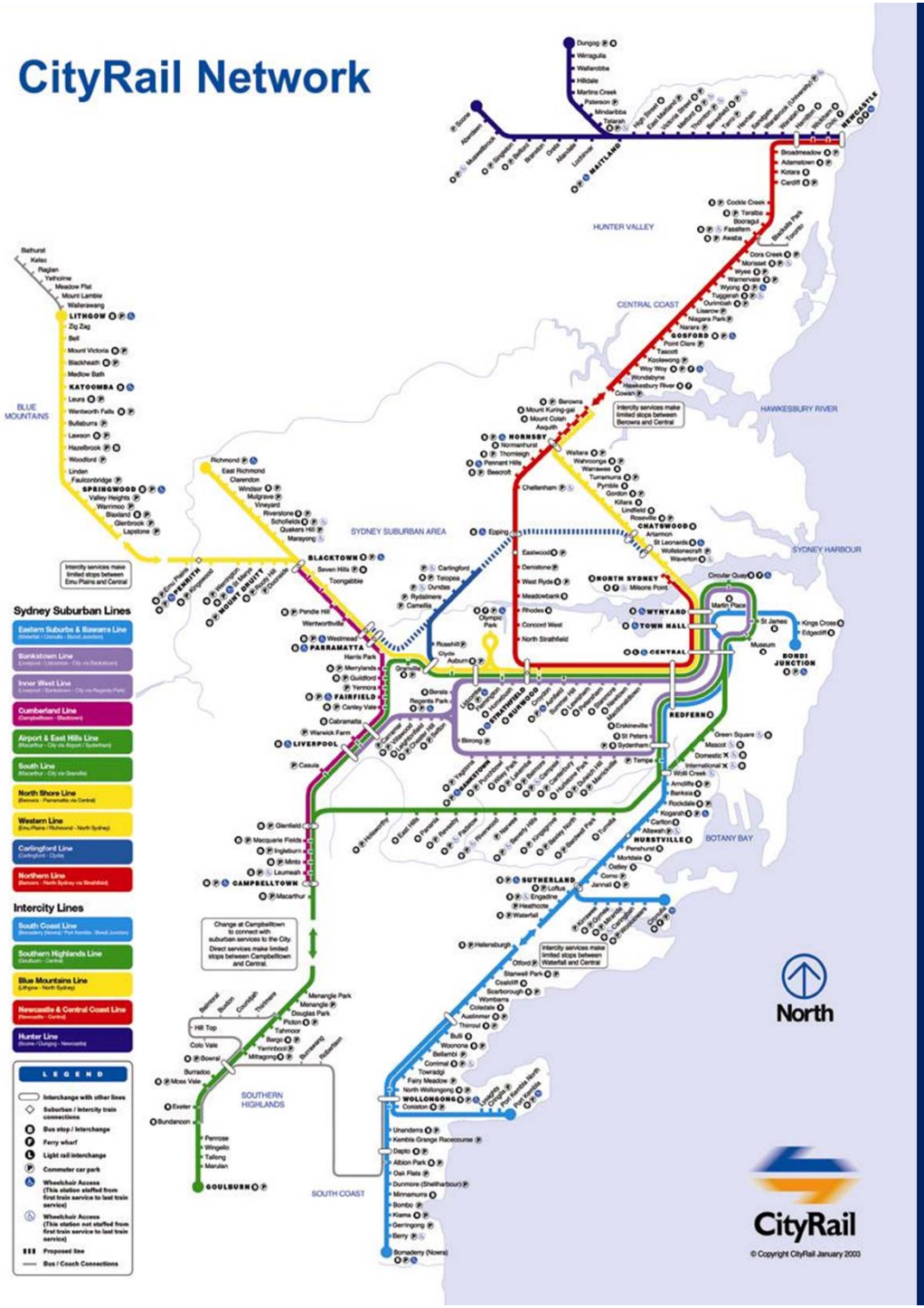
The Metro Map Problem

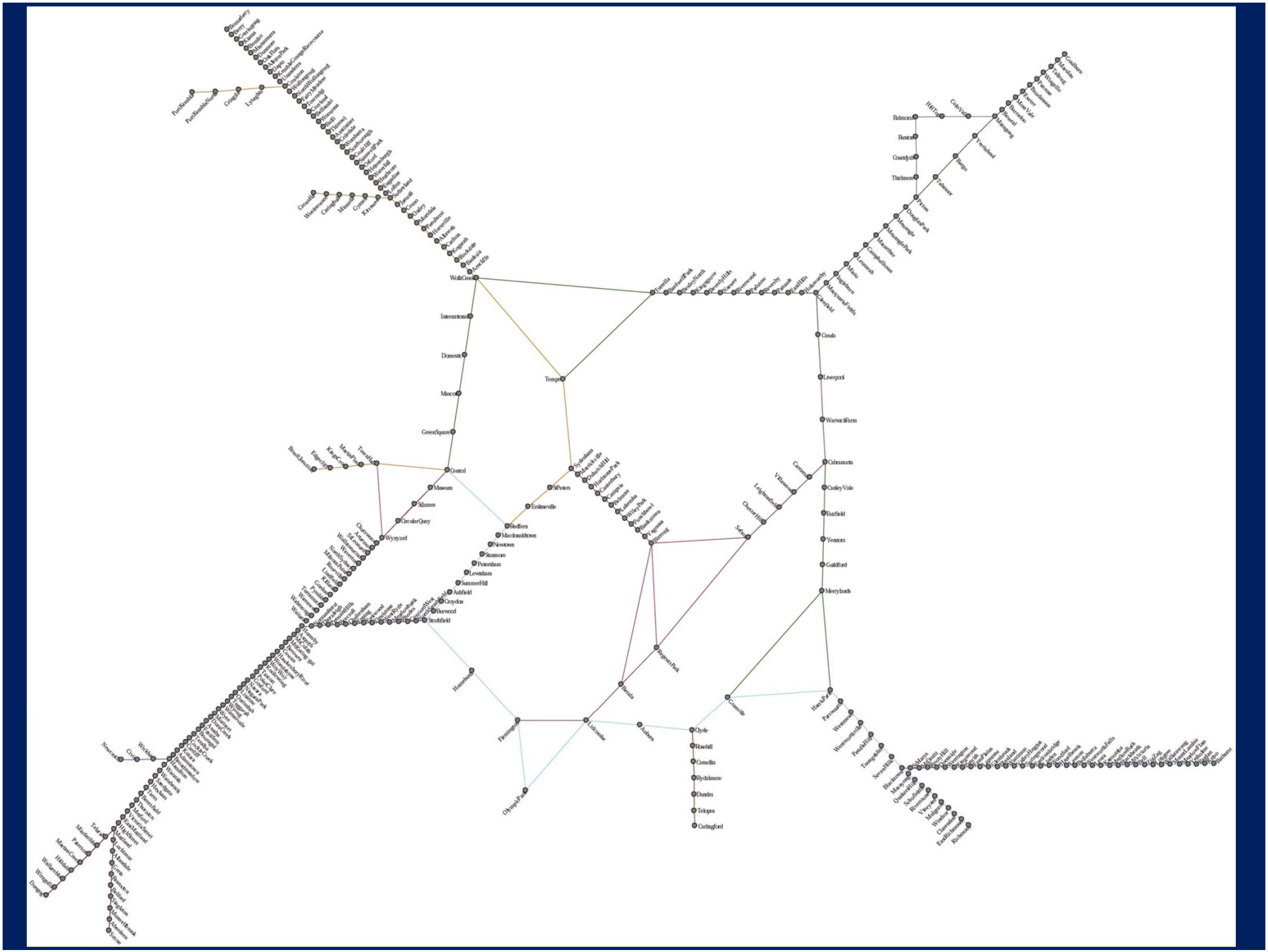
- Existing metro maps, produced by professional graphic artists, are ***excellent examples*** of network visualization

- *Challenge: Can we produce good metro maps* ***automatically?***

- ***Possible solution:***
  - Use a force-directed approach
  - Define forces that map good layout to low energy

CityRail Network





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L5 Cornellà
L1 Feixa Llarga
L3 Canyelles
L5 Horta
L11 Trinitat Nova
L4 Trinitat Nova
L11 Can Cuiàs
L1 Fondo
L4 La Pau
L2 Pep Ventura
L2 Paral·lel
Parc de Montjuïc
Esplugues de Llobregat
Cornellà de Llobregat
l'Hospitalet de Llobregat
Santa Coloma de Gramenet
Badalona
Sant Adrià de Besòs
Gavarra
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Jaume I
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Poblenou
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El Maresme Fòrum
Besòs Mar
Besòs
La Pau
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Gorg
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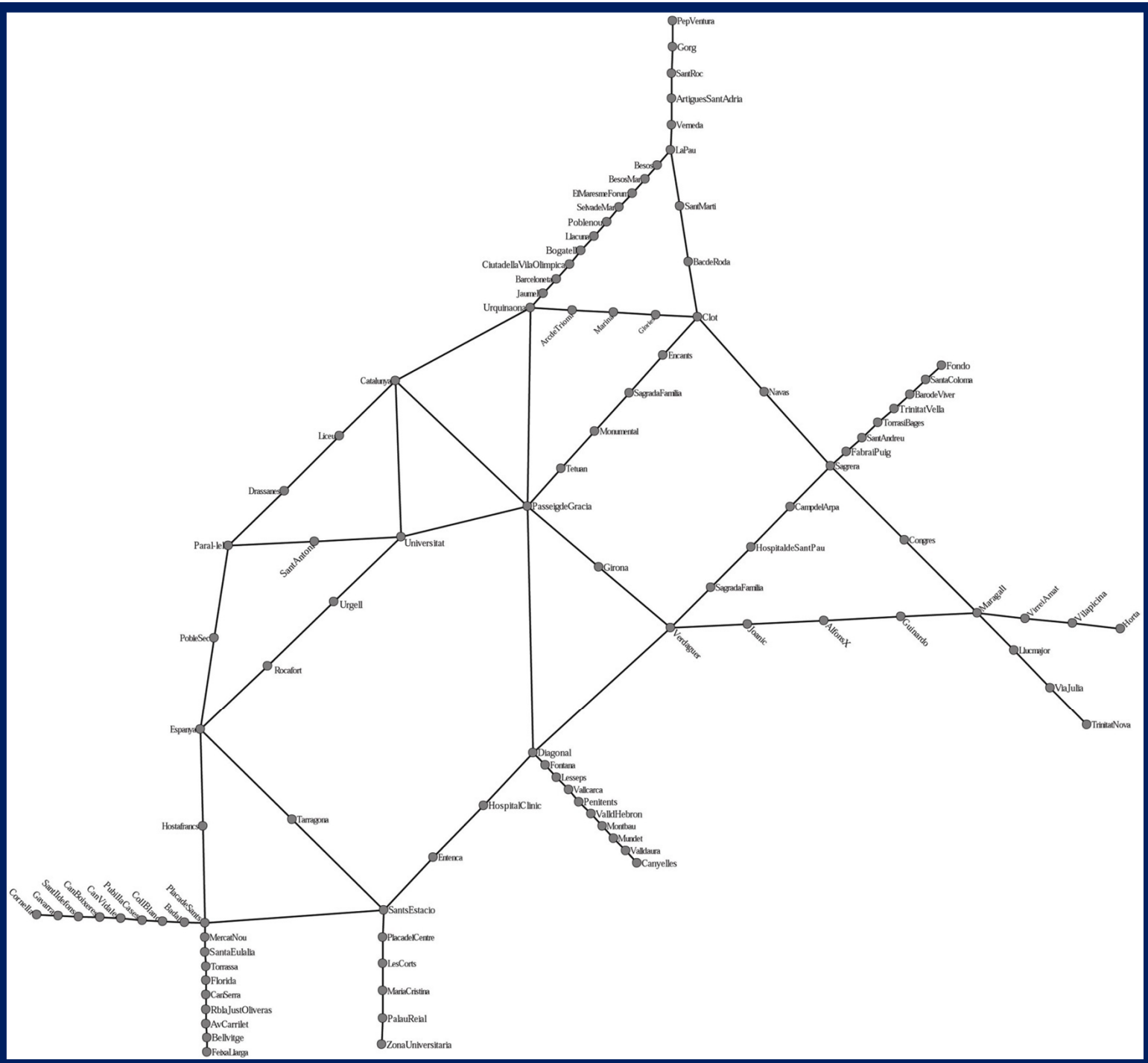
Paral·lel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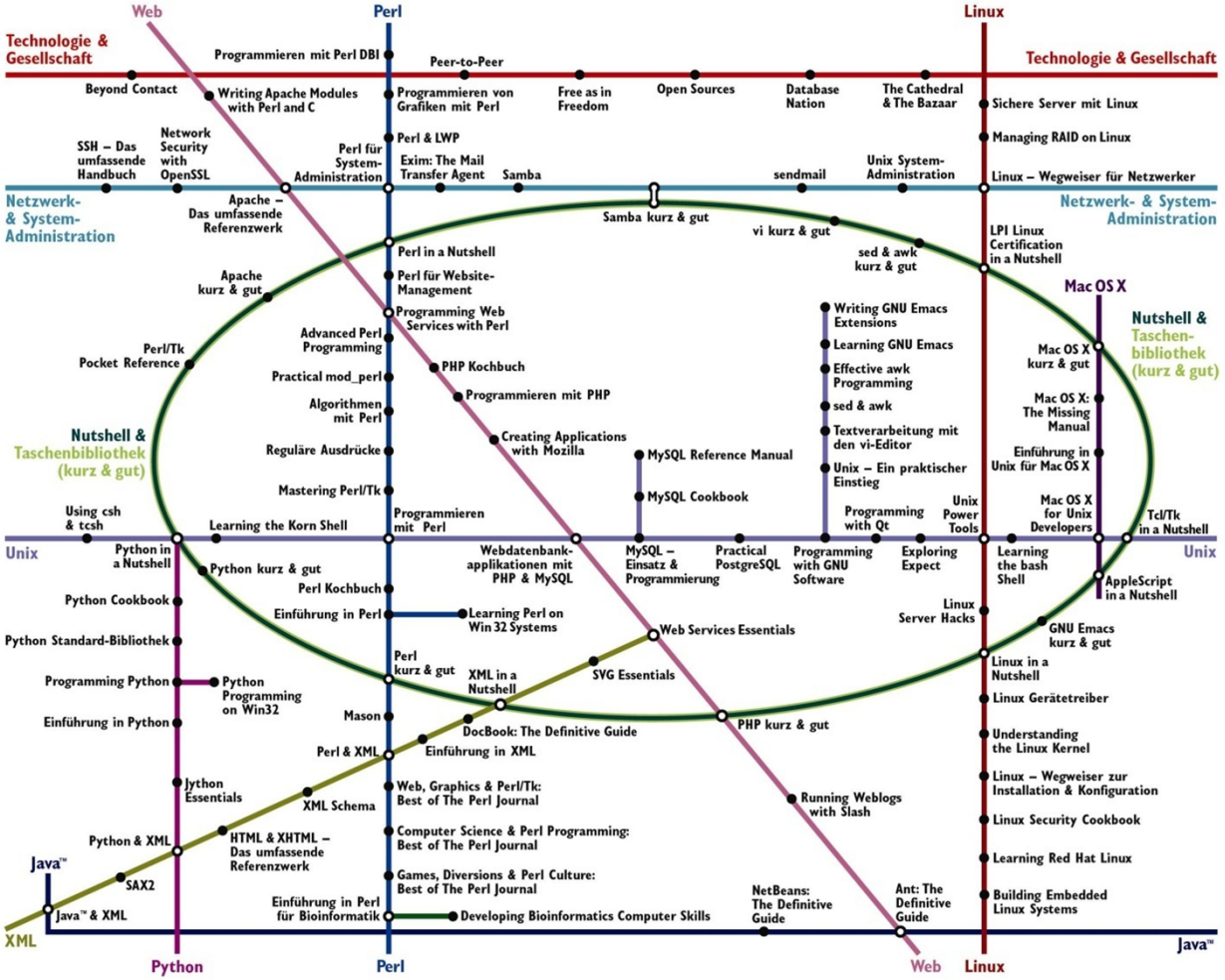

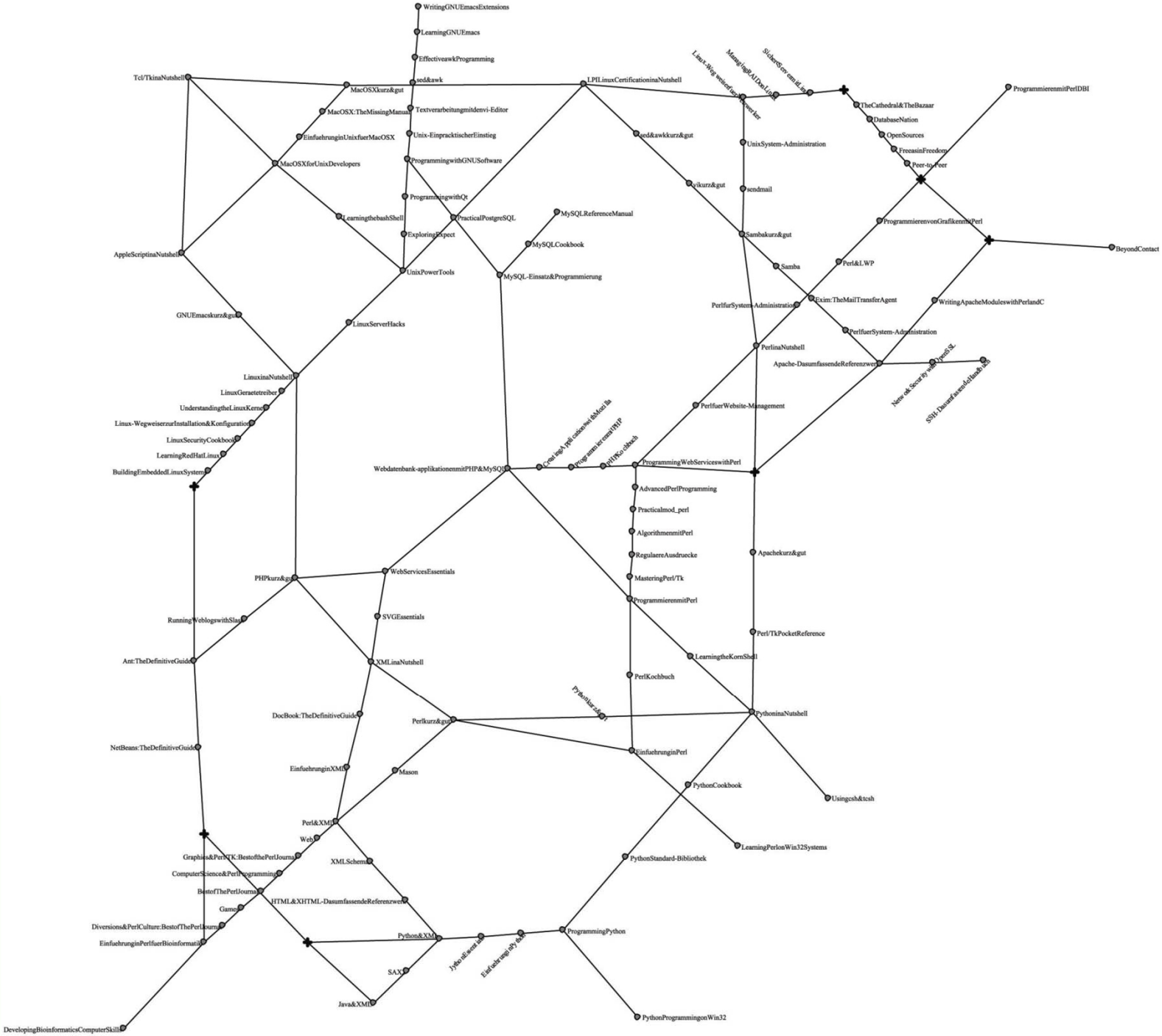
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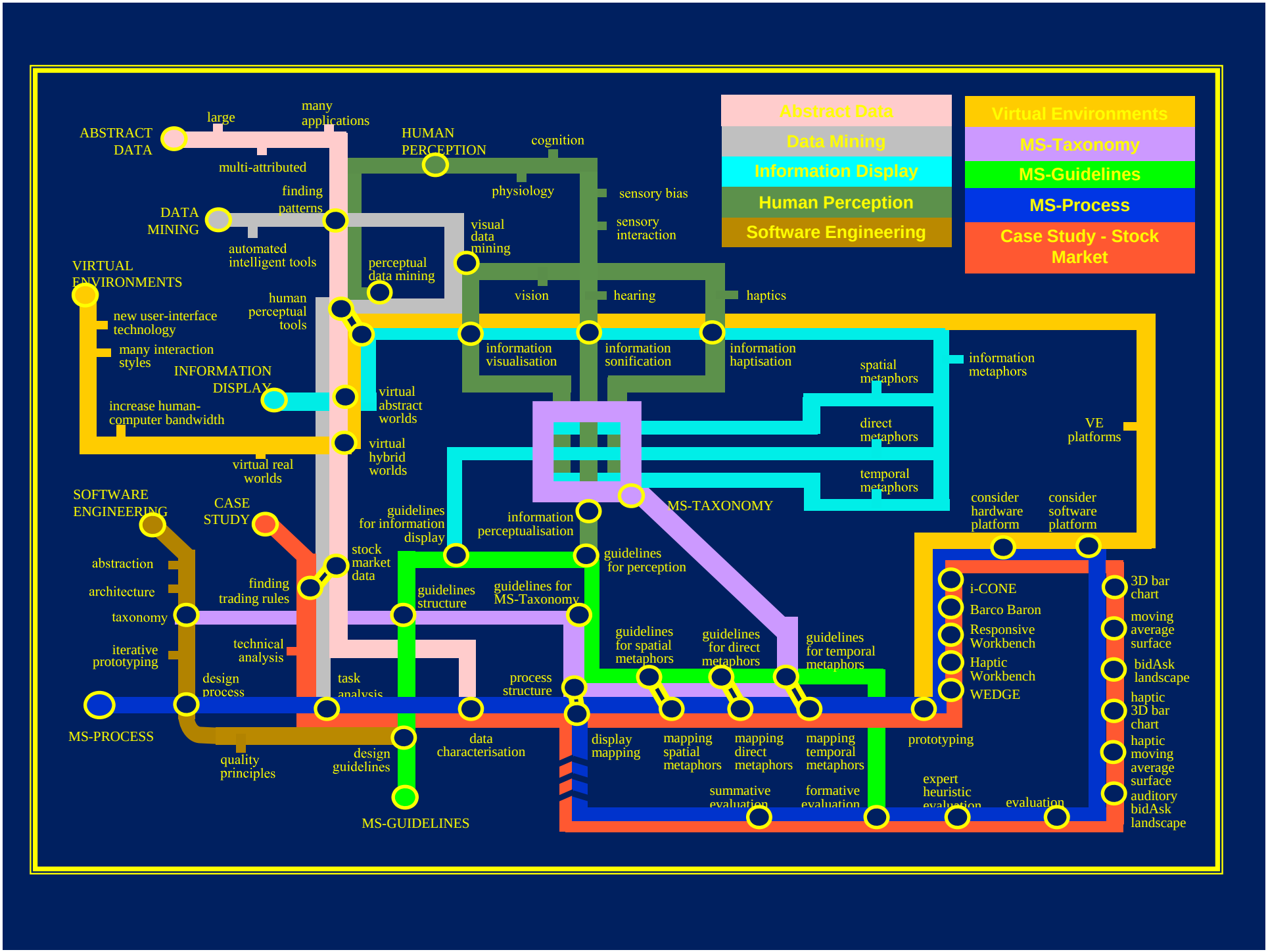
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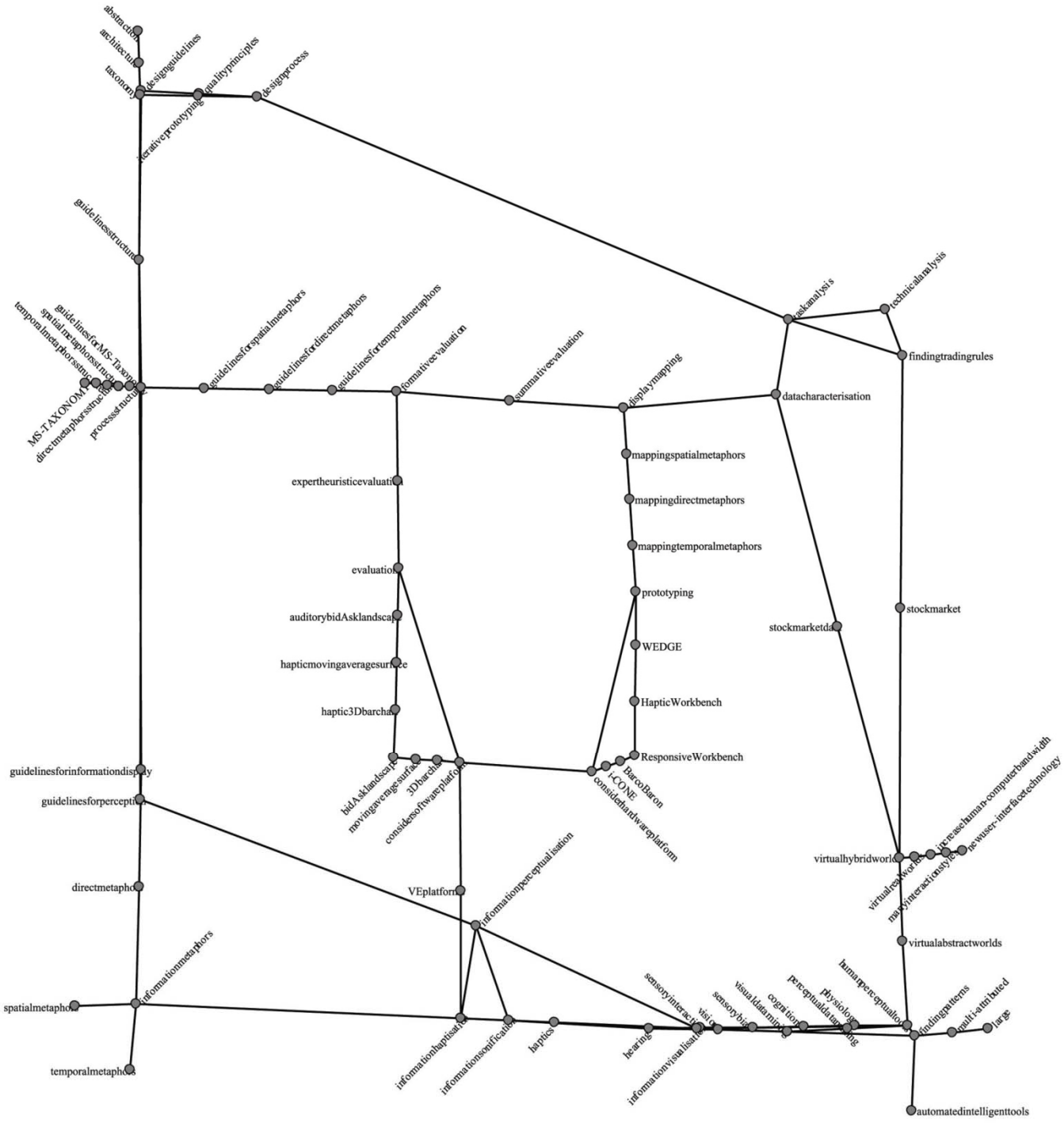
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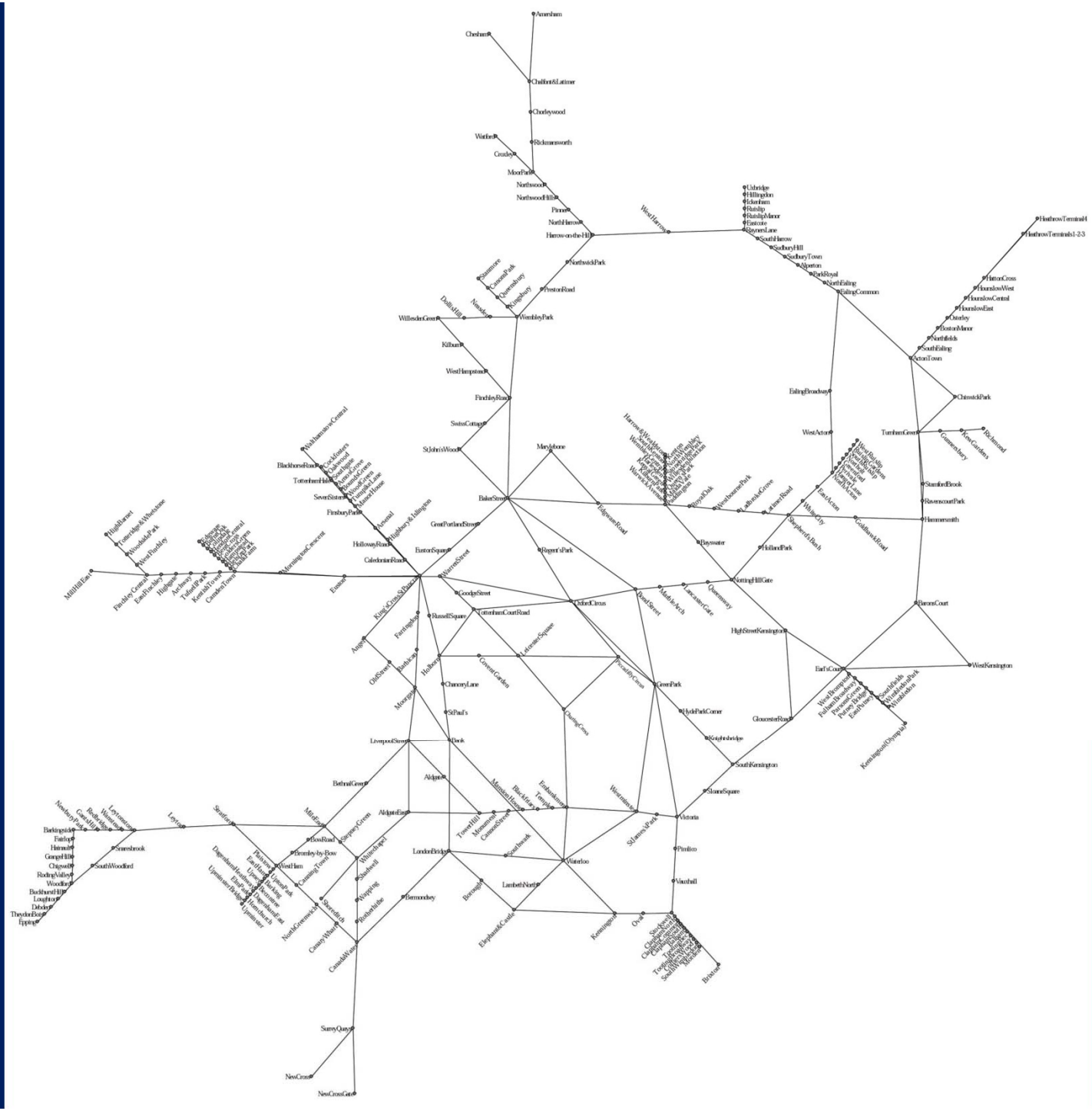
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HendonCentral
ColindaleHendon
BrentCross
CricklewoodHarlesden
WillesdenJunction
KensalGreen
WestBrompton
SouthKensington
Knightsbridge
HydeParkCorner
GreenPark
PiccadillyCircus
LeicesterSquare
CoventGarden
ChanceryLane
StPaul's
Bank
LiverpoolStreet
Monument
MansionHouse
BlackFriars
Temple
Embankment
Westminster
Aldgate
AldgateEast
TowerHill
CannonStreet
LondonBridge
Borough
LambethNorth
ElephantCastle
Kennington
Oval
Stockwell
ClaphamNorth
ClaphamCommon
ClaphamSouth
TootingBec
TootingBroadway
SouthWimbledon
Morden
Victoria
StJames'sPark
Pimlico
Vauxhall
Waterloo
SouthBank
Bermondsey
Canada Water
SurreyQuays
NewCross
NewCrossGate
Stratford
LeytonstoneLeyton
Wanstead
RedbridgeGantsHill
NewbeyPark
Barkingside
Fairlop
Hainault
GrangeHill
Chigwell
RodingValley
Woodford
BuckhurstHill
Loughton
Debden
TheydonBois
Epping
SouthWoodford
Snaresbrook
MileEnd
BowRoad
Bromley-by-Bow
WestHam
CanningTown
PlaistowUpton Park
EastHam
BecktonPark
BeckonSE
CustomHouse
GallionsReach
DagenhamHeathway
DagenhamEast
UpminsterBridge
Upminster
NorthGreenwich
CuttyWharf
Shadwell
Wapping
RotherhitheCanada Water
Shortlands
Stepney Green
WhitechapelShadwell
Bermondsey
Borough
LambethNorth
ElephantCastle
Kennington
Oval
Stockwell
ClaphamNorth
ClaphamCommon
ClaphamSouth
TootingBec
TootingBroadway
SouthWimbledon
Morden
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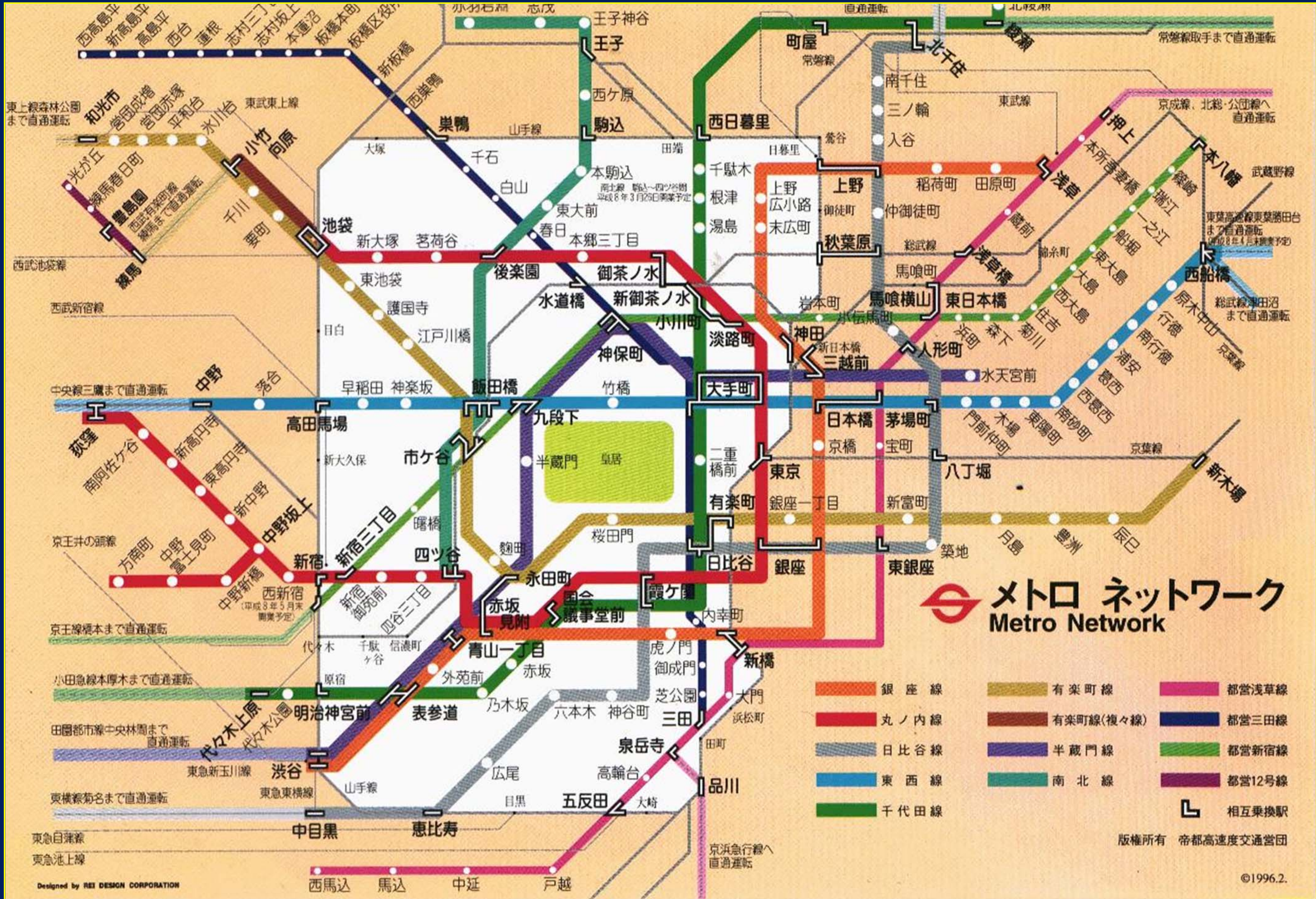
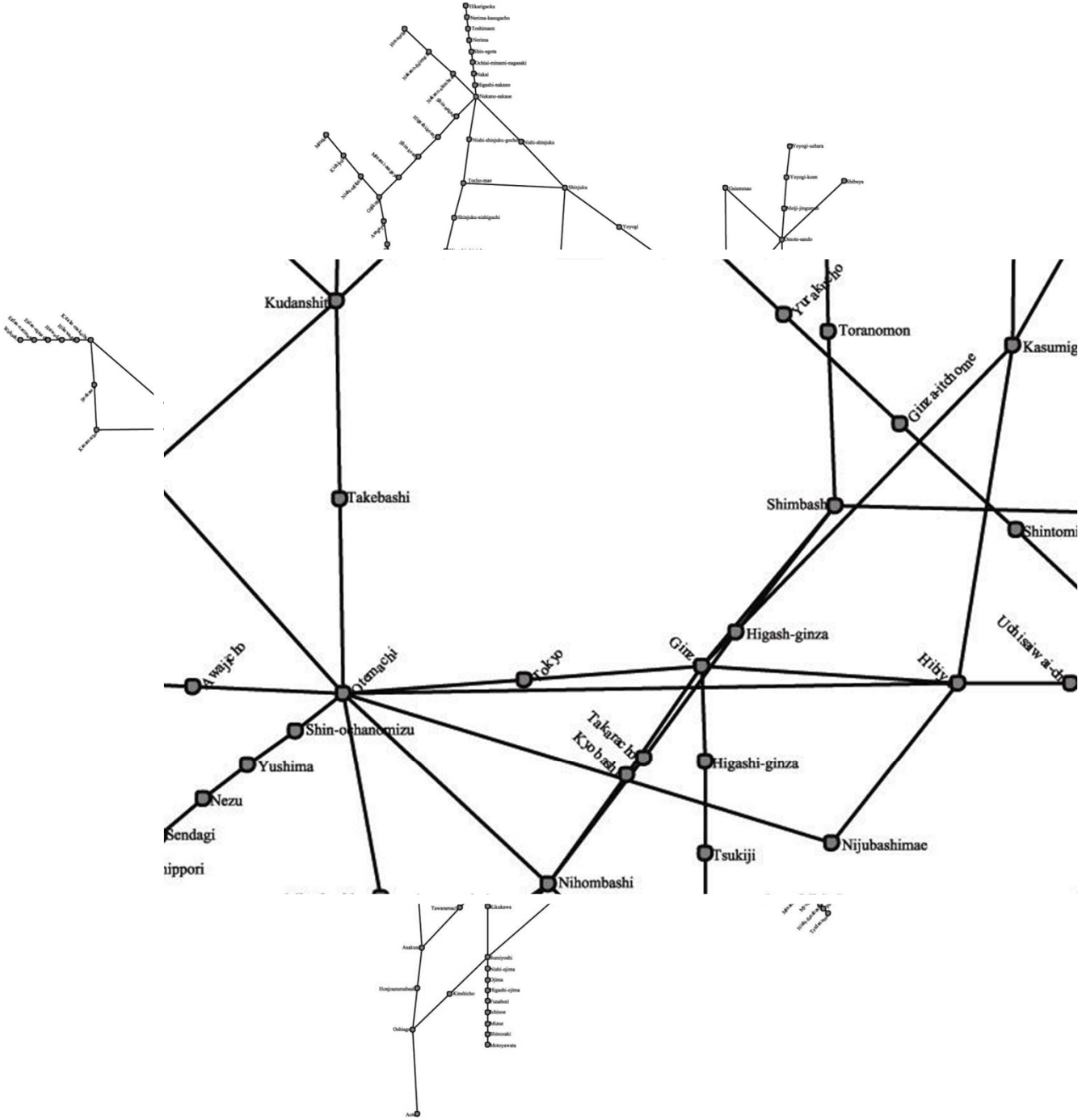
Morden

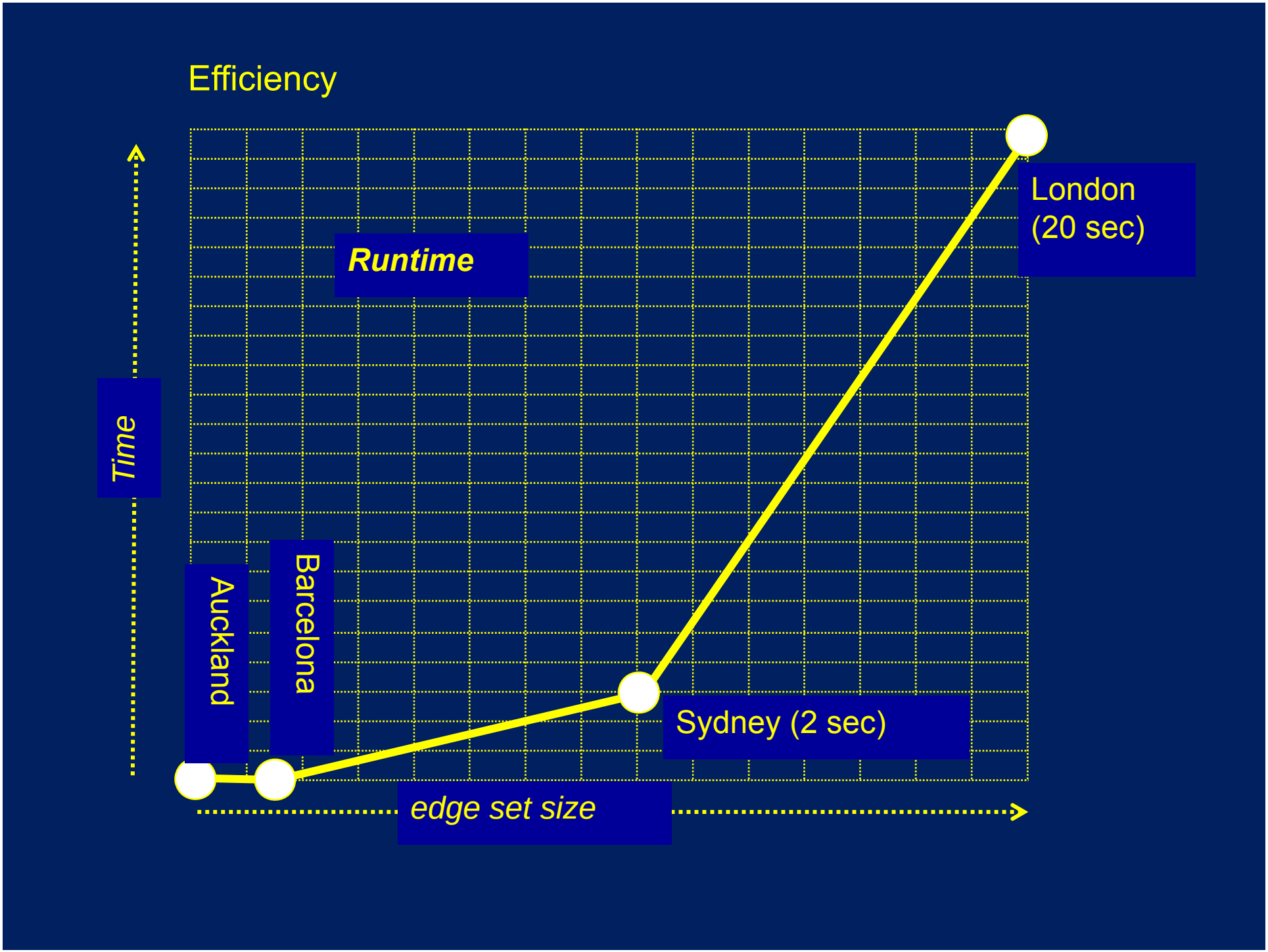


メトロ ネットワーク
Metro Network
銀座線
丸ノ内線
日比谷線
東西線
千代田線
有楽町線
有楽町線(複々線)
半蔵門線
南北線
都営浅草線
都営三田線
都営新宿線
都営12号線
相互乗換駅
版権所有 帝都高速度交通営団
©1996.2.
Designed by REI DESIGN CORPORATION
常磐線取手まで直通運転
京成線、北総・公団線へ直通運転
東上線森林公園まで直通運転
中央線三鷹まで直通運転
京王線橋本まで直通運転
小田急線本厚木まで直通運転
田園都市線中央林間まで直通運転
東横線菊名まで直通運転
京浜急行線へ直通運転
総武線津田沼まで直通運転
西武池袋線
西武新宿線
東武東上線
東武線
京葉線
京王井の頭線
東急新玉川線
東急東横線
東急目蒲線
東急池上線
山手線
王子
駒込
西日暮里
北千住
町屋
綾瀬
和光市
小竹向原
池袋
巣鴨
上野
浅草
秋葉原
押上
本八幡
西船橋
中野
高田馬場
飯田橋
九段下
大手町
日本橋
茅場町
八丁堀
新木場
荻窪
中野坂上
新宿
新宿三丁目
市ケ谷
四ツ谷
永田町
赤坂見附
国会議事堂前
霞ケ関
日比谷
有楽町
銀座
東銀座
東京
新橋
三田
品川
五反田
泉岳寺
渋谷
表参道
明治神宮前
代々木上原
中目黒
恵比寿
青山一丁目
後楽園
御茶ノ水
新御茶ノ水
水道橋
神保町
淡路町
小川町
神田
三越前
人形町
馬喰横山
東日本橋
浅草橋
水天宮前
千駄木
根津
湯島
上野広小路
末広町
稲荷町
田原町
仲御徒町
南千住
三ノ輪
入谷
千石
白山
春日
本郷三丁目
東大前
本駒込
西ケ原
王子神谷
新大塚
茗荷谷
東池袋
護国寺
江戸川橋
早稲田
神楽坂
竹橋
落合
半蔵門
皇居
二重橋前
桜田門
麹町
曙橋
京橋
宝町
銀座一丁目
新富町
築地
月島
辰巳
門前仲町
木場
東陽町
南砂町
西葛西
葛西
浦安
南行徳
行徳
原木中山
虎ノ門
御成門
芝公園
神谷町
六本木
乃木坂
外苑前
広尾
白金台
高輪台
大門
西馬込
馬込
中延
戸越
西新宿
新中野
東高円寺
新高円寺
南阿佐ケ谷
方南町
中野富士見町
中野新橋
光が丘
練馬春日町
豊島園
練馬
平和台
氷川台
千川
要町
西高島平
新高島平
高島平
西台
蓮根
本蓮沼
板橋本町
板橋区役所前
新板橋
西巣鴨
森下
菊川
住吉
西大島
大島
東大島
船堀
一之江
瑞江
篠崎
内幸町


Kudanshita
Takebashi
Toranomon
Shimbashi
Shintomi
Higash-ginza
Otemachi
Tokyo
Hibiya
Shin-ochanomizu
Yushima
Nezu
Sendagi
Nihombashi
Higashi-ginza
Tsukiji
Nijubashimae

Informal conclusion:

- The force directed method is a *little bit effective*, but *not very effective*.


Efficiency
Time
Runtime
London
(20 sec)
Sydney (2 sec)
Barcelona
Auckland
edge set size

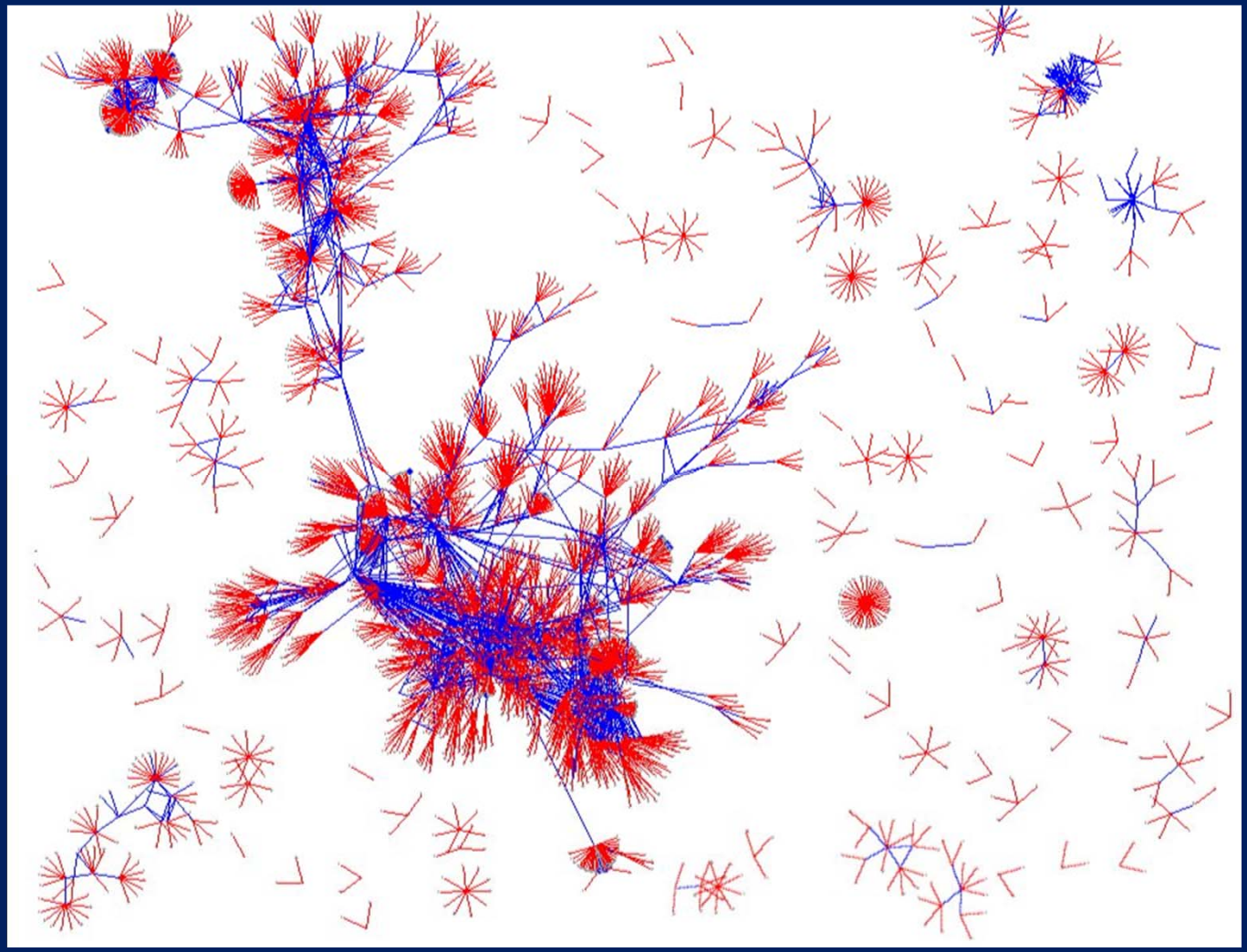
Informal conclusion:

- The force directed method for metro maps is *not computationally efficient*.

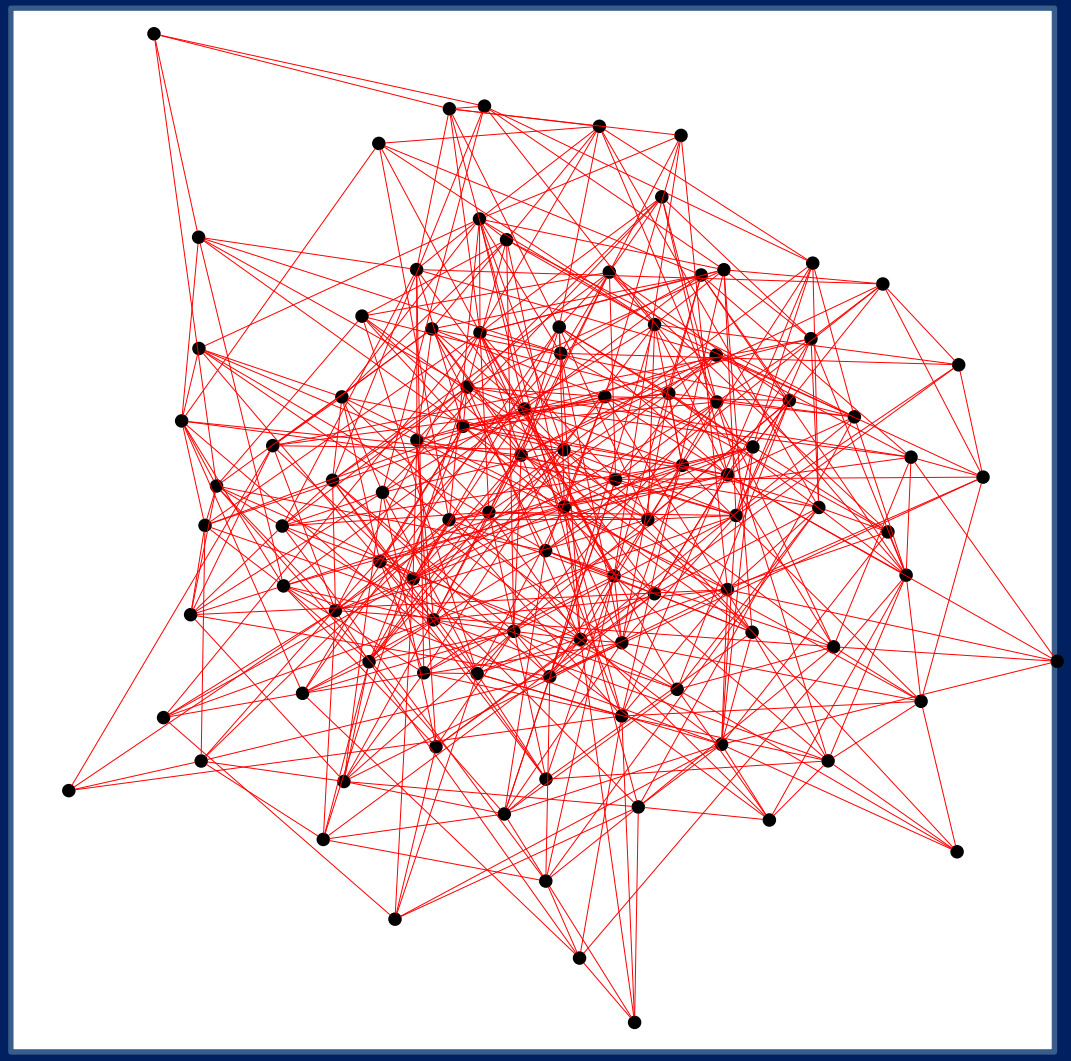
The performance of force directed methods on metro maps is typical force directed methods:

- Elegant and easy to implement
- Effective but not very effective
- Not very efficient

For some data sets, force directed methods give reasonably good drawings.



For some data sets, force directed methods give bad drawings.



How good are current force directed methods?

Elegance:

- Many simple methods, easy to implement
- Numerical software often available

Effectiveness:

- Very flexible ☑
- Straight-line edges ☑
- Planar graphs are *not* drawn planar ☒
- Very poor untangling for large graphs ☒

Efficiency:

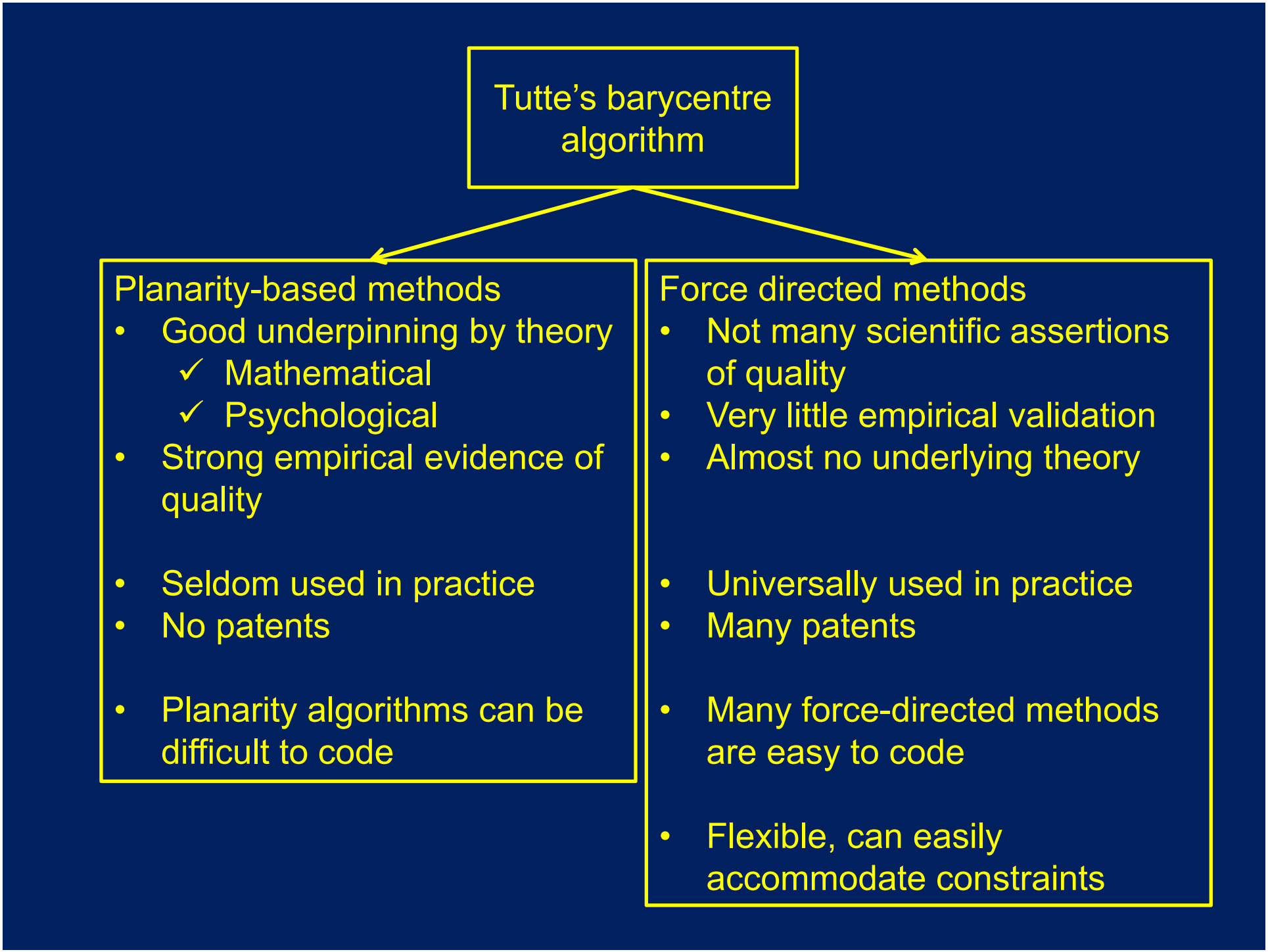
- OK for small graphs
- Sometimes OK for larger graphs (using sophisticated numerical methods)

The commercial state-of-the-art for force directed methods:

- Many commercial force-directed tools graph drawing methods are available
  - IBM (ILOG)
  - TomSawyer Software
  - yWorks
- Much free software available
  - GEOMI
  - GraphVis
- Many patents on variations of force-directed methods
- Force-directed methods account for 90% of commercial and free graph drawing software for undirected graphs

The scientific state-of-the-art for force directed methods

- Very few scientific human experiments have been done on the results of force directed methods.
- Very few theorems have been proven about force directed methods
  - Tutte's theorem
  - A theorem on symmetry of the output
  - Some theorems on multidimensional scaling can be applied.
- Some empirical comparisons (in terms of hard metrics) have been done.
- Many informal (unscientific?) investigations have been done
  - Appeal to developer intuition
  - Case studies in context


Tutte's barycentre algorithm
Planarity-based methods
• Good underpinning by theory
✓ Mathematical
✓ Psychological
• Strong empirical evidence of quality
• Seldom used in practice
• No patents
• Planarity algorithms can be difficult to code
Force directed methods
• Not many scientific assertions of quality
• Very little empirical validation
• Almost no underlying theory
• Universally used in practice
• Many patents
• Many force-directed methods are easy to code
• Flexible, can easily accommodate constraints

2. How to draw a planar graph?
   a) Before Tutte: 1920s – 1950s
   b) Tutte: 1960s
   c) After Tutte: 1970s – 1990s
   d) Recent work

Recent work

- *slightly non-planar graphs*

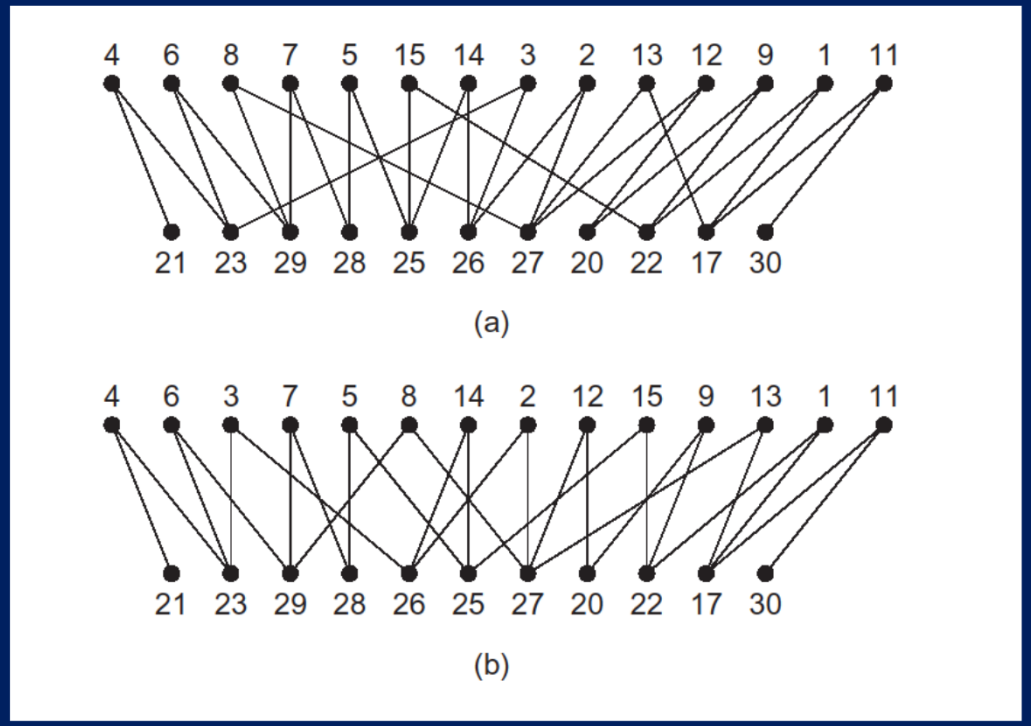
Motivation

Mutzel experiment 1997 - 98

- Informal “experiment”, performed at a talk
- Audience members were the “subjects”

Results

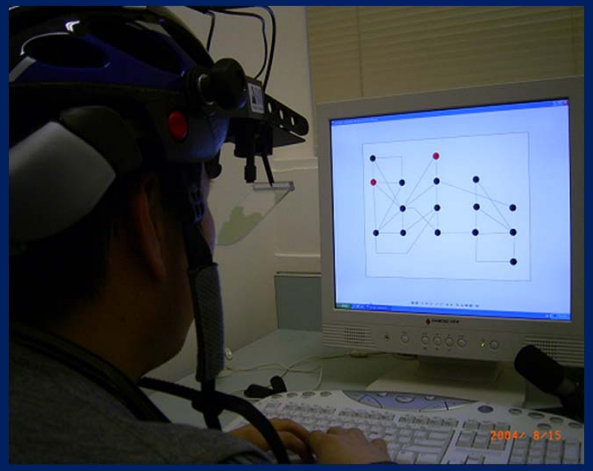
- People prefer (a) over (b)
- People erroneously see (a) as having fewer crossings than (b)



Motivation

Tony Huang 2003+

- Series of formal human experiments using eye-tracking.



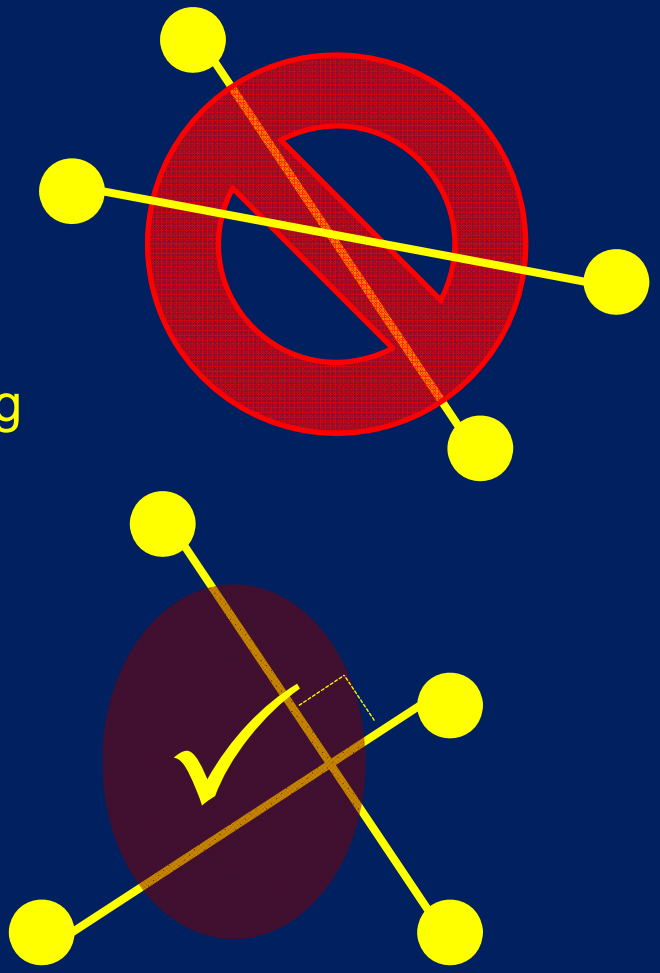
Huang's thesis

If the crossing angles are large, then non-planar drawings are OK.

Slightly non-planar graphs

Right-Angle Crossing (RAC) graphs:

- Straight-line edges
- If two edges cross, then the crossing makes a right angle



Questions for slightly non-planar graphs:

- How dense can a RAC graph be?

Theorem (Liotta, Didimo, Eades, 2009)

Suppose that ***G*** is a RAC graph with ***n*** vertices and ***m*** edges.
Then $m \leq 4n-10$.

Questions for slightly non-planar graphs:

- How dense can a RAC graph be?
- How can you compute a drawing of a RAC graph?

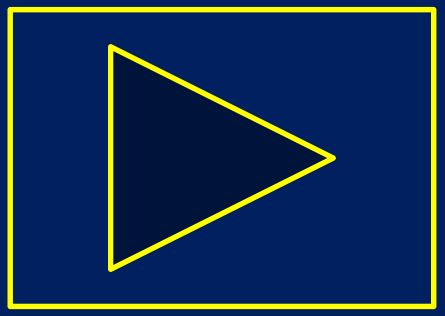
Theorem (Liotta, Eades, unpublished)*

The following problem is NP-hard:

*Input: A graph **G***

*Question: Is there a straight-line RAC drawing of **G**?*

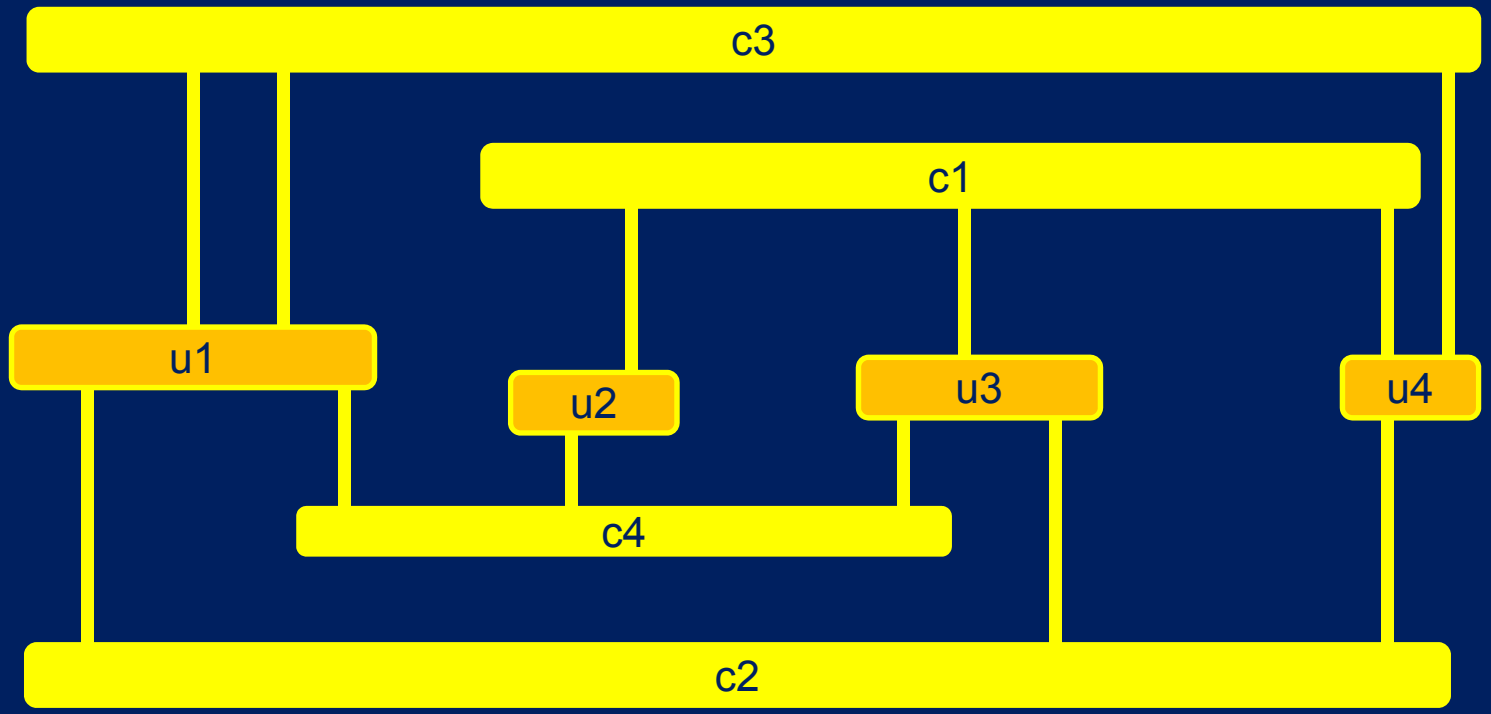
*Independently proved and published by Argyriou, Bekos and Symvonis



Proof

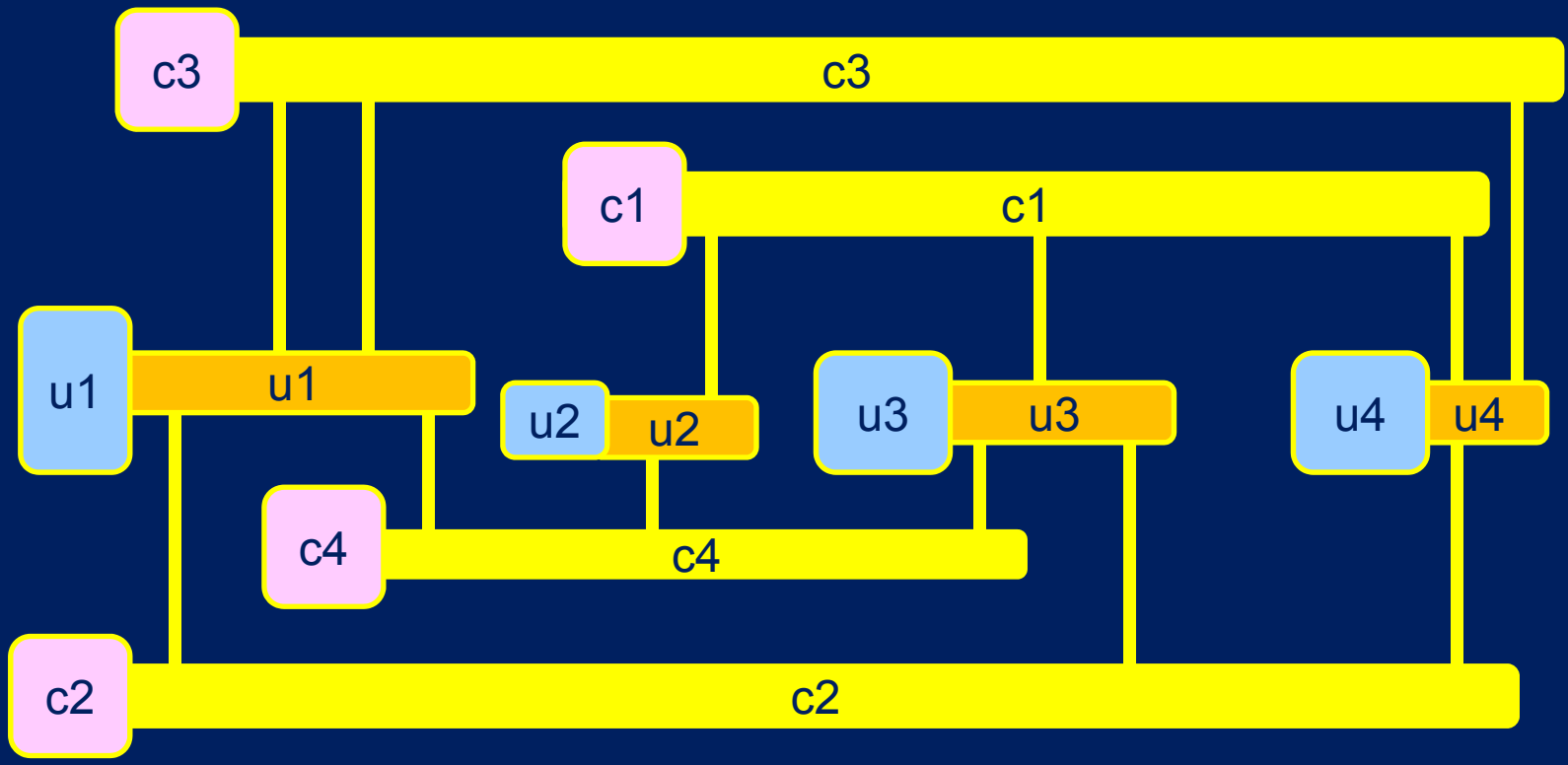
- Reduction from planar-3-sat
- Draw the instance ***H*** of planar-3-sat as a template
- Fill in details of the template ***H*** to form a graph ***G*** that has a RAC drawing if and only if ***H*** is satisfiable.

- Fairly generic proof strategy for NP-hardness for layout problems.

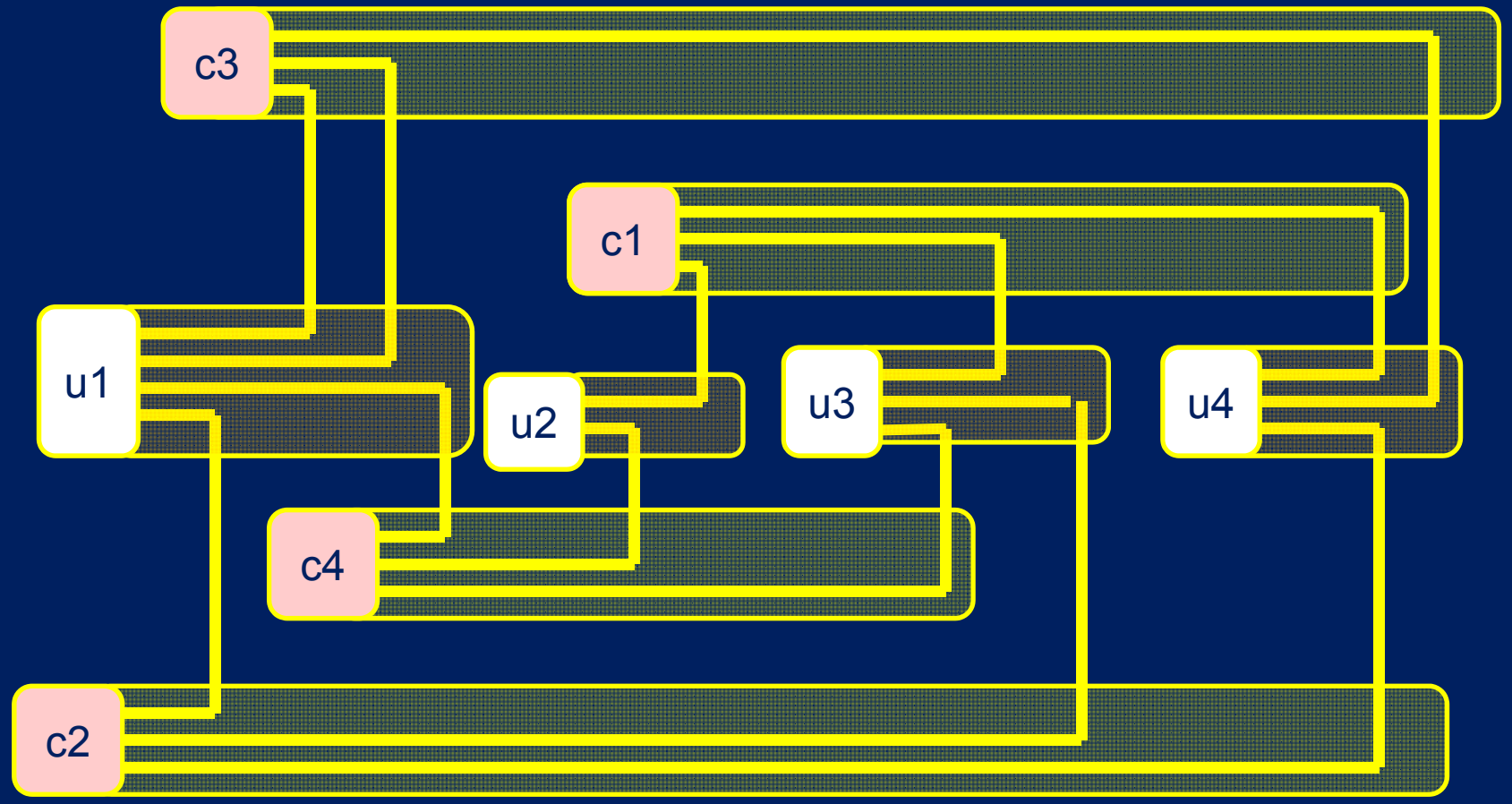


Instance ***H*** of planar 3-sat graph

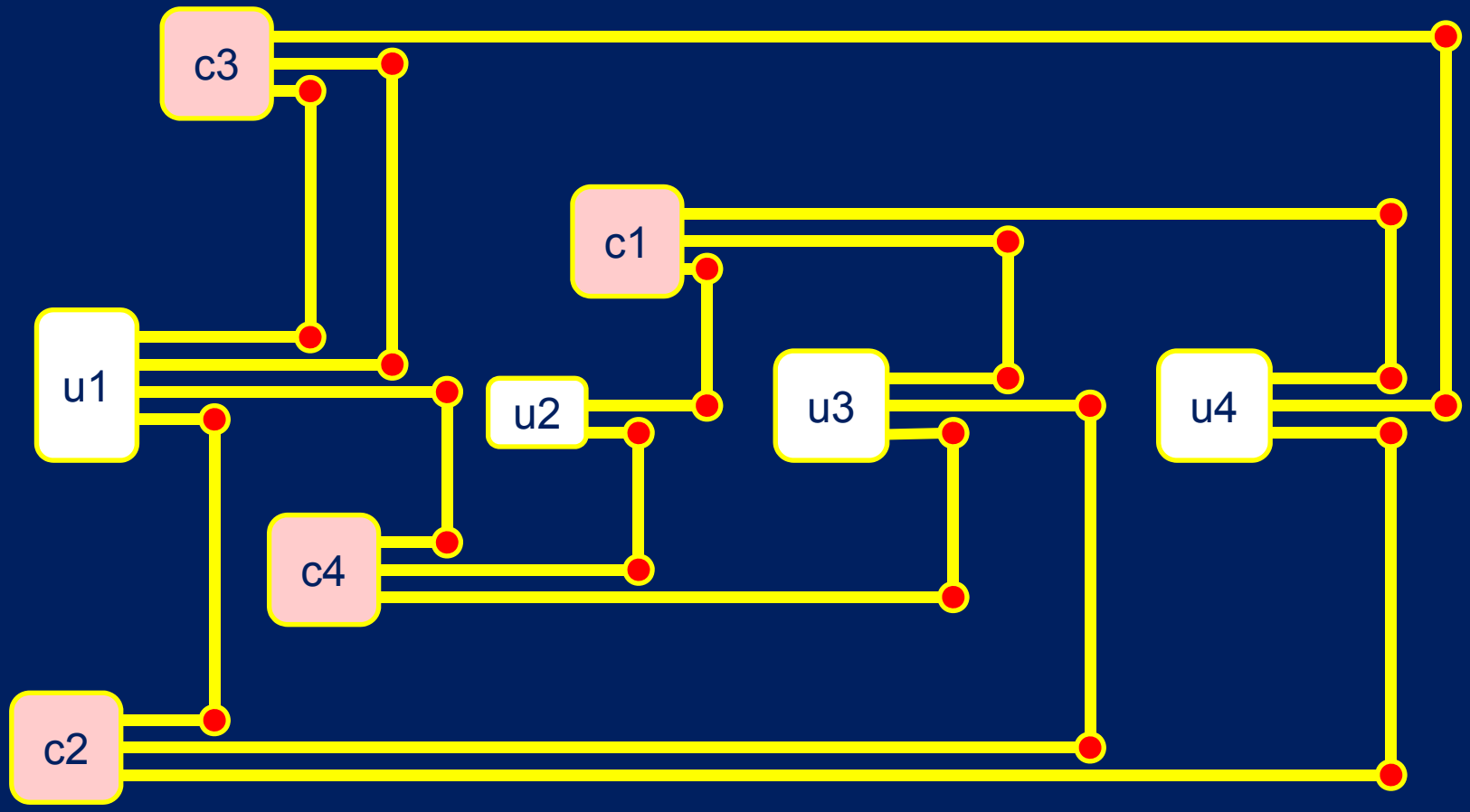
1. Draw ***H*** as a visibility drawing



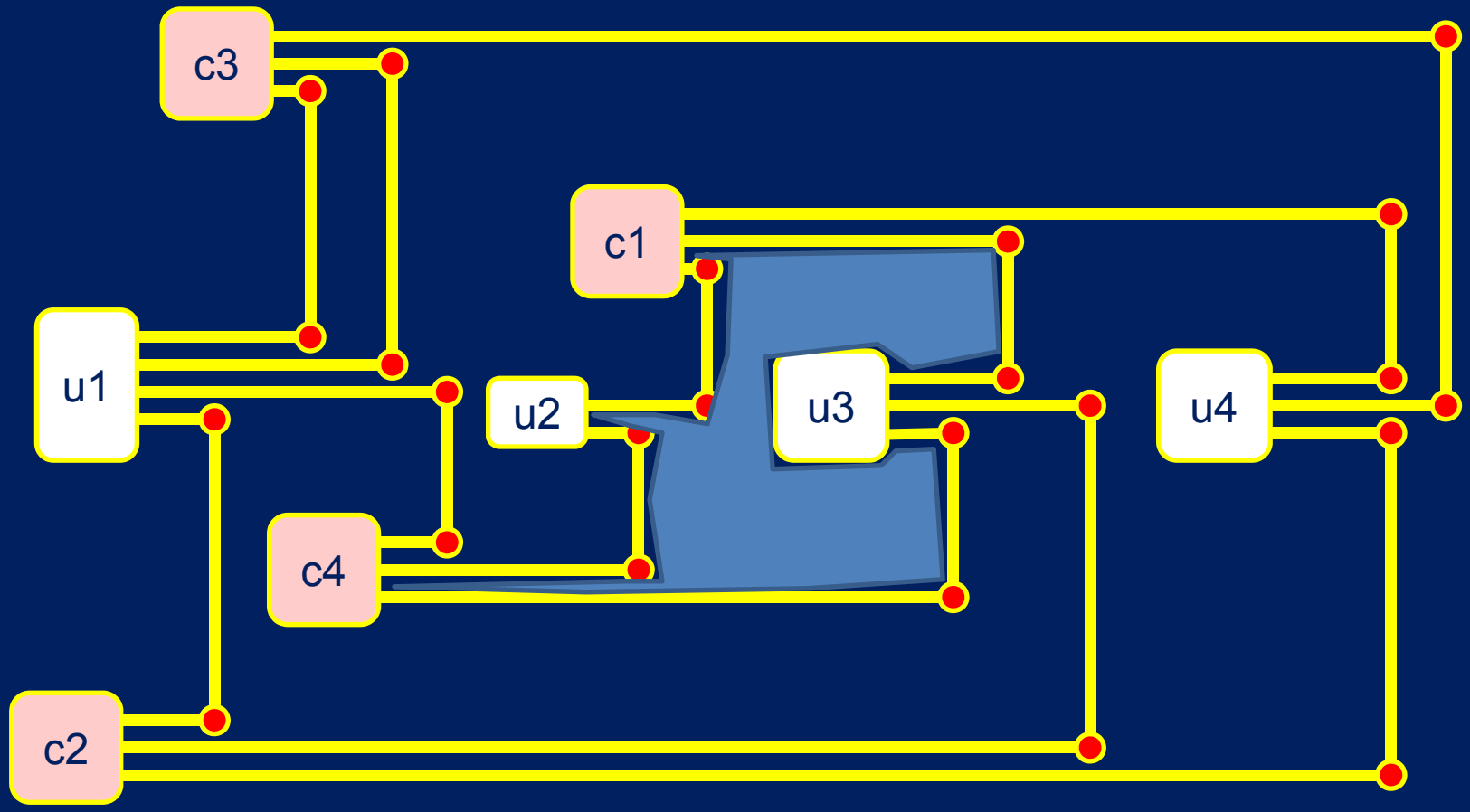
2. Enhance the drawing:
   - "node boxes" for
     - clauses c1, c2, …
     - variables u1, u2, …



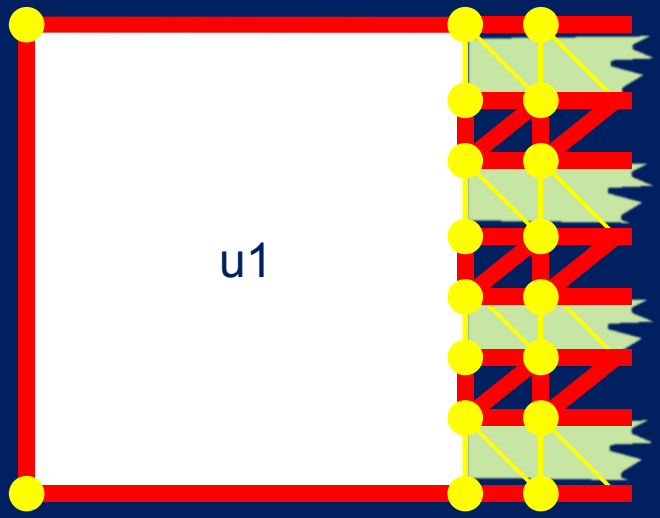
3. Transform to a 2-bend drawing
   - “pipes” to communicate between variables and clauses



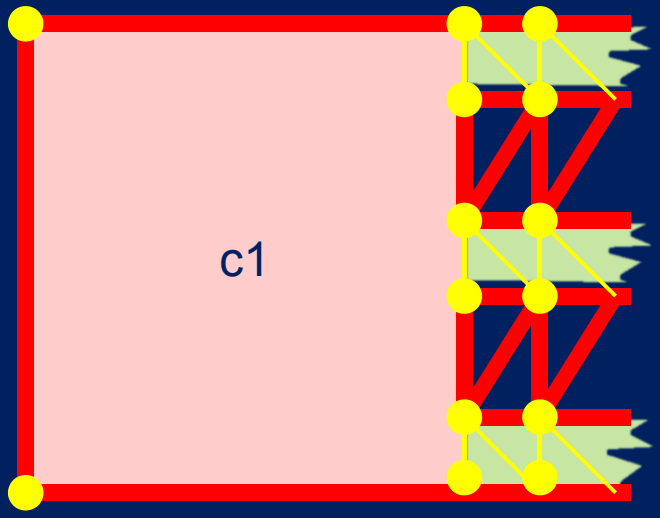
4. Transform to a no-bend drawing
   - extra nodes at bend points



5. Triangulate every face to make it impassable

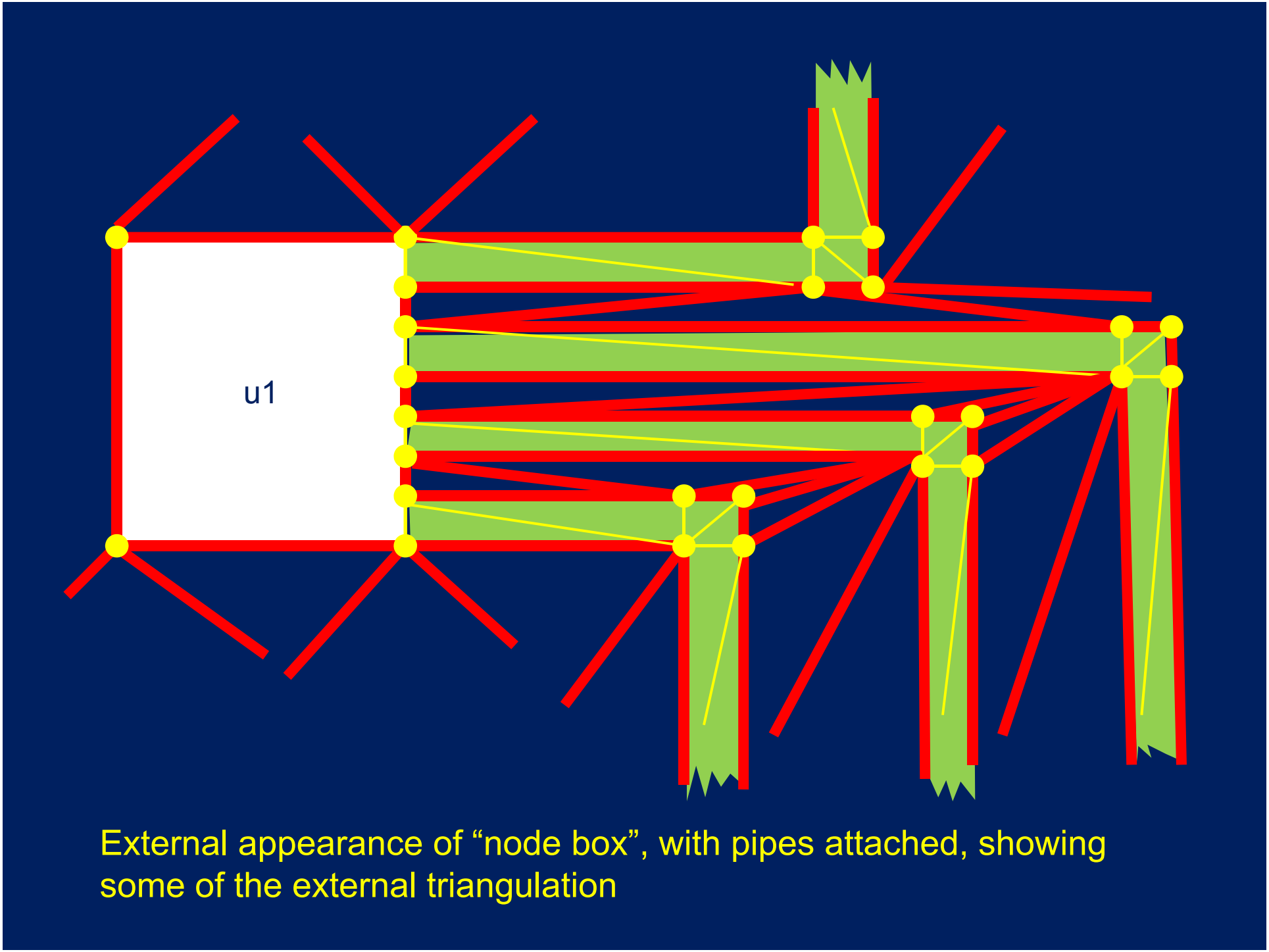


variable

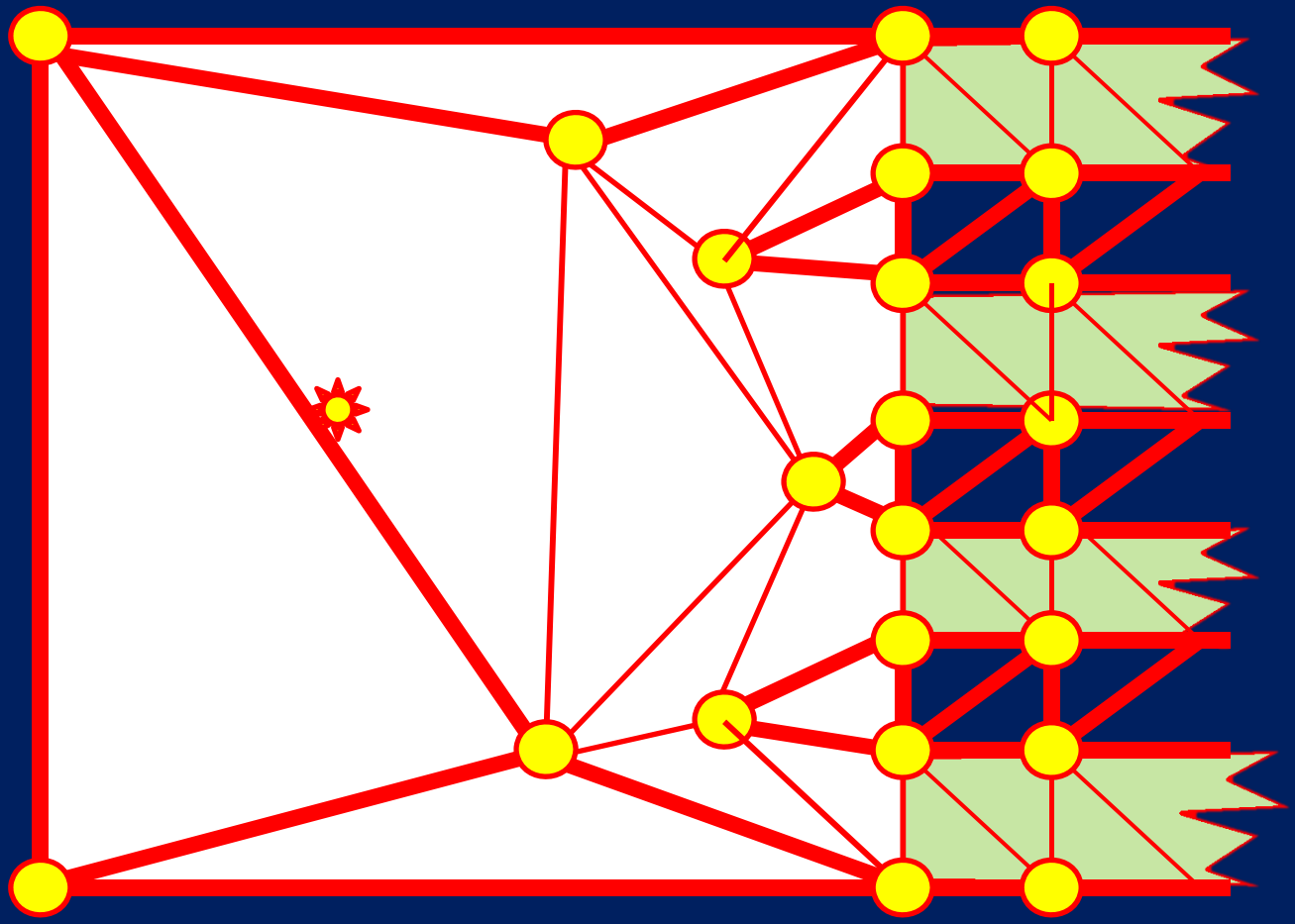


clause

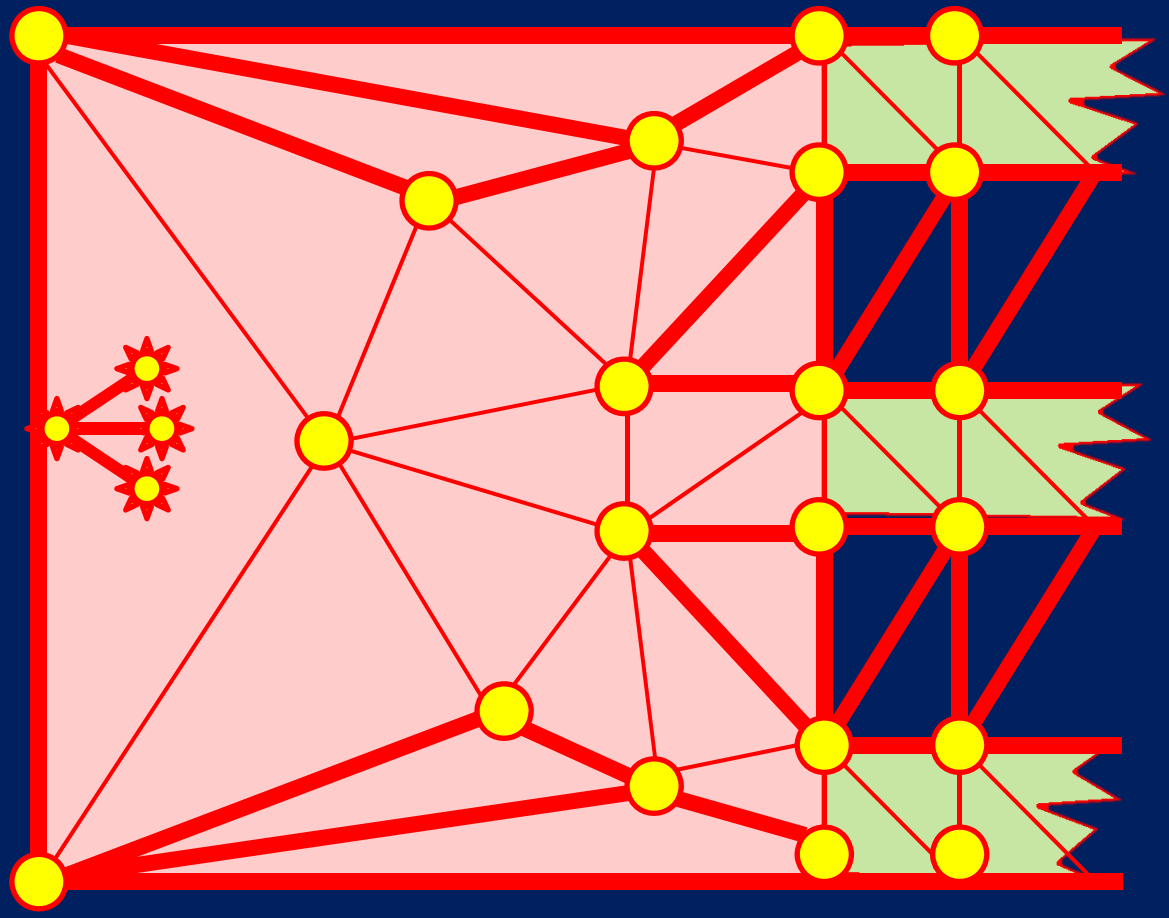
External appearance of “node boxes”, with “pipes” attached



External appearance of “node box”, with pipes attached, showing some of the external triangulation

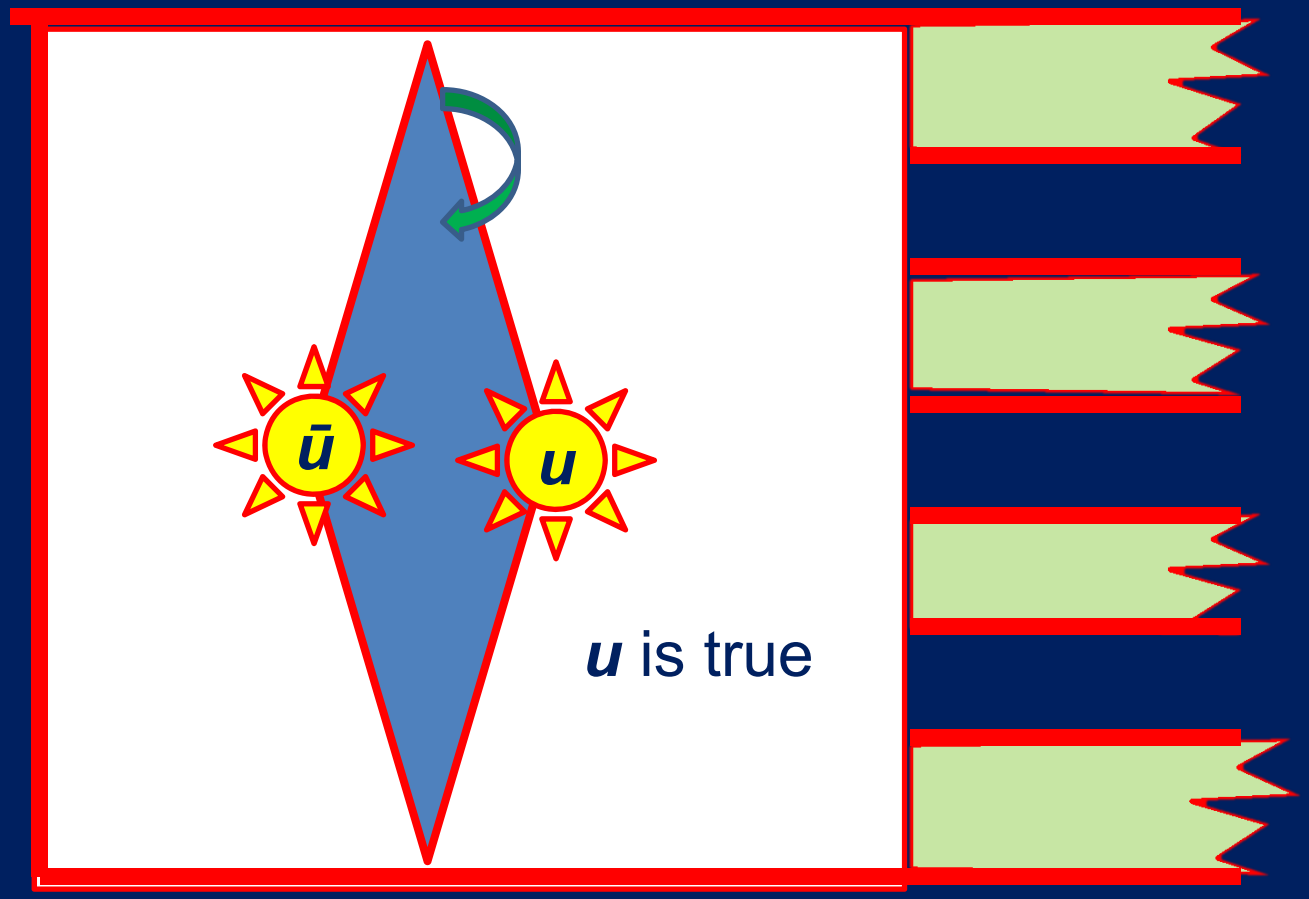


Variable gadget with pipes attached

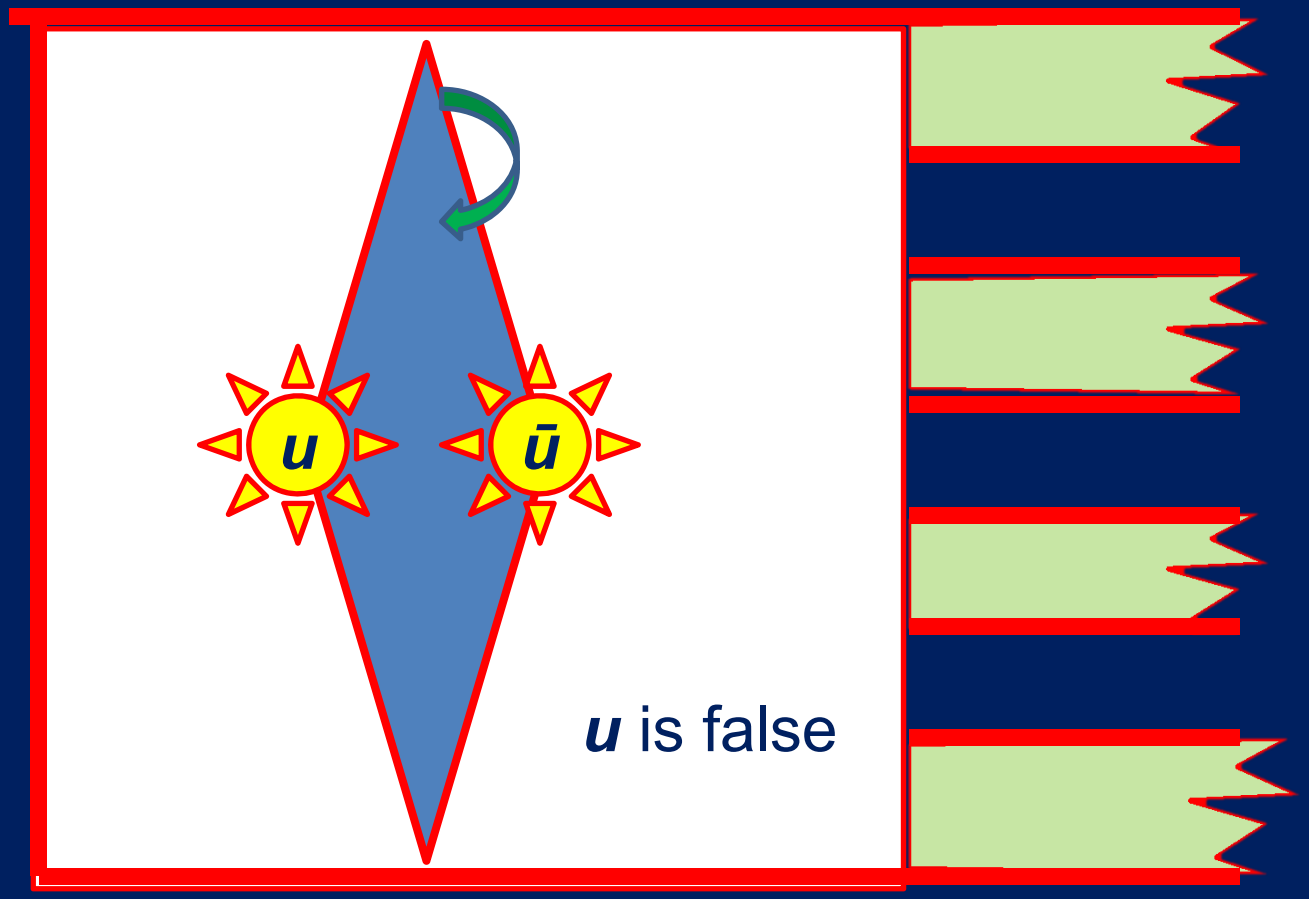


Clause gadget with pipes attached

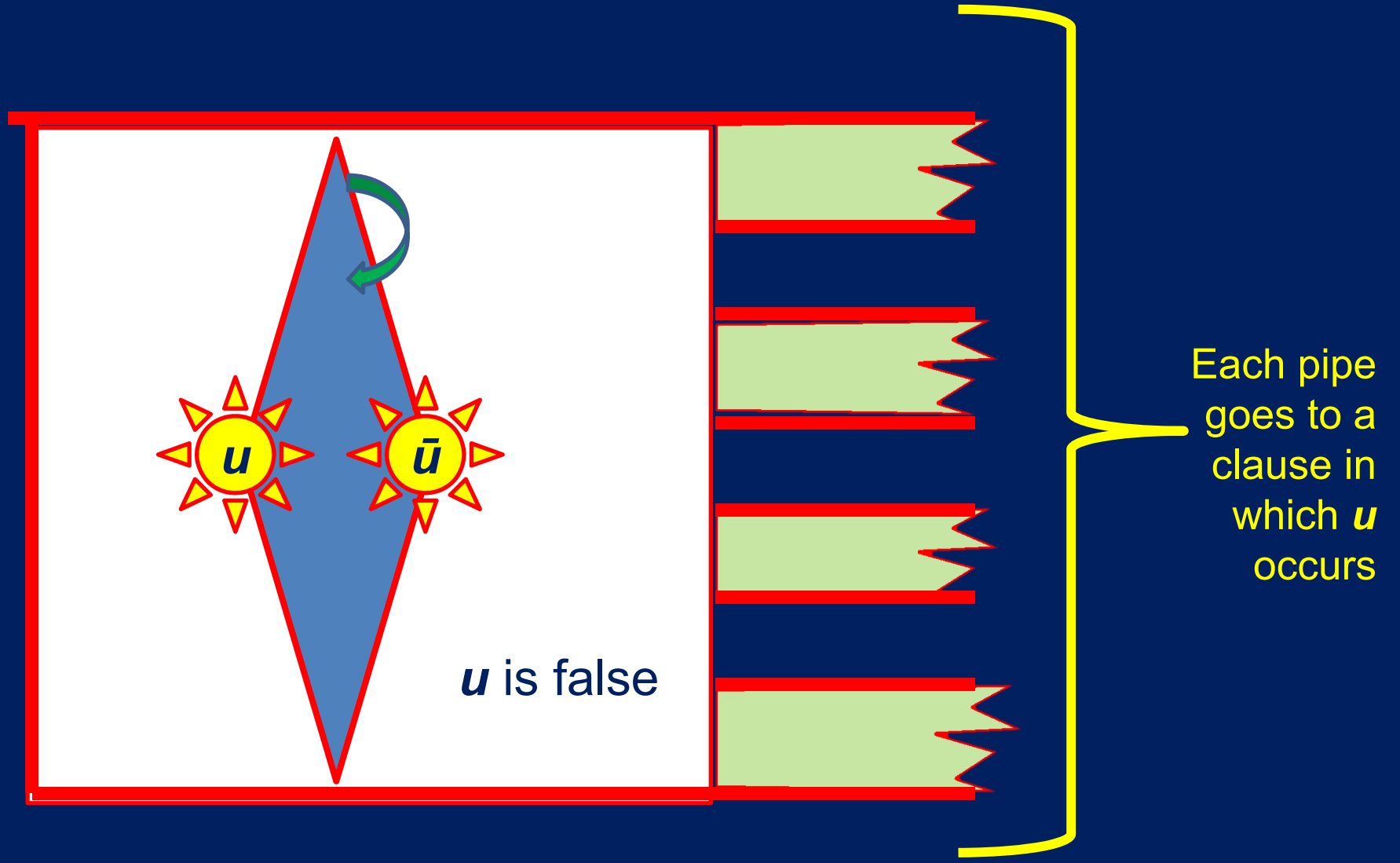
Logical view of variable gadget



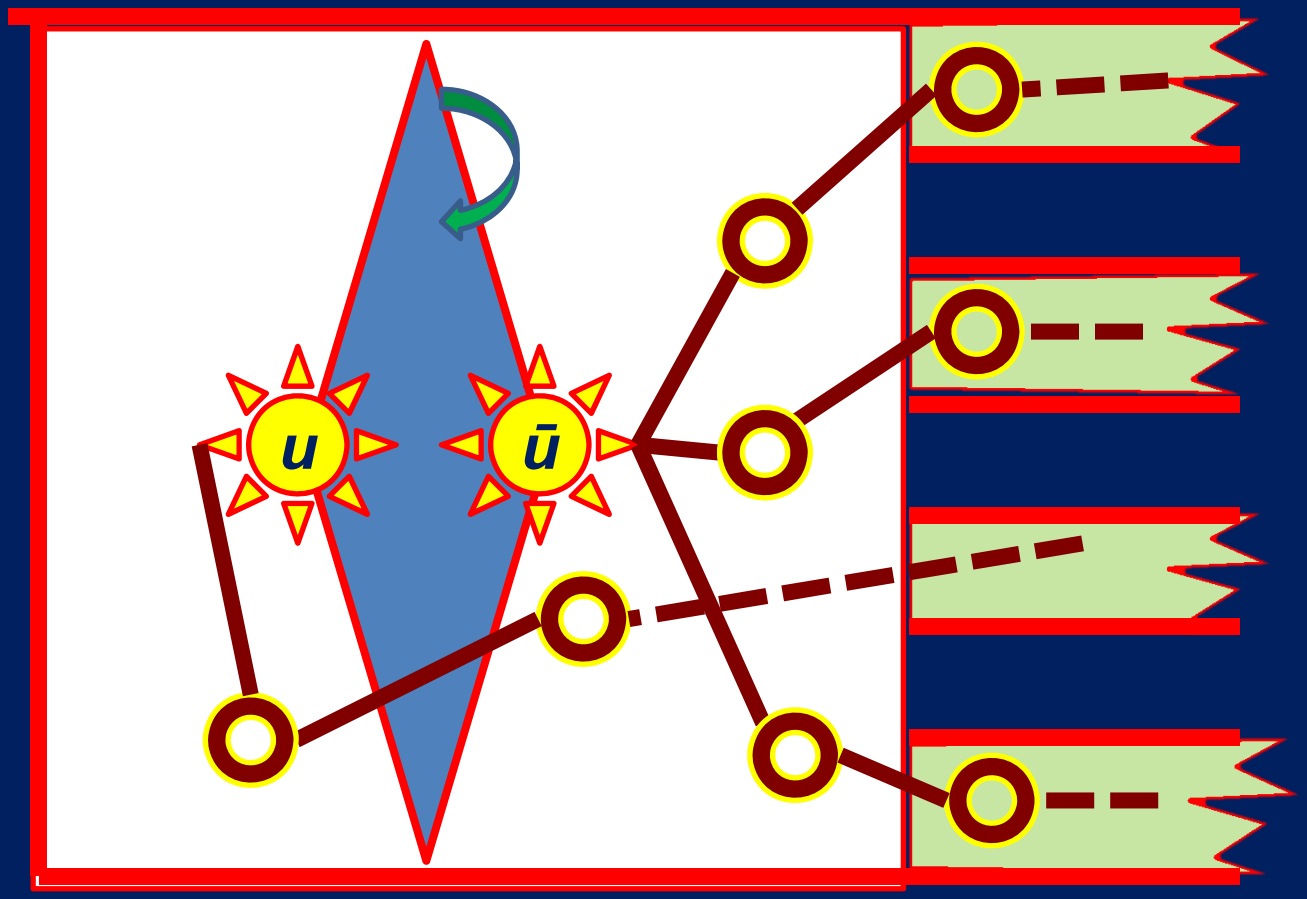
Logical view of variable gadget



Logical view of variable gadget

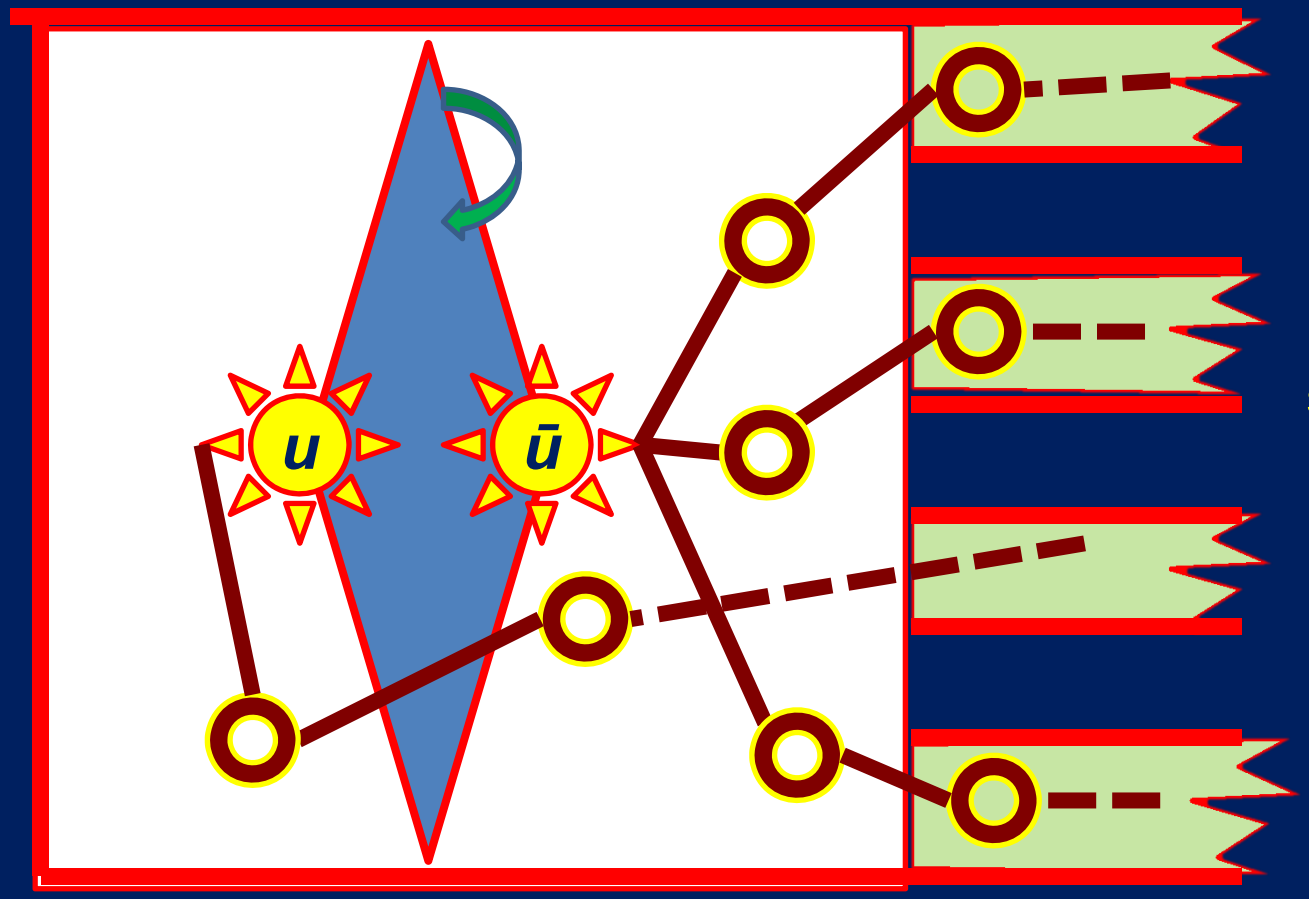


Logical view of variable gadget

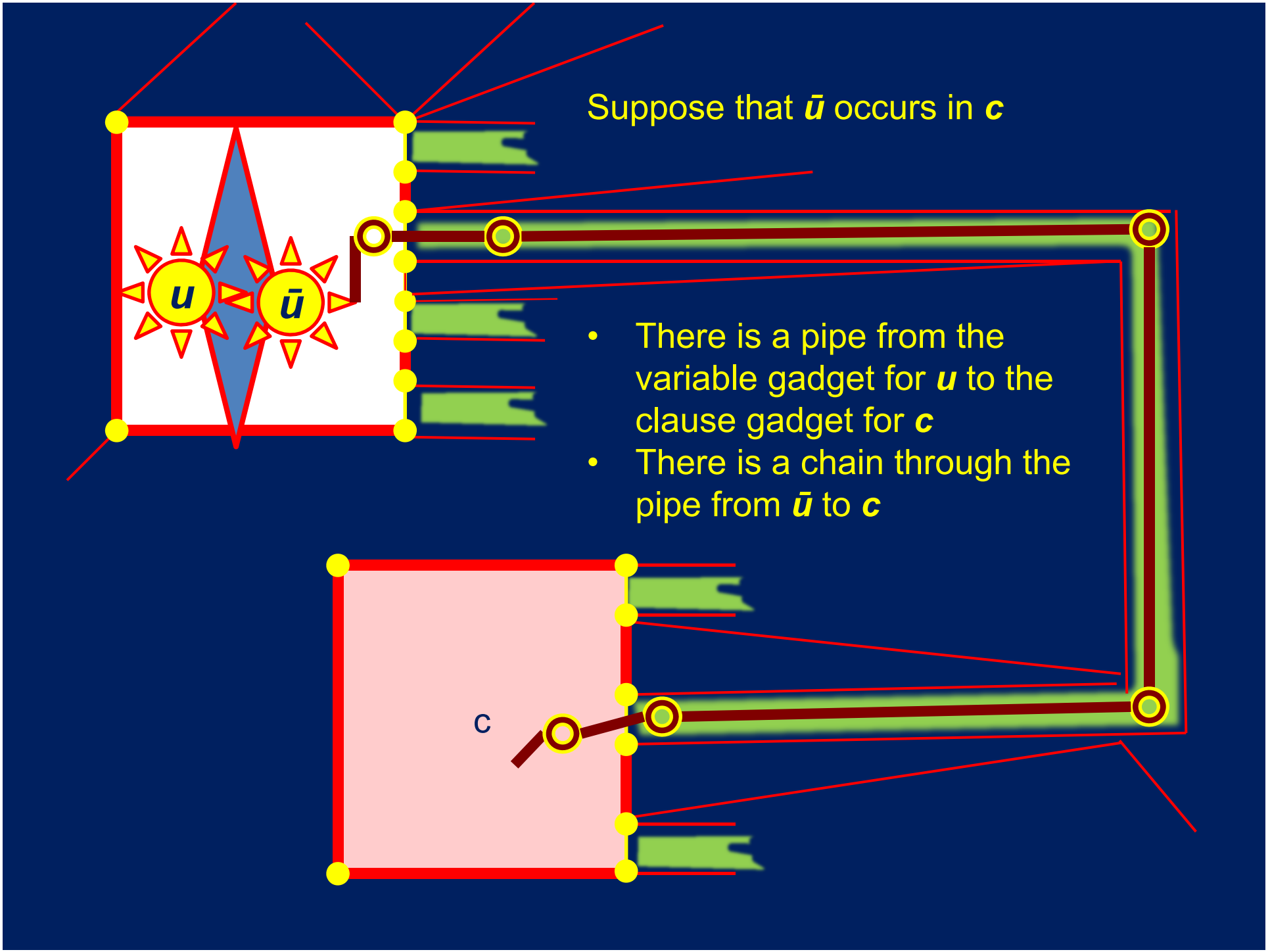


Literals are attached to the clauses in which they occur, using chains threaded through the pipes

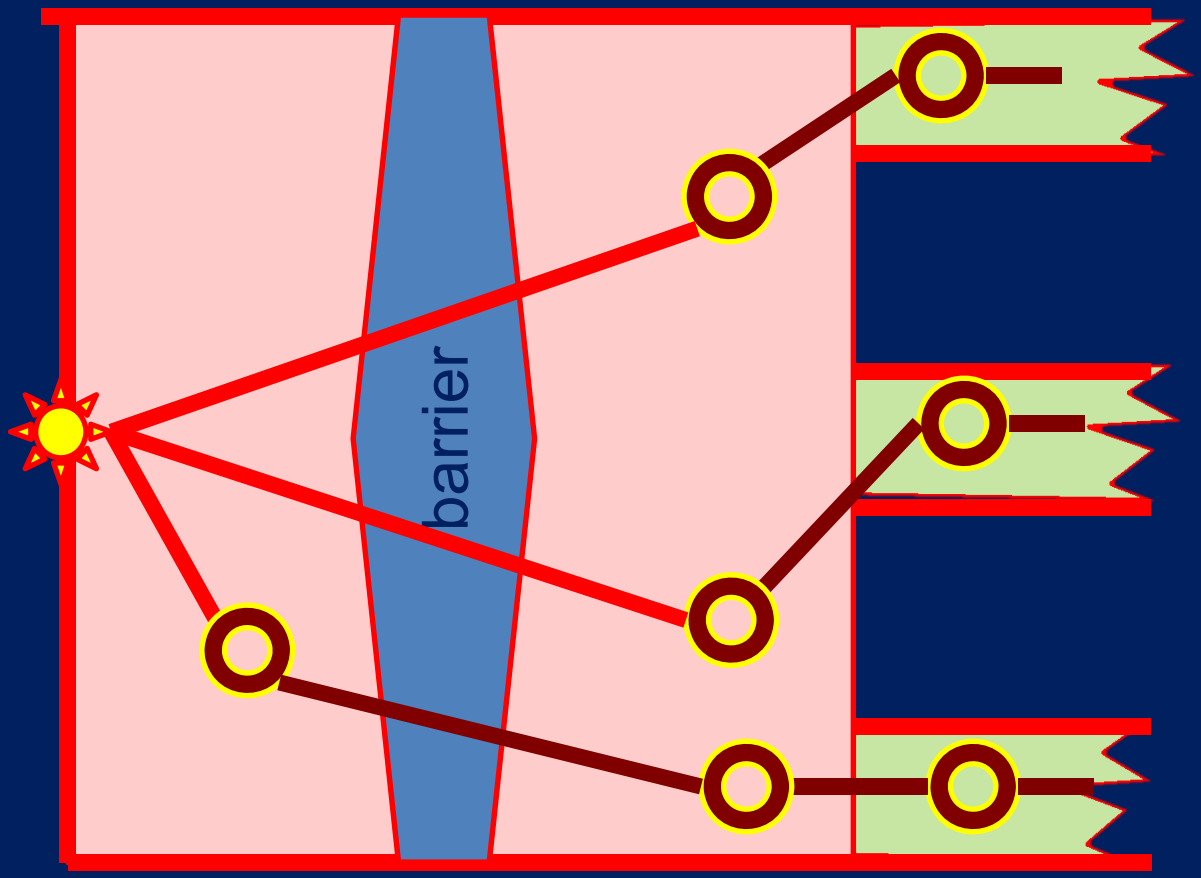
Logical view of variable gadget



Chains attached to the rear literal spend an extra link before getting into the pipe.


Suppose that $\bar{u}$ occurs in c
u
$\bar{u}$
• There is a pipe from the variable gadget for u to the clause gadget for c
• There is a chain through the pipe from $\bar{u}$ to c
c

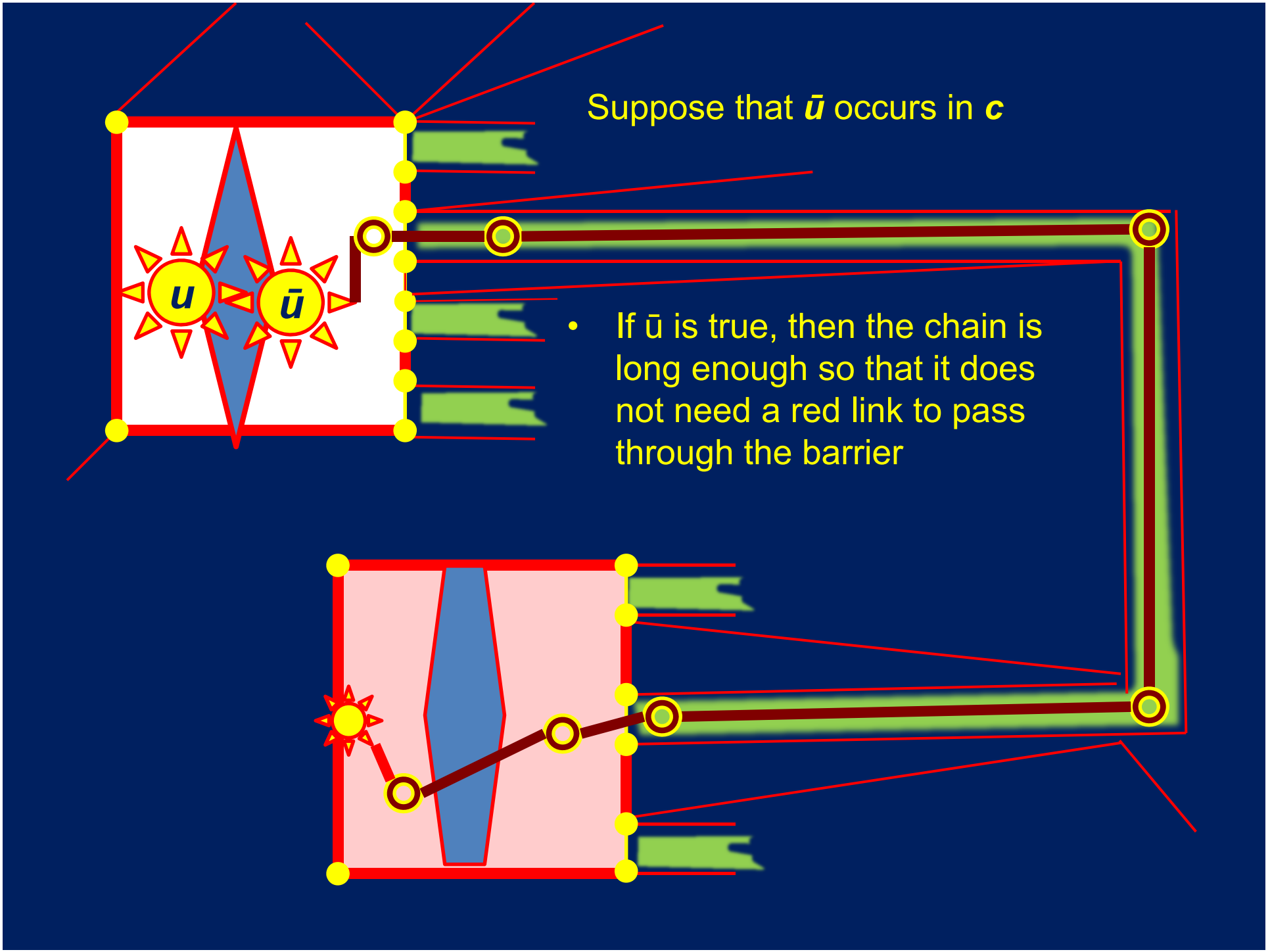
Logical view of clause gadget

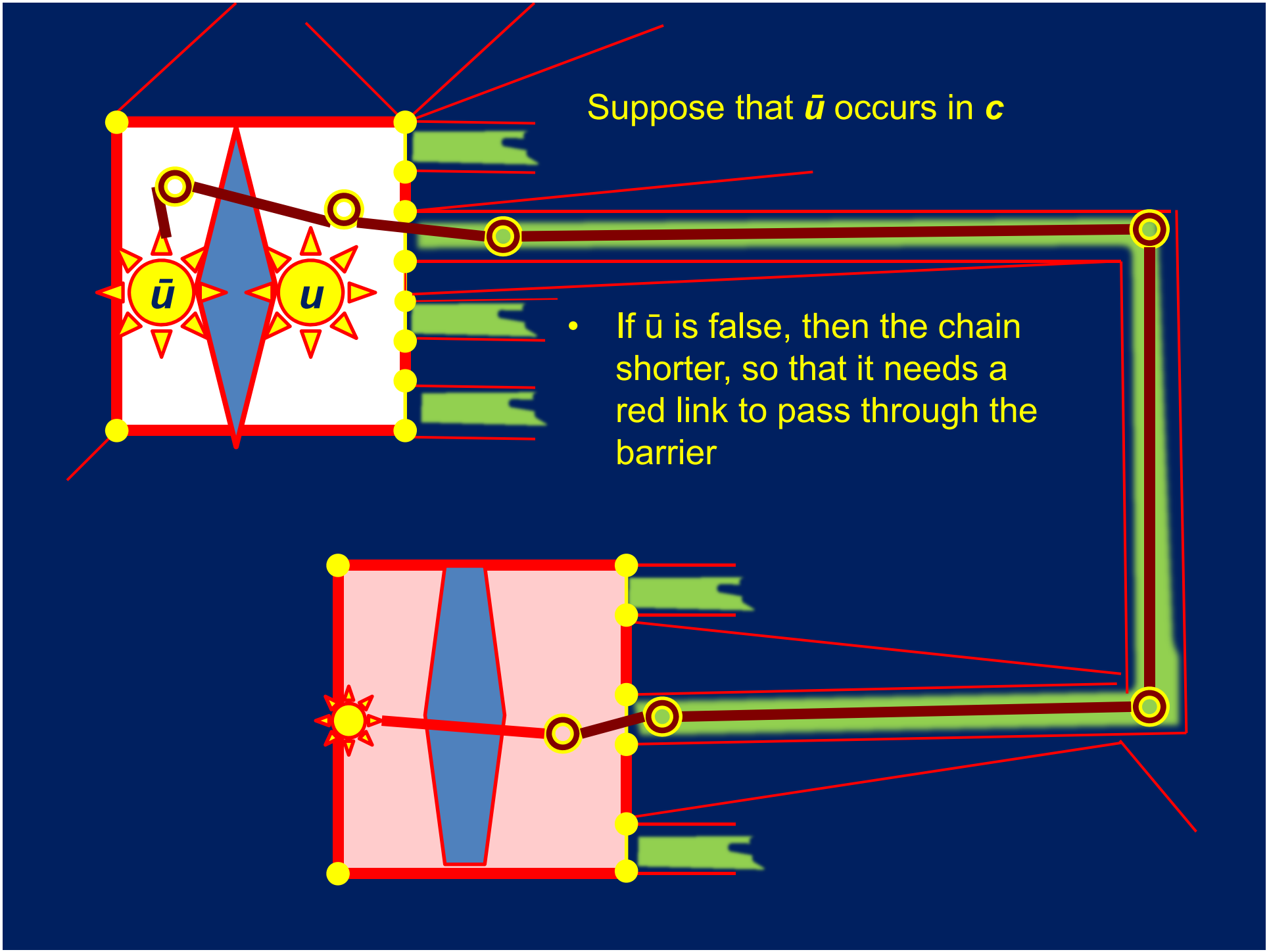


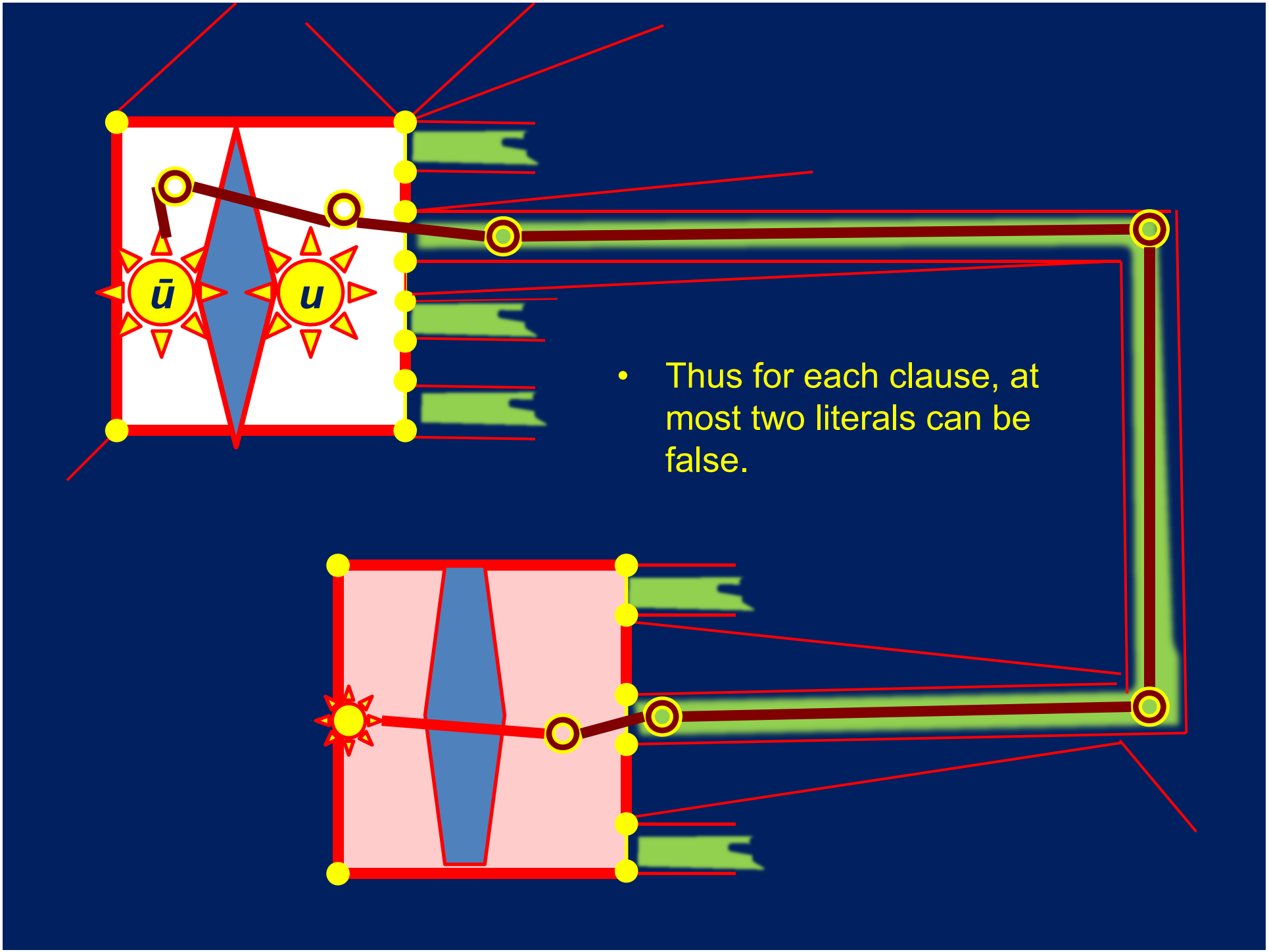
The barrier allows

- Any number of brown links to pass through
- At most two red links to pass through

Thus at least one chain needs to be long enough to reach past the barrier


Suppose that $\bar{u}$ occurs in c
u
ū
• If ū is true, then the chain is long enough so that it does not need a red link to pass through the barrier


Suppose that $\bar{u}$ occurs in c
$\bar{u}$
u
• If ū is false, then the chain shorter, so that it needs a red link to pass through the barrier



- Thus for each clause, at most two literals can be false.

Notes

- This is a *fairly generic proof strategy* for NP-hardness for layout problems.
- Details of clause and variable gadgets are straightforward but tedious
- The same proof works for *1-planar graphs*: just choose different gadgets for clauses and variables.

What about heuristics?

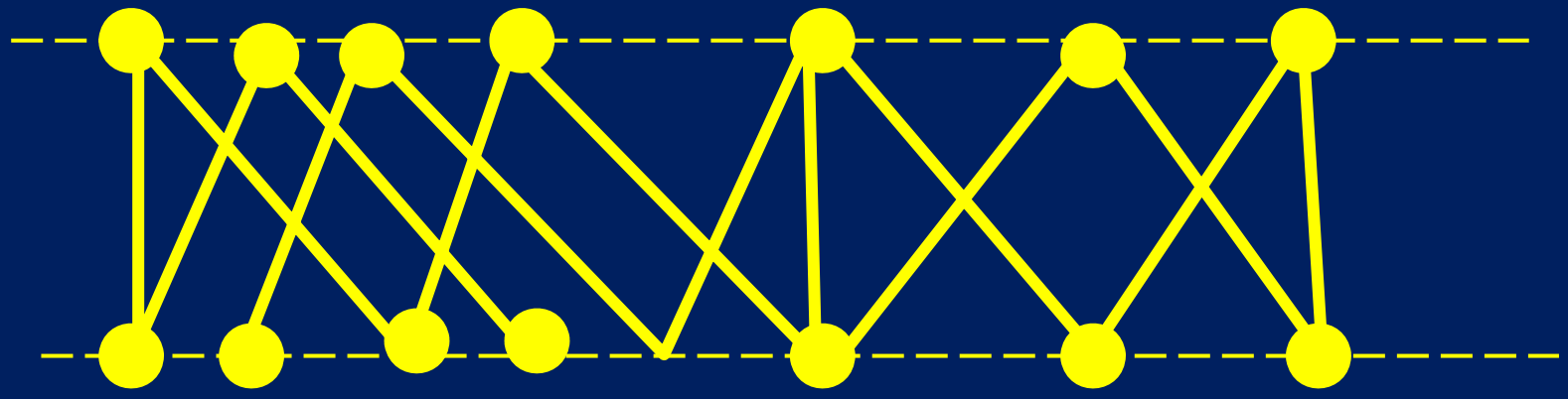
- Several force-directed methods (Huang, Nguyen, Hong, et al.) have been tested and seem to work OK to produce larger crossing angles than regular force-directed methods.

- No combinatorial heuristics known

What about special classes of graphs?

Theorem (Liotta, Didimo, di Giacomo, Eades)

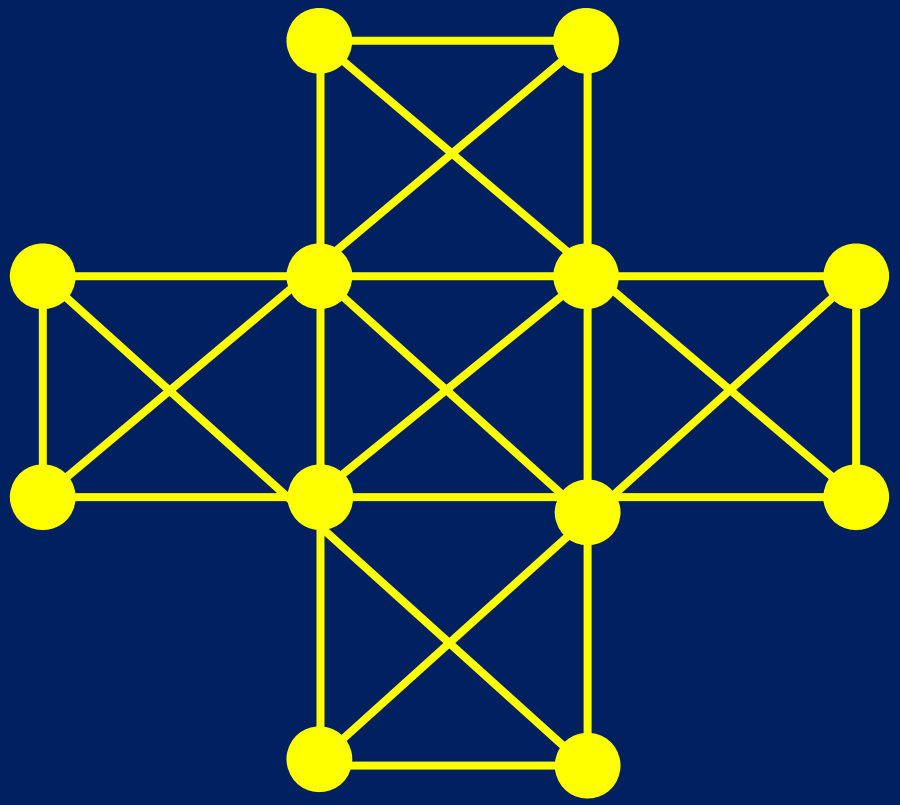
There is a linear time algorithm to test and draw graphs restricted to two layers with right-angle crossings.



What about special classes of graphs?

Theorem (Reisi)

There is a linear time algorithm to test and draw outer-1-planar graphs with right-angle crossings.



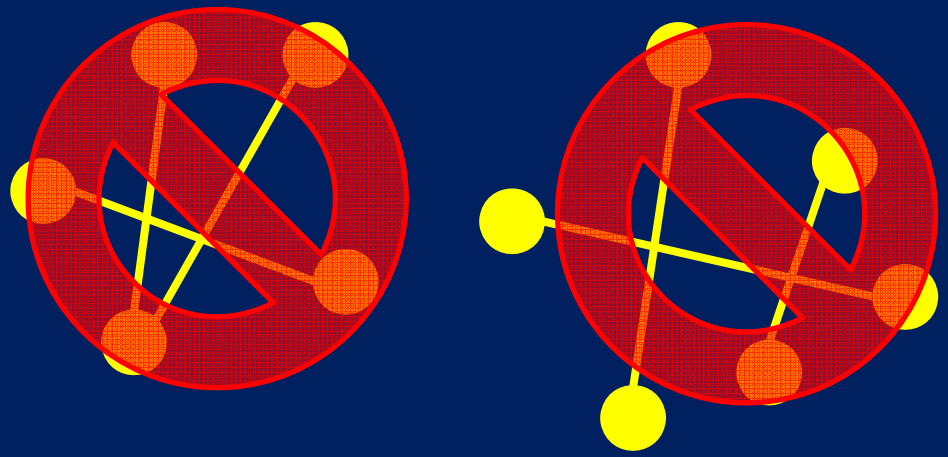
However:

- the problem of drawing graphs with large crossing angles remains mostly open, both from both practical and theoretical points of view.

More slightly-non-planar graphs

1-planar:

- No edge has more than one crossing



Questions for 1-planar graphs:

- How dense can a 1-planar graph be?

4n-8 (Ackerman-Tardos)

- How can you compute the topology of 1-planar graph?

NP-hard (Korzhik- Mohar)

- *Given the topology, how can you compute a drawing of a 1-planar graph?*

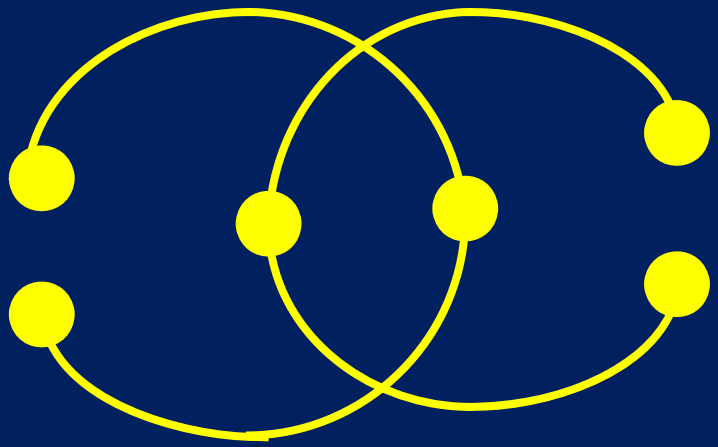
Recent algorithm (Hong-Eades-Liotta-Poon)

- *What is the relationship between 1-planar graphs and RAC graphs?*

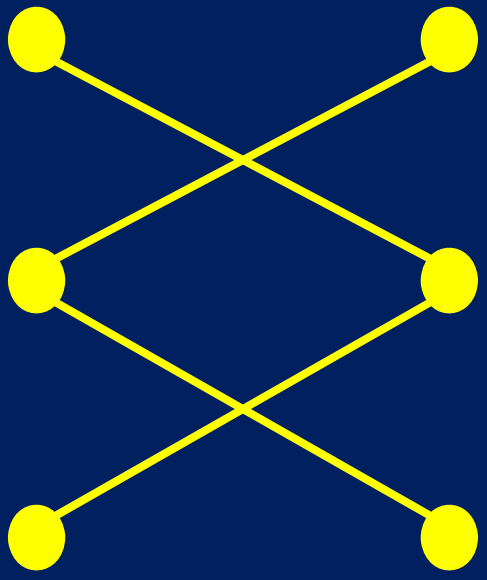
Recent theorem (Eades-Liotta)

Some 1-planar graphs have a 1-planar straight-line drawing,
others have no 1-planar straight-line drawing.

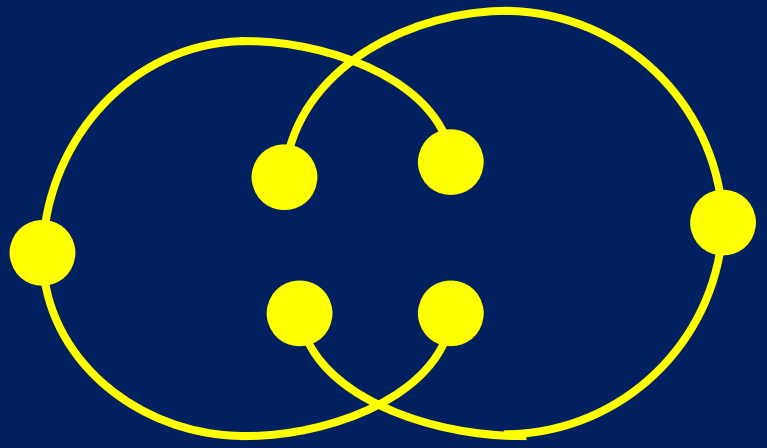
Chanel graph



The chanel graph has a straight-line drawing



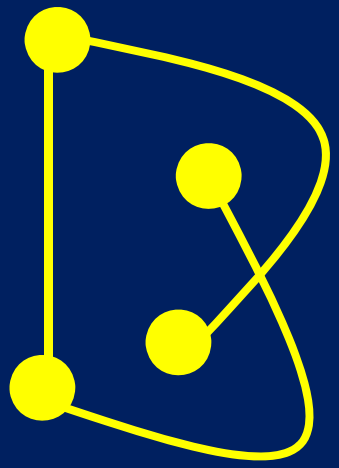
Gucci graph



Lemma:

The gucci graph has no straight-line drawing

Bulgari graph

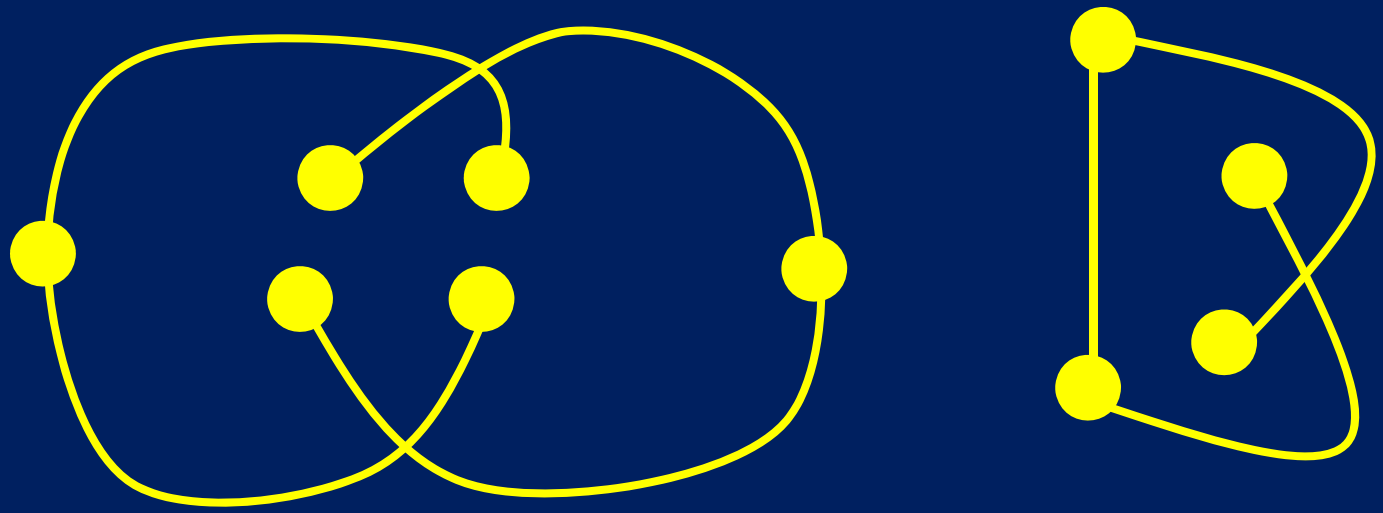


Lemma:

The bulgari graph has no straight-line drawing

Theorem (Hong, Liotta, Poon, Eades, 2011)

Suppose that ***G*** is a 1-planar topological embedding. Then ***G*** has a straight-line drawing if and only if ***G*** has no gucci subgraph and no bulgari subgraph.

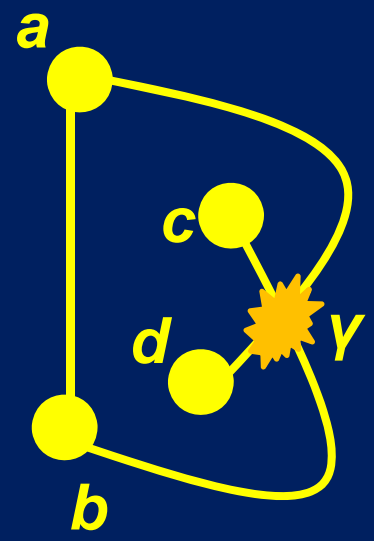


Theorem (Hong, Liotta, Poon, Eades, 2011)

Suppose that ***G*** is a 1-planar topological embedding. Then ***G*** has a straight-line drawing if and only if ***G*** has no gucci subgraph and no bulgari subgraph.

Proof

- *Necessity* for bulgari graph:
  - Angles in the triangle ***abγ*** add up to ***π***
  - Thus 3 angles at ***γ*** add up to strictly less than ***π***
  - This is impossible if ***ad*** and ***bc*** are straight lines.
- Similar argument for the gucci graph



Theorem (Hong, Liotta, Poon, Eades, 2011)

Suppose that ***G*** is a 1-planar topological embedding. Then ***G*** has a straight-line drawing if and only if ***G*** has no gucci subgraph and no bulgari subgraph.

Proof

- *Sufficiency*
  - Much more complicated
  - Provides an algorithm:

    Input: a 1-planar topological embedding ***G*** with no gucci subgraph and no bulgari subgraph.

    Output: a straight line drawing of ***G***

Algorithm for Straight-line 1-planar graph drawing

Input: a 1-planar topological embedding ***G***

Output: a straight line drawing of ***G***, if it exists

1. *Testing:* Test whether ***G*** has a gucci subgraph or a bulgari subgraph.
2. *Augmentation:* Add non-crossing edges to make an augmented graph G^+, with higher connectivity.
3. *Drawing*: Apply a convex drawing algorithm together with an SPQR trees to G^+.

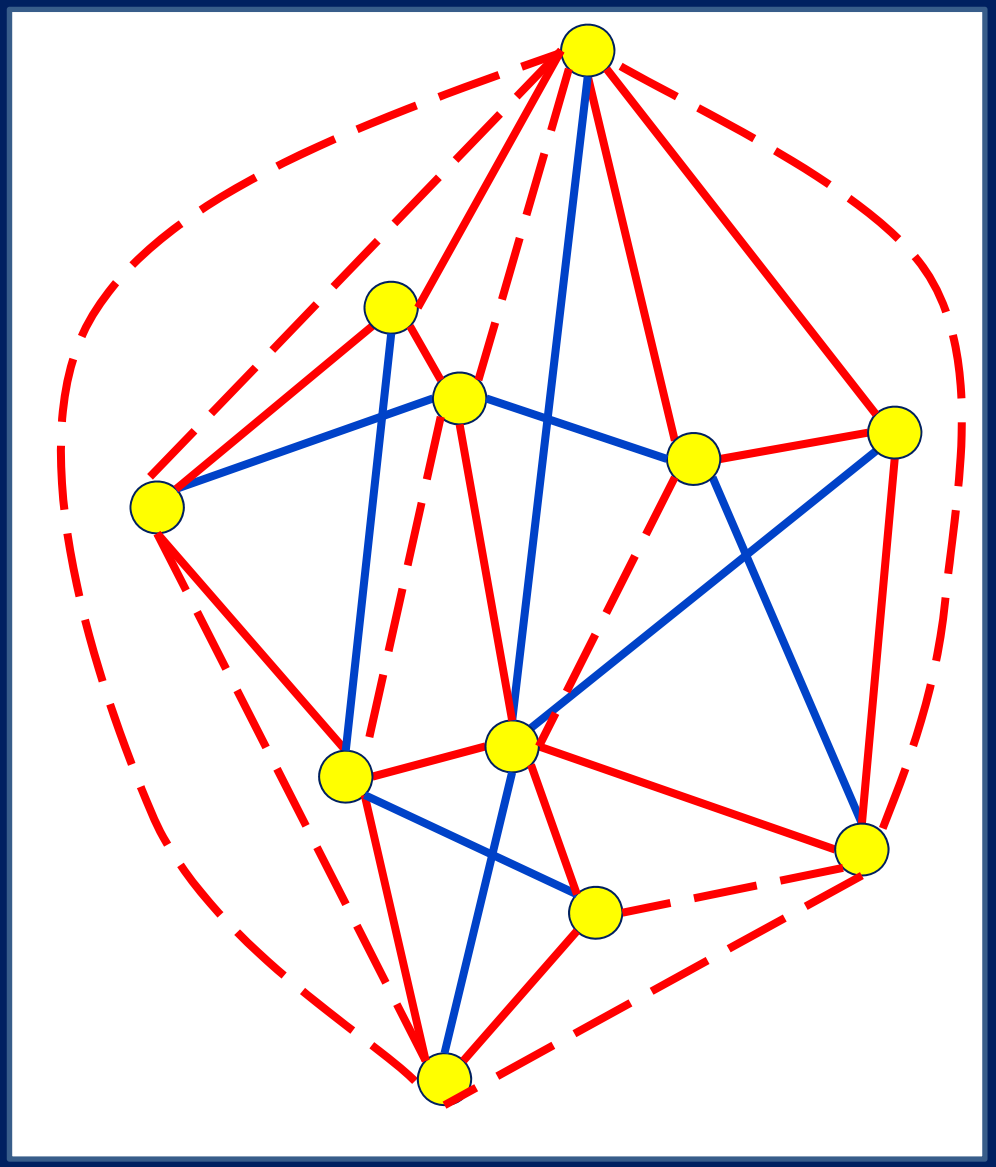
1. *Testing:* Test whether ***G*** has a gucci subgraph and no bulgari subgraph.
2. *Augmentation:* Add non-crossing edges to make an augmented graph G^+, with higher connectivity.
3. *Drawing*: Use a convex drawing algorithm together with an SPQR trees to G^+.

Red and blue edges

- *Red* edges: no crossings
- *Blue* edges: 1 crossing

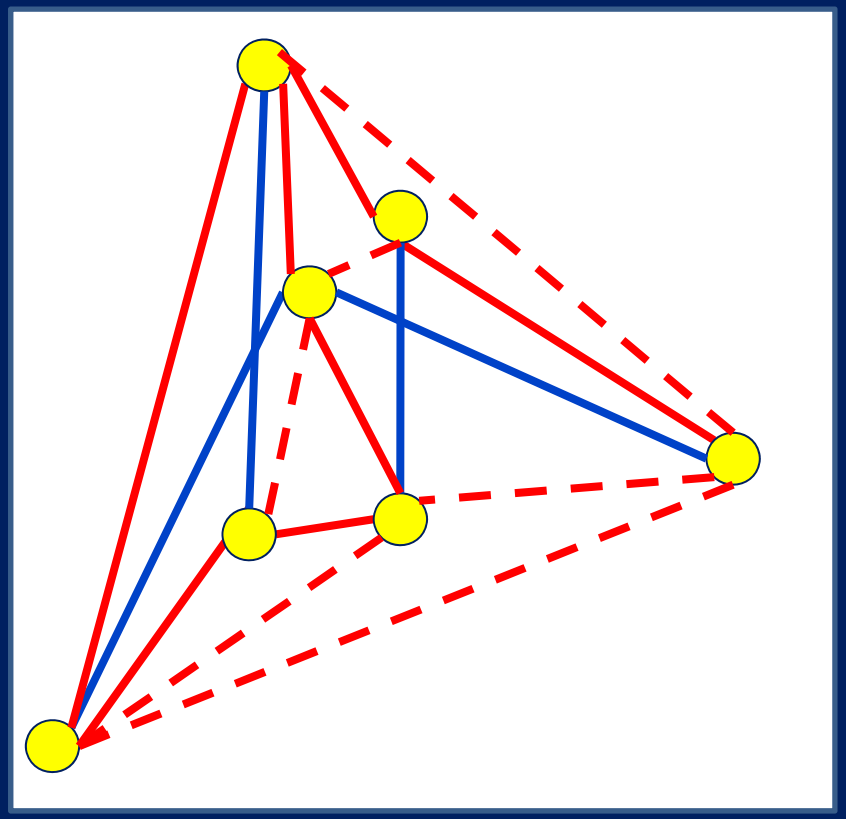
Red-maximal:

- A graph is *red-maximal* if adding an edge causes a crossing
- ie, a graph is *red-maximal* if whenever a pair ***a, b*** of vertices share a face, then ***(a,b)*** is an edge.



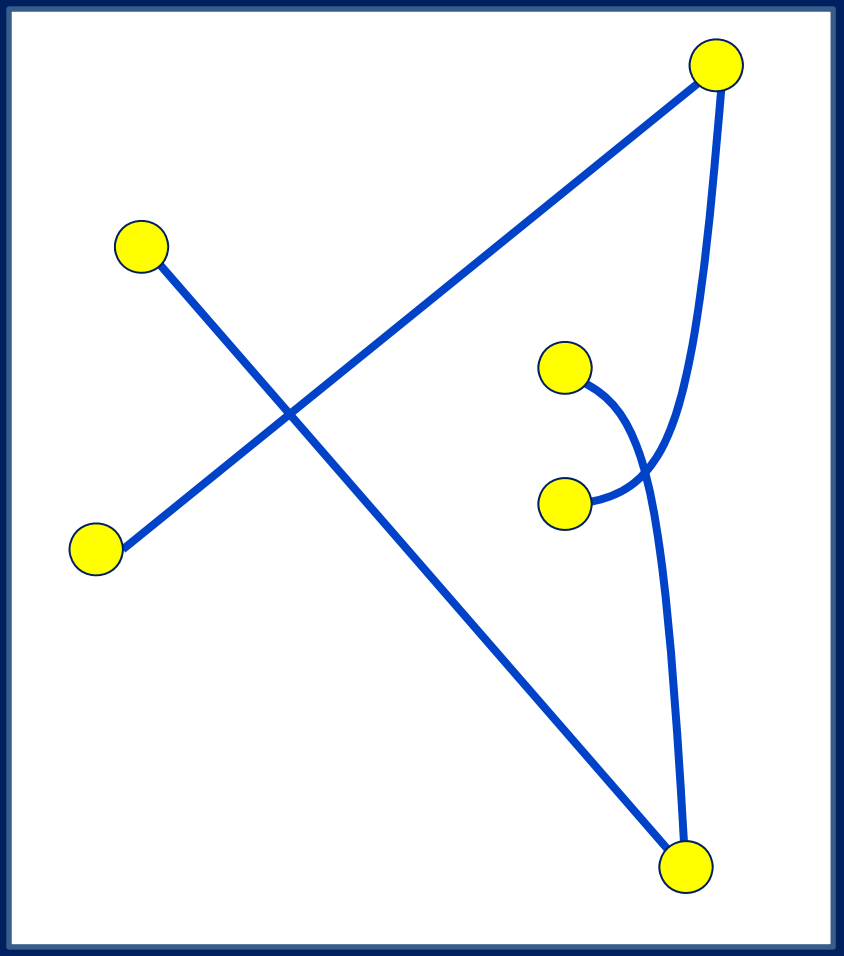
Naive augmentation algorithm
to obtain red-maximality:

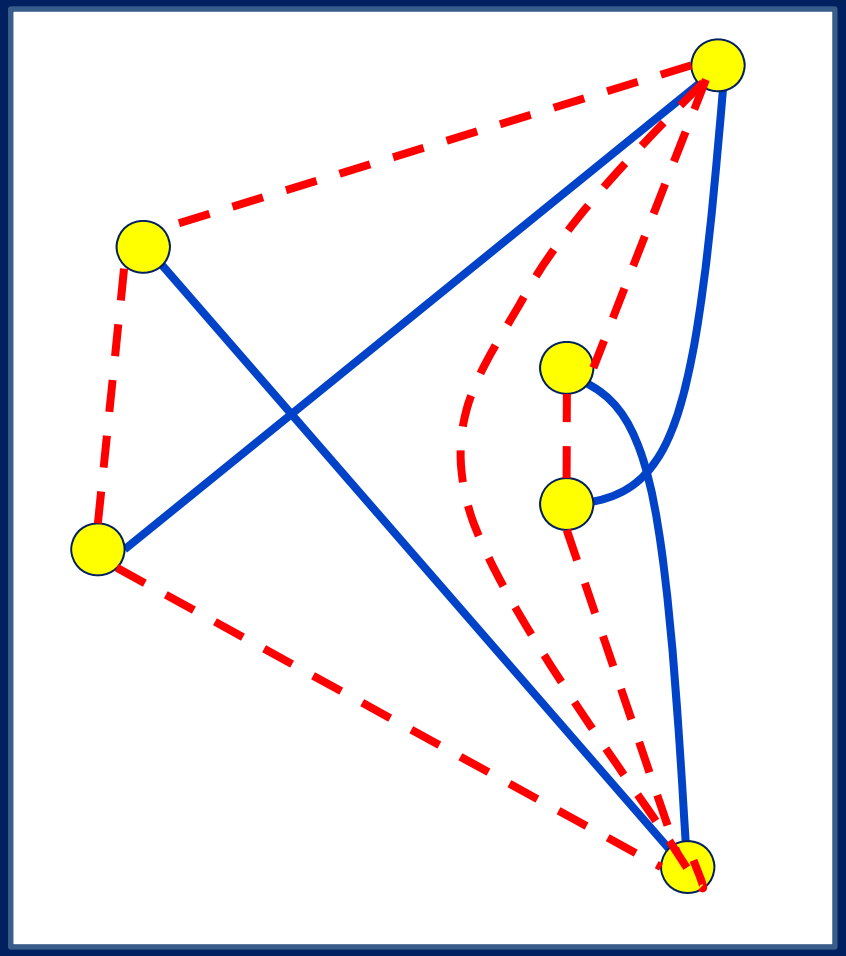
For each nonadjacent pair
a,b *of vertices that share*
a face, add the edge
(a,b)



- But: the naive algorithm does not work ... ...

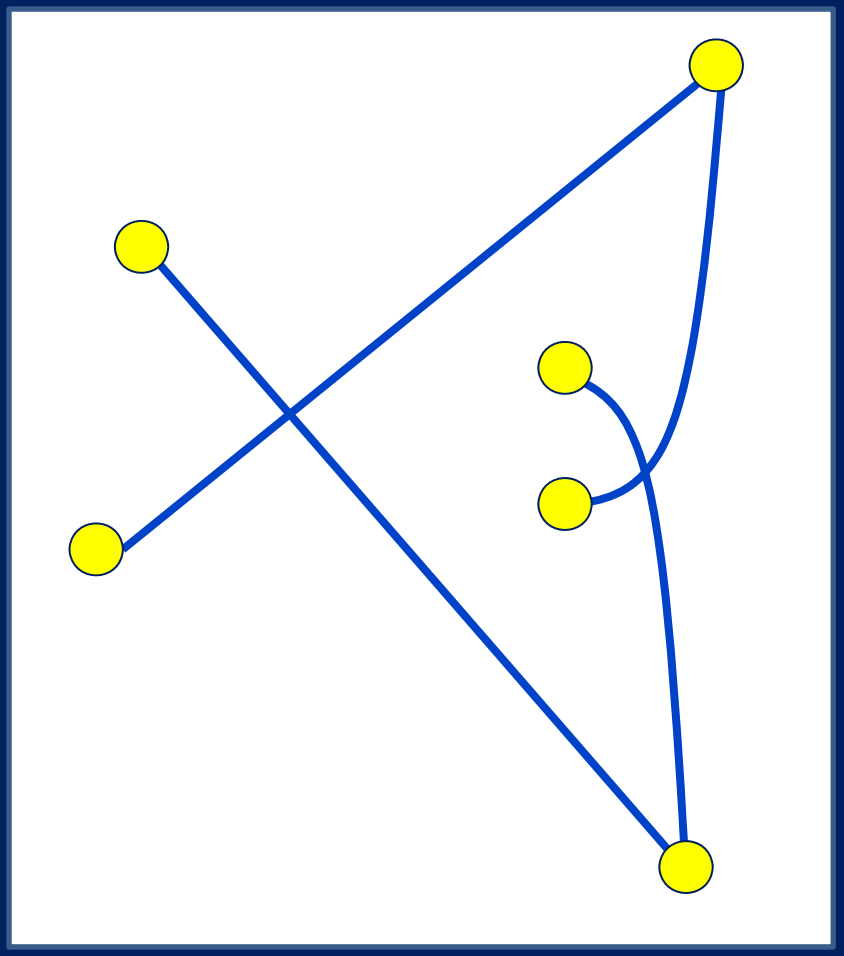
Some augmentations may not be correct

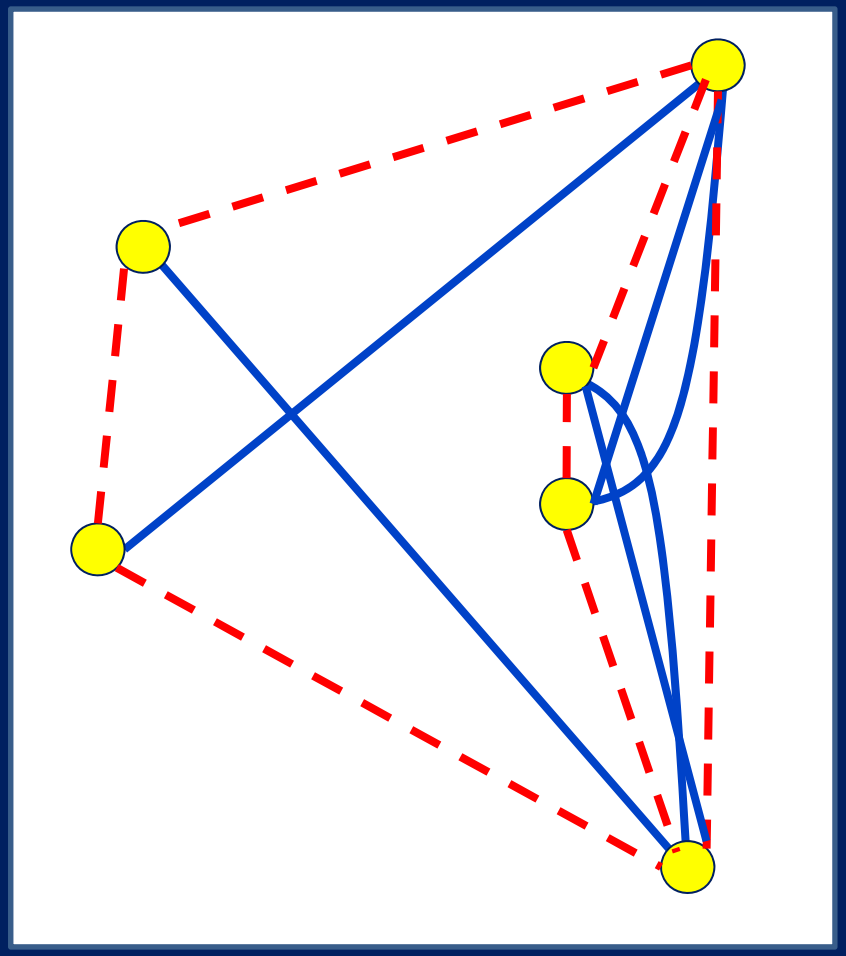




Augmentation 1: introduces a bulgari subgraph

Some augmentations may not be correct





Augmentation 2: no bulgari subgraph

1. *Testing:* Test whether ***G*** has a gucci subgraph and no bulgari subgraph.
2. *Augmentation:* Add non-crossing edges to make an augmented graph G^+, with higher connectivity.
3. *Drawing*: Use a convex drawing algorithm together with an SPQR trees to G^+.

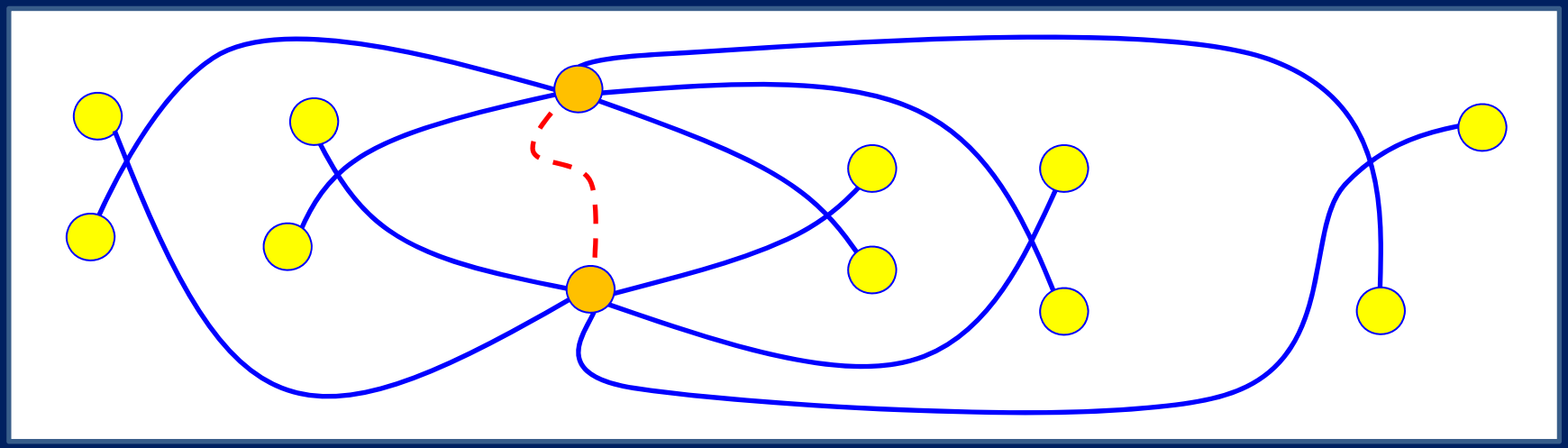
Problem:

- Naive algorithm does not work.
- Red-maximal supergraph is not unique
  - We need to choose a red-maximal supergraph with no bulgari/gucci subgraphs

1. *Testing:* Test whether ***G*** has a gucci subgraph and no bulgari subgraph.
2. *Augmentation:* Add non-crossing edges to make an augmented graph G^+, with higher connectivity.
3. *Drawing*: Use a convex drawing algorithm together with an SPQR trees to G^+.

Augmentation algorithm

- Augmentation can be done an edge at a time, but when two crossings share two endpoints, we must be carefully order the edge insertions.
- Can be done in linear time.



1. *Testing:* Test whether ***G*** has a gucci subgraph and no bulgari subgraph.
2. *Augmentation:* Add non-crossing edges to make an augmented graph G^+, with higher connectivity.
3. *Drawing*: Use a convex drawing algorithm together with an SPQR trees to G^+.

Drawing Algorithm

If the red subgraph ***R*** of G^+ is triconnected

Then

- Draw ***R*** using a convex drawing algorithm
- Insert crossing edges

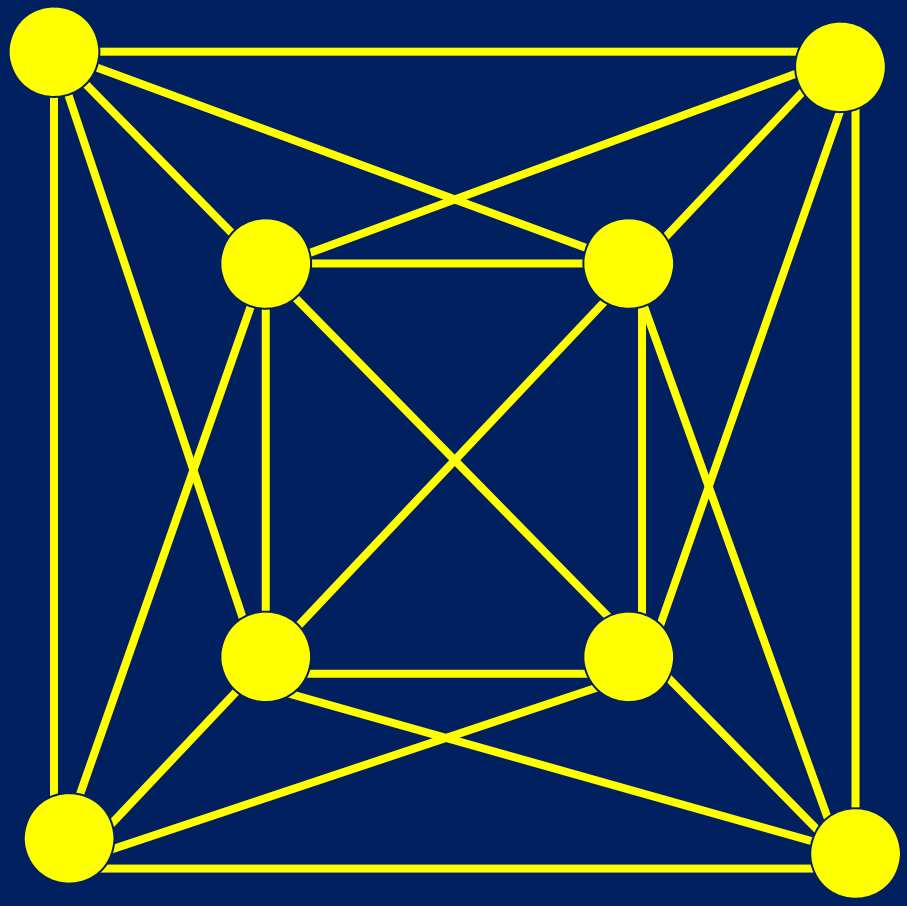
Else

- Use SPQR tree in a recursive way to build a drawing.

Relationships between the various classes of slightly non-planar graphs:-

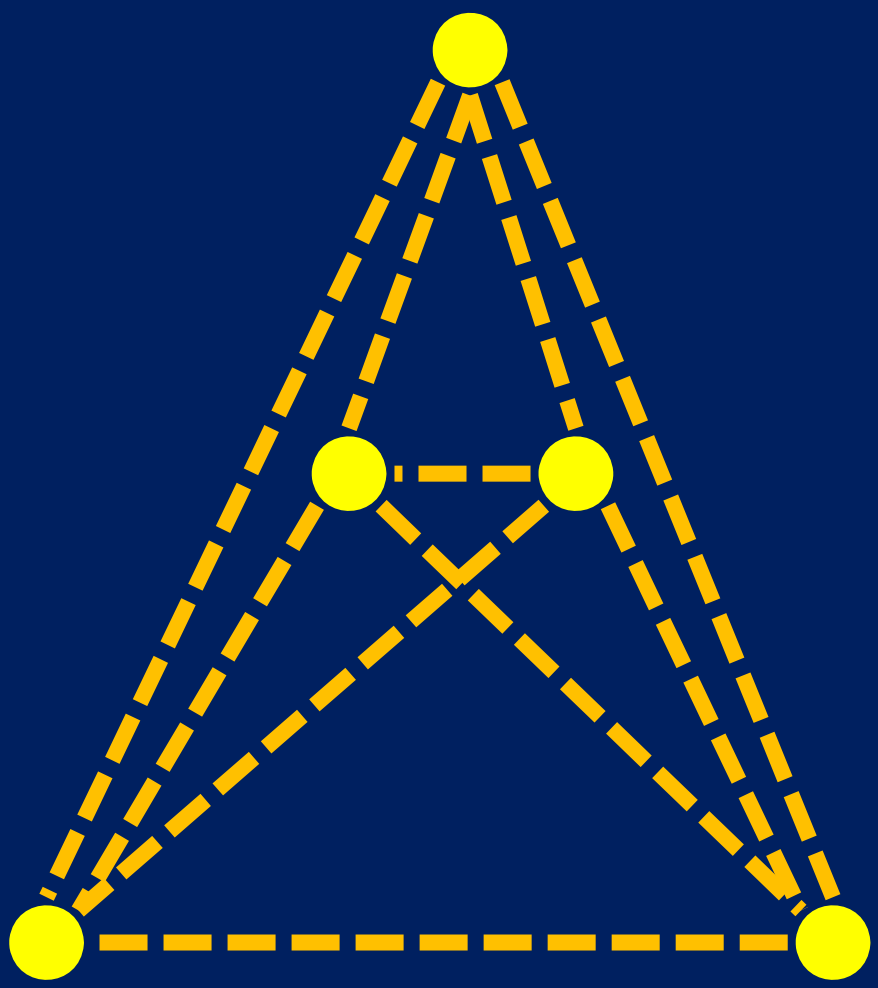
Observation1

There are 1-planar graphs that are not RAC.

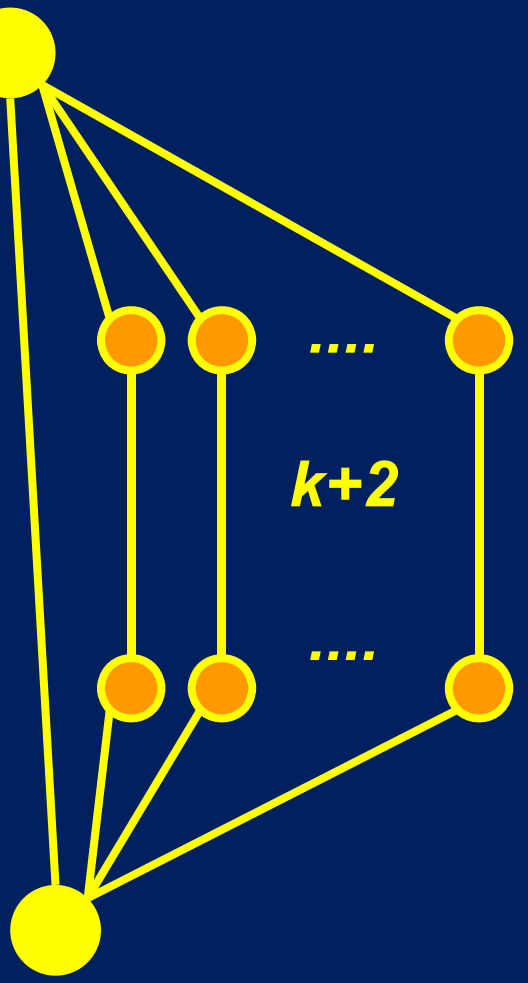


Observation 2

There are RAC graphs that are not 1-planar.

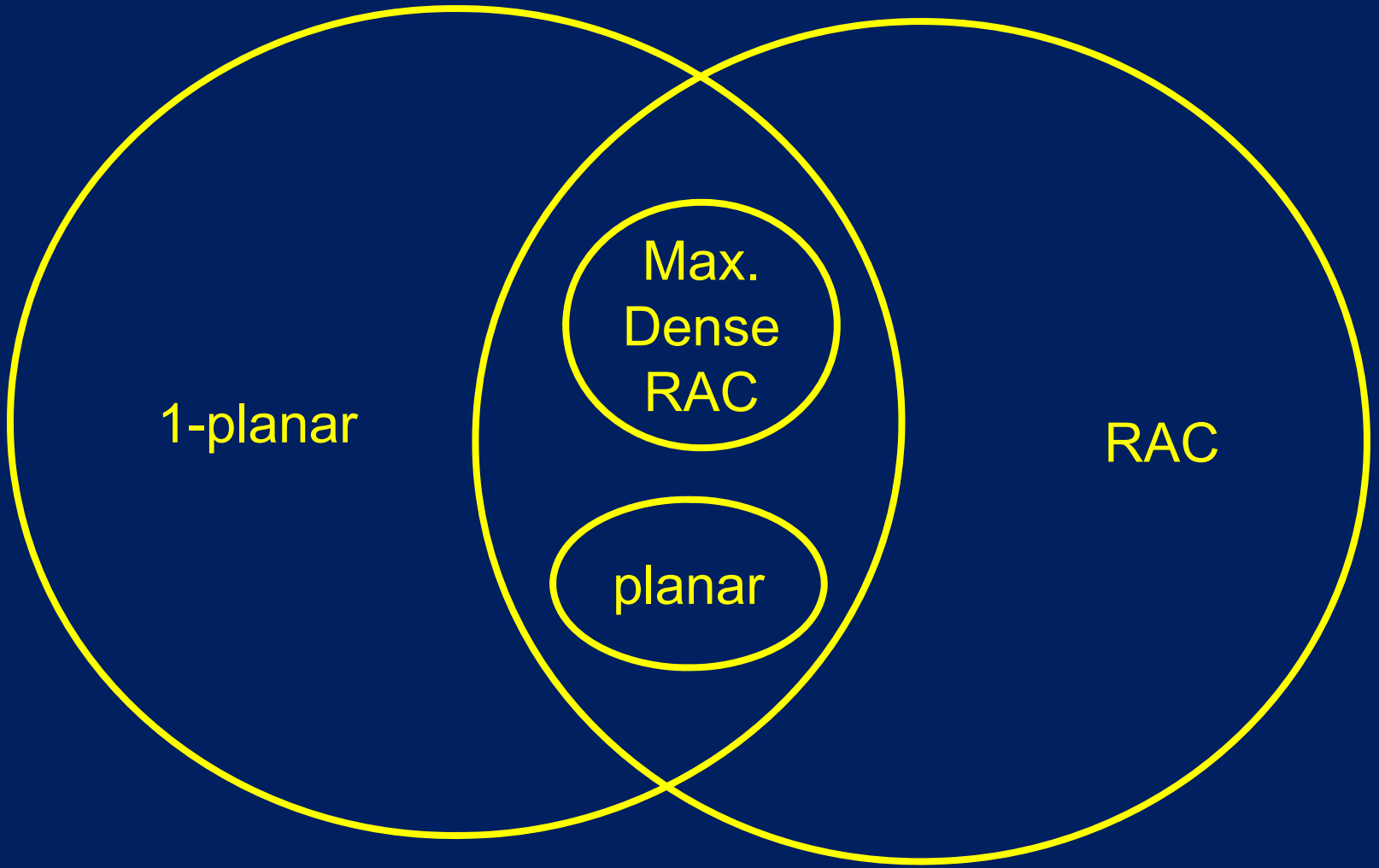


Replace each edge in K_5 with:



Theorem (Liotta, Eades)

If ***G*** is RAC and has ***4n-10*** edges, then ***G*** is 1-planar.



Final remarks

Other classes of *slightly-non-planar* graphs

- *2-planar*: each edge has at most 2 crossings
- *Quasi-planar*: no three edges mutually cross each other
- *Fan-crossing-free*: no pair of incident edges cross another edge
- *Twist-crossing-free*: no path of length 3 crosses itself
- *1-skew*: deletion of one edge makes it planar

Open problems for *curious* people

Class	***Maximum density***	***Computing topology***	***Computing straight-line drawing***
Planar	3n-6	Linear time	Linear time
1-planar	4n-8	NP-hard	Linear time*
Fan-crossing-free	?	?	?
Twist-crossing-free	?	?	?
Quasi-planar	6.5n-20	?	?
Right-angle-crossing (RAC)	4n-10	NP-hard	?
1-skew	3n-5	Quadratic(?)	Linear time*
...	...	...	...

Wild conjecture

Suppose that ***H*** is a topological graph, other than a pair of crossing edges.

Then the following problem is NP-hard.

*Instance: A graph **G***

*Question: Does **G** have a topological embedding in which **H** is not a subgraph?*

Open problems for *practical* people with some mathematical skills

We say that a graph drawing ***D*** has *crossing resolution* ***Θ(D)*** if each crossing angle is at least ***Θ.***

1. Investigate graph drawings ***D*** with ***Θ(D) ≥ π/3.***

2. Consider the following problem:

   Maximum Crossing Resolution
   Input: a graph G
   Output: a drawing D of G with maximum crossing resolution ***Θ(D).***

   Isolate the continuous and discrete parts of this optimisation problem.

Open problems for people with some HCI/Psych skills

Investigate the perceptual and mathematical relationships between crossing resolution and the number of crossings.

Open problems for practical people with lots of mathematical skills:

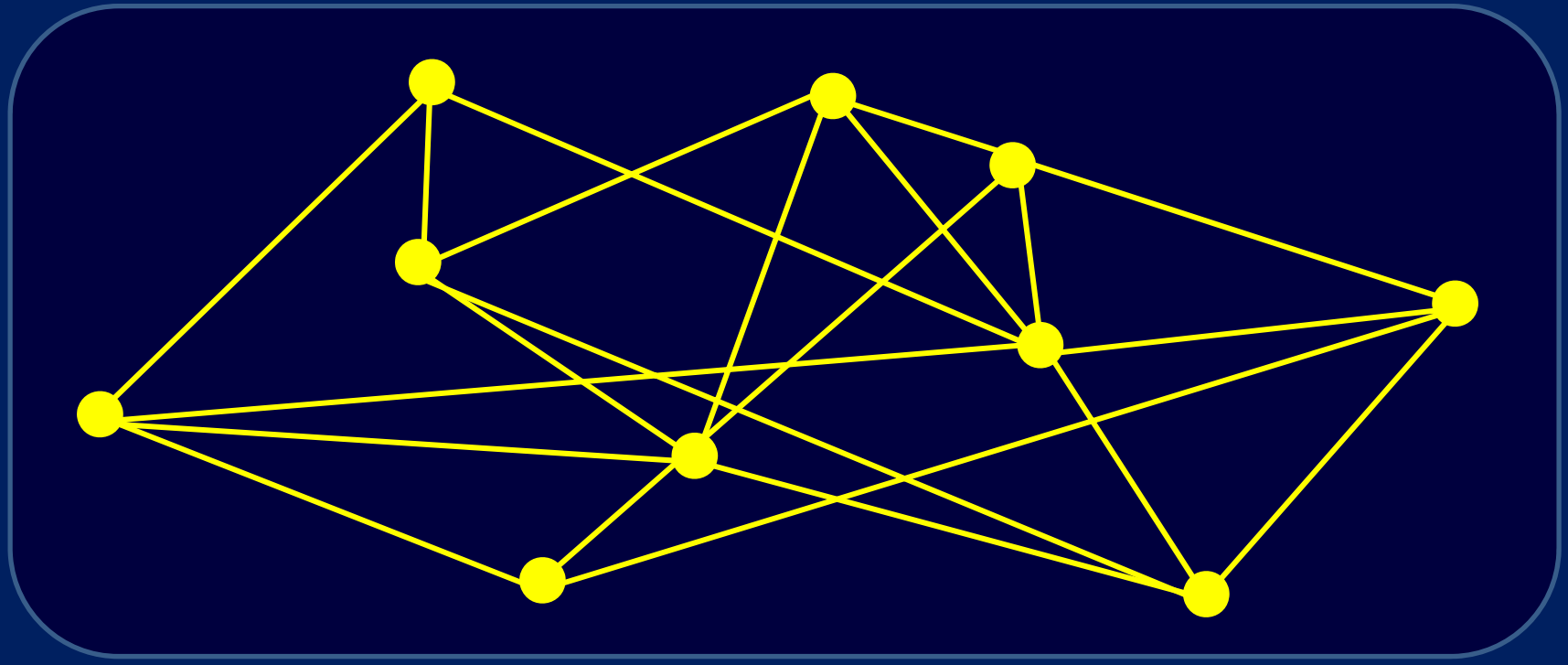
Randomisation

- In a “random” graph drawing with “short” edges:
  - How many edge crossings?
  - What is the crossing resolution?
  - What is the vertex resolution?
  - How do these parameters vary with graph theoretic parameters, such as diameter?

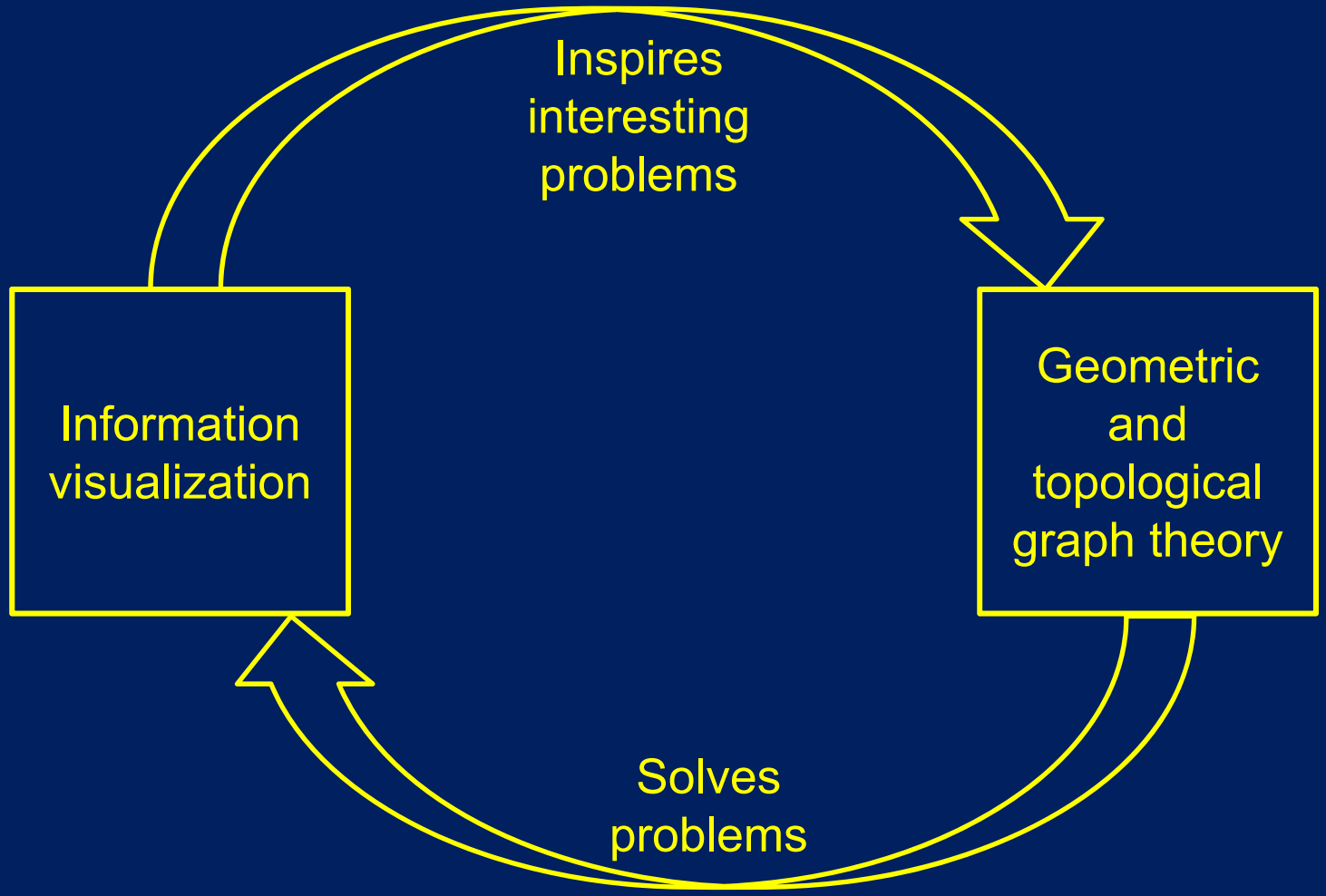
- Implications for graph drawing algorithms
  - Can we design algorithms that exploit randomization?
  - Can we provide stochastic guarantees of performance?

Open problem for computational geometry people:-

- Given a straight-line graph drawing, what is the smallest crossing angle?

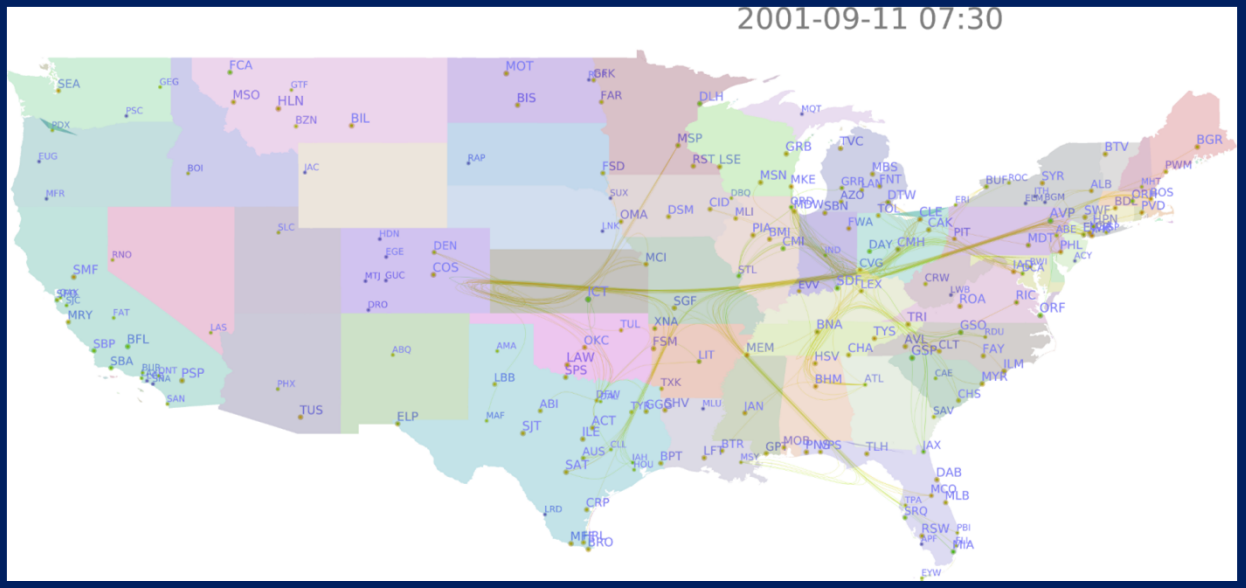


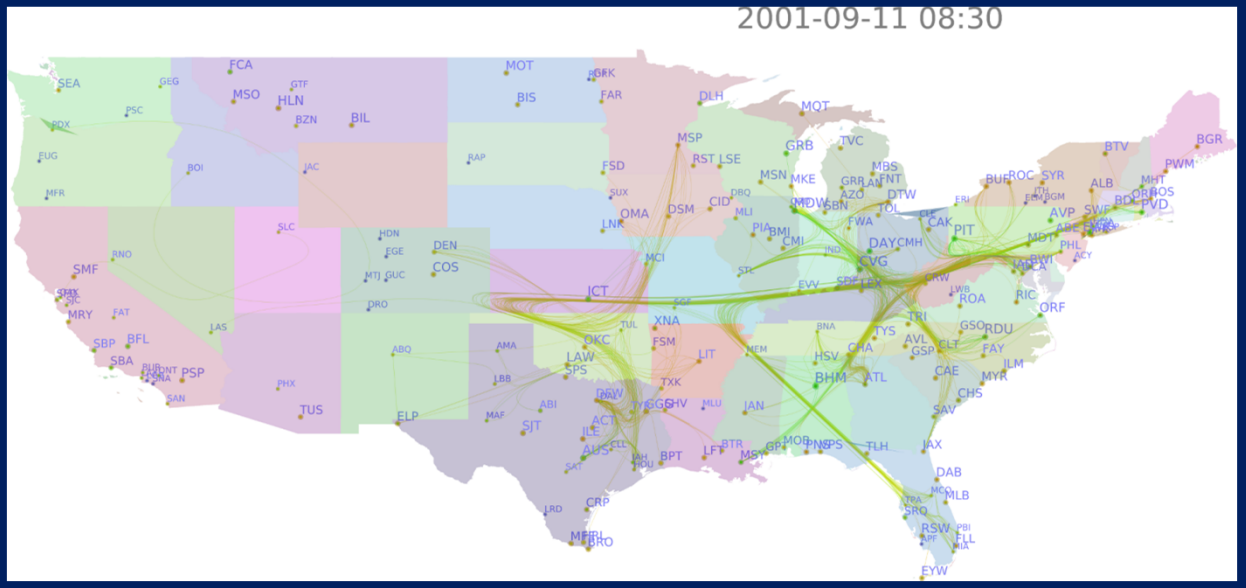
General open problems



Play video …

Quan Nguyen: edge bundling problems

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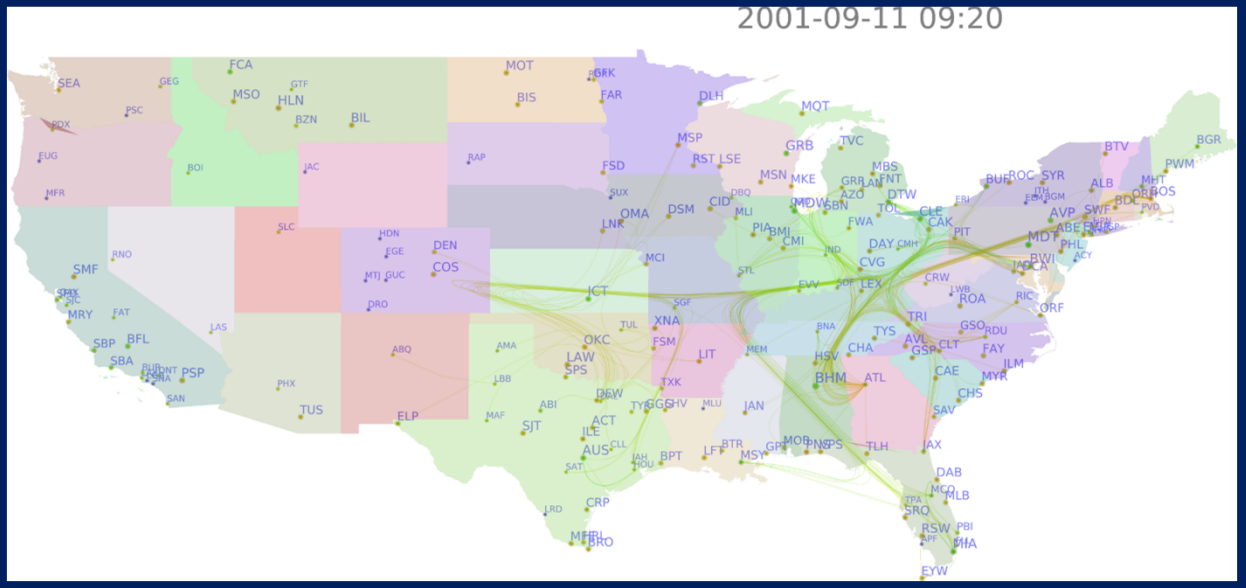
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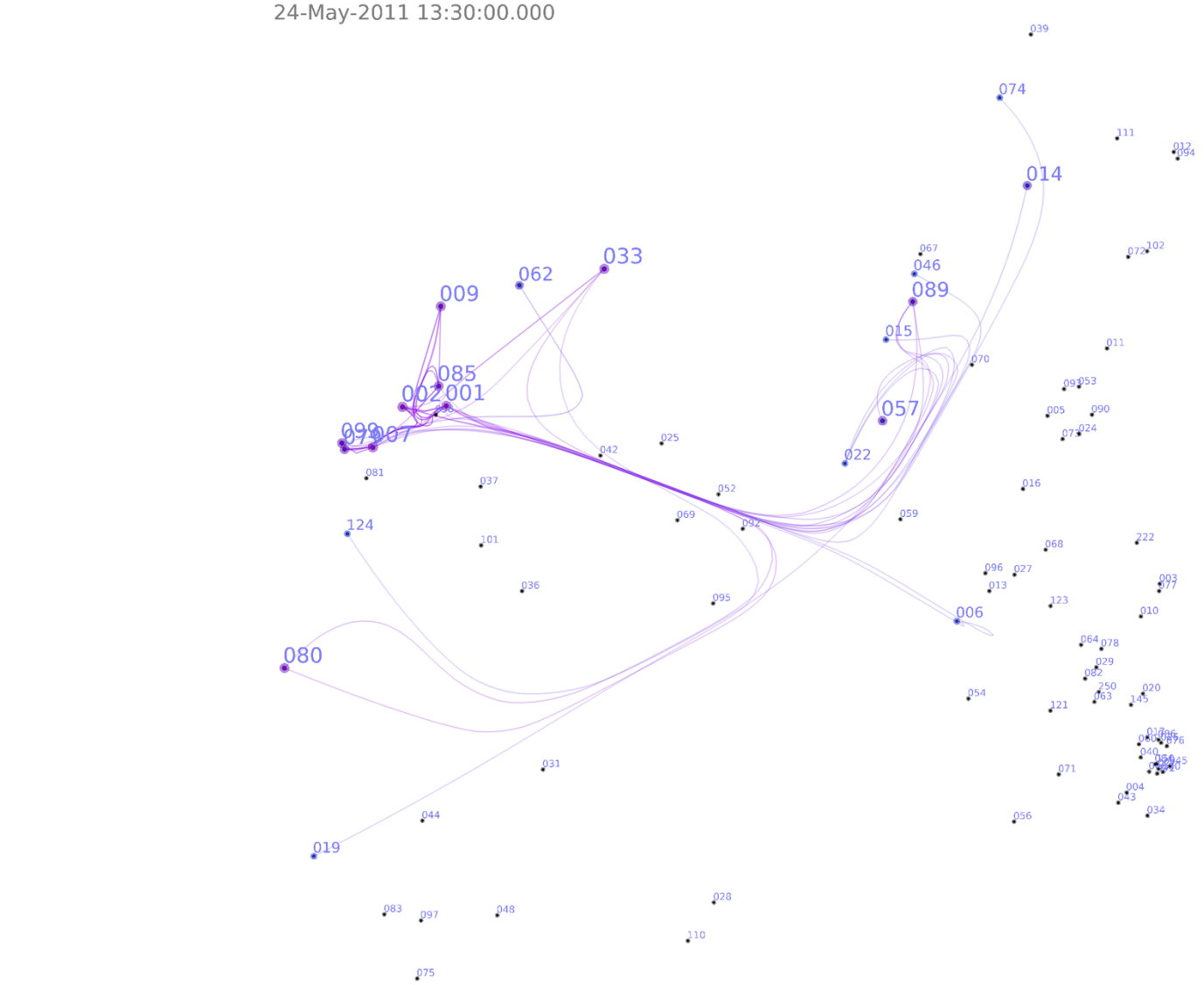
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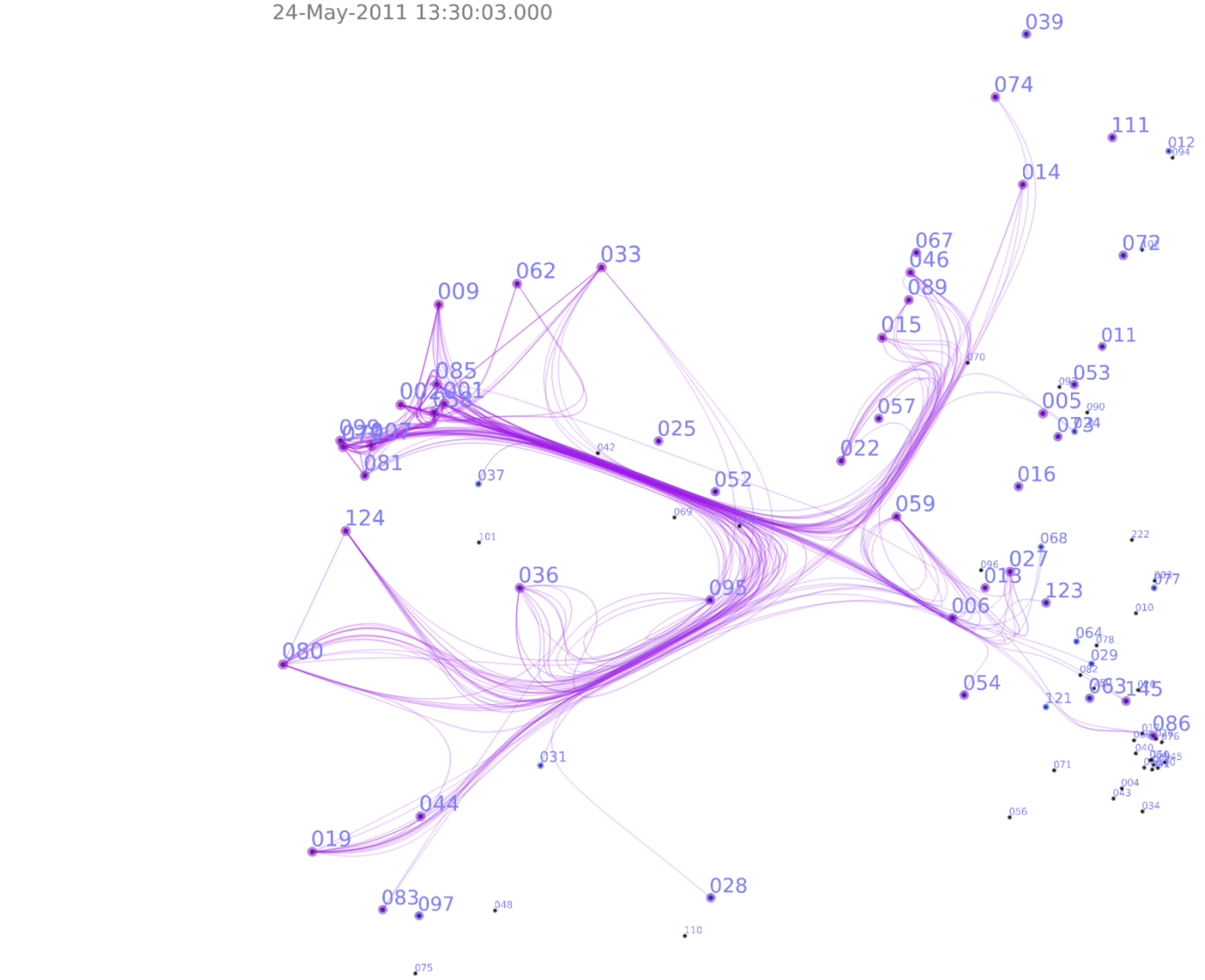
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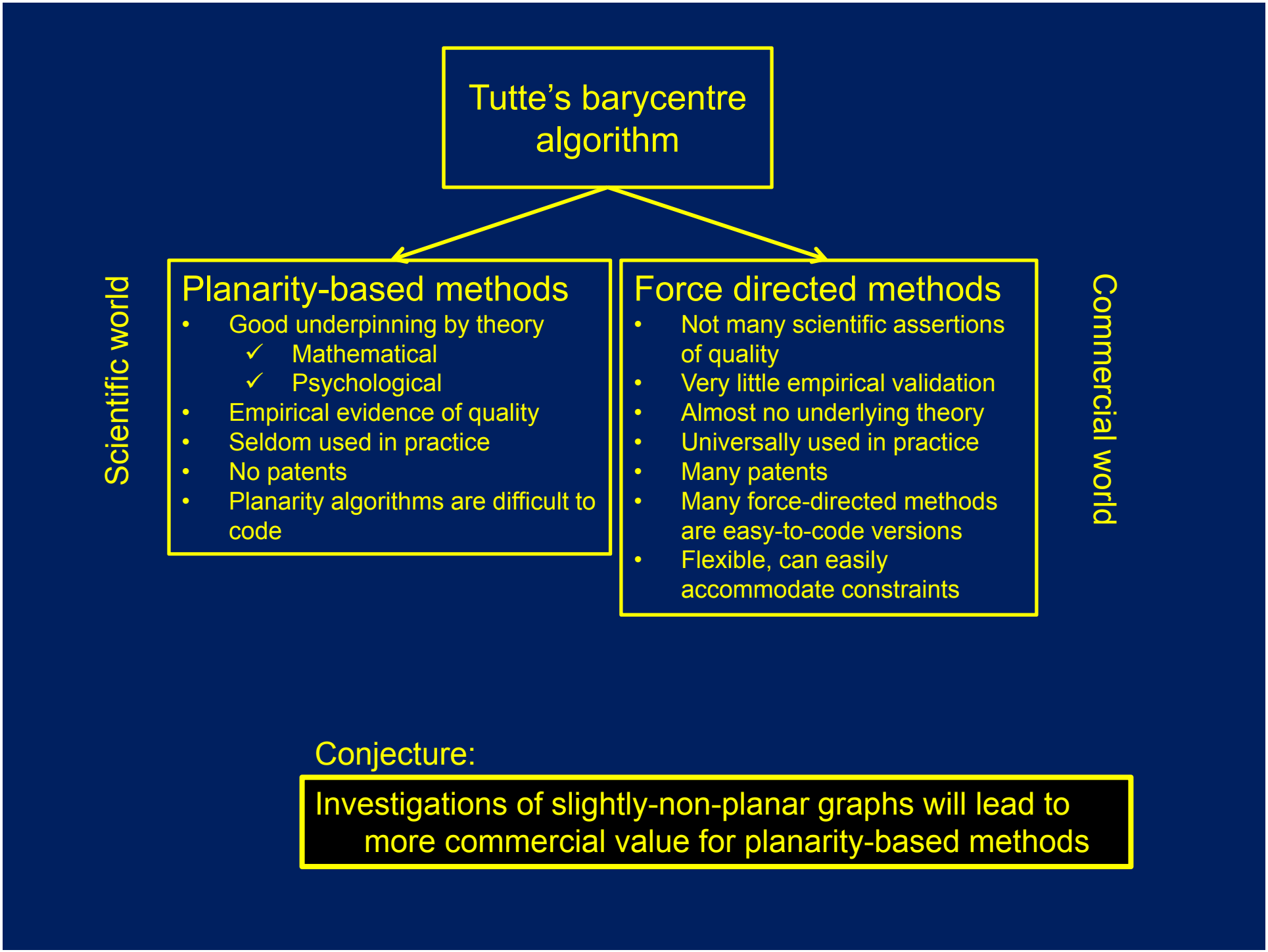
Problems:

- What mathematically-sound measures of edge bundling pictures define good visualizations?
- What algorithms can be used to optimise these measures?
- What are the mathematical limits on geometric graphs with as a function of these measures?

Final final conjecture ….


Tutte's barycentre algorithm
Scientific world
Planarity-based methods
• Good underpinning by theory
✓ Mathematical
✓ Psychological
• Empirical evidence of quality
• Seldom used in practice
• No patents
• Planarity algorithms are difficult to code
Force directed methods
• Not many scientific assertions of quality
• Very little empirical validation
• Almost no underlying theory
• Universally used in practice
• Many patents
• Many force-directed methods are easy-to-code versions
• Flexible, can easily accommodate constraints
Commercial world
Conjecture:
Investigations of slightly-non-planar graphs will lead to more commercial value for planarity-based methods