k-dismantlability in graphs

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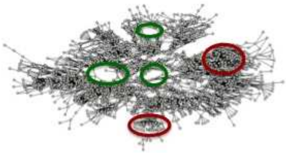
Between graph theory and complex networks

- For analyzing complex networks, most of the tools focus on dense parts (= communities). We propose to look at some non-dense parts - "holes" - and the way they are organized in the network.
- Sociological concept -> "structural holes" of R.S. Burt (1982) which are places with a low density of links and that an individual must hold to increase his influence.



Between graph theory and complex networks

- For analyzing complex networks, most of the tools focus on dense parts (= communities). We propose to look at some non-dense parts - "holes" - and the way they are organized in the network.
- Sociological concept -> "structural holes" of R.S. Burt (1982) which are places with a low density of links and that an individual must hold to increase his influence.
- The idea is to peel the graph, vertex after vertex, to reduce the network to the skeleton of its holes. A vertex will be "peelable" if its neighborhood verify some given properties.

The aim of this talk is to explore several mathematical ways of pealing.

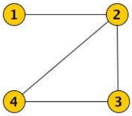
Notations

- G denote a finite undirected graph without loop.
- The open neighborhood of a vertex i in G : $N_G(i) = \{j; j \sim i\}$
- The closed neighborhood of a vertex i in G : $N_G[i] = N_G(i) \cup \{i\}$

Definition

A vertex i is dominated in G if there exists $j \neq i$ such that $N_G[i] \subseteq N_G[j]$. We note $i \vdash j$.

Example :



We have $N_G[4] = \{2, 3, 4\} \subset N_G[2] = V(G)$ and then $4 \vdash 2$. Note for example that $1 \nvdash 3$

Let us now explore some examples of peeling and their relations to cycles in graphs...

From simplicial to 1-dismantlable graph

Simplicial vertex

Definition

- A vertex i is simplicial if $N_G[i]$ is complete.
- A graph G is simplicial if there is a linear ordering $1, 2, \cdots, n$ of its vertices st. $i < n$ is simplicial in $G - \{1, 2, \cdots, i-1\}$.

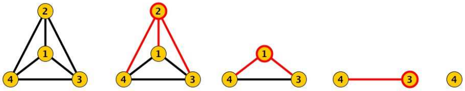
A graph G is chordal if it contains no induced cycle of length ≥ 4

Theorem (Dirac, 1961)

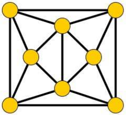
A finite graph is chordal iff it is simplicial.

Idea of the proof : by induction on the number of vertices of G since if G is chordal then $G - v$ is also chordal. ■

Example :



no simplicial vertex



From simplicial to 1-dismantlable graph

Isometric vertex

Definition

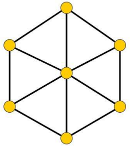
- A vertex i is isometric if the distances between the vertices of $G - i$ are equal to those between corresponding vertices in G.
- A graph G is isometric if there is a linear ordering $1, 2, \cdots, n$ of its vertices st. $i < n$ is isometric in $G - \{1, 2, \cdots, i-1\}$.

A graph G is bridged if any cycle C of length ≥ 4 has a shortcut (*ie.* a pair of vertices whose distance in G is strictly smaller than in C)

Theorem (Anstee and Farber, 1988)

A finite graph is bridged iff it is isometric and has no induced C_4 or C_5.

Remark : Chordal $\Rightarrow$ Bridged since there is no induced cycles of length ≥ 4 in a chordal graph.



From simplicial to 1-dismantlable graph

Dismantlable vertex

Definition

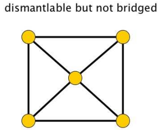
- A vertex i is dismantlable in G if it is dominated in G.
- A graph G is dismantlable if there is a linear ordering $1, 2, \cdots, n$ of its vertices st. $i < n$ is dismantlable in $G - \{1, 2, \cdots, i-1\}$.

The Cop-Rob game : The players begin the game by selecting their initial positions in the graph (the cop must choose his vertex first). They then move alternatively, according to the following rule : a player at vertex i can either remain at i or move to any neighbour of i. The cop wins when the cop and robber occupy the same vertex.

Theorem (Quillot, 1983 ; Nowakowski and Winkler, 1983)

A finite graph is dismantlable iff it is cop-win.

Remark : Bridged $\Rightarrow$ Dismantlable
(Anstee and Farber, 1988)



From simplicial to 1-dismantlable graph

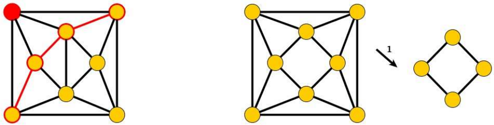
1-dismantlable vertex

Definition (Boulet Fieux J., 2008, 2010)

- A vertex i is 1-dismantlable if $N_G(i)$ is dismantlable.
- A graph G is 1-dismantlable if there is an ordering $1, 2, \cdots, n$ of its vertices st. $i < n$ is 1-dismantlable in $G - \{1, 2, \cdots, i-1\}$

Example :

1-dismantlable but not dismantlable



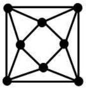
G :	simplicial	$\Rightarrow$	isometric	$\Rightarrow$	dismantlable	$\Rightarrow$	1-dismantlable
	chordal	$\Rightarrow$	bridge	$\Rightarrow$	cop-win	$\Rightarrow$	? ? ?
$N(i)$:	complete	$\Rightarrow$	convex	$\Leftarrow$	cone	$\Rightarrow$	dismantlable

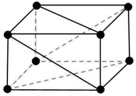
Remark : A subgraph H is convex if H includes every shortest path with end-vertices in H

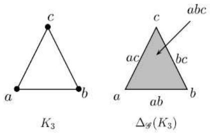
1-dismantlability and simplicial complexes

- A simplicial complex K, with vertex set V, is a collection of finite non empty subsets σ of V (the simplices) s.t. : $V = \bigcup_{\sigma \in K} \sigma$ and if $(\sigma \in K, x \in \sigma)$ then $\sigma - \{x\} \in K$.
- $\Delta(G)$ denote the simplicial complex whose k-simplices are the complete subgraphs with k vertices (flag complexes)
- An elementary reduction in $\Delta(G)$ is the suppression of a pair of simplices (σ, τ) with τ a proper maximal face of σ and τ is not a face of another simplex.

Example :







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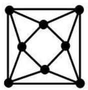

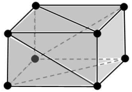

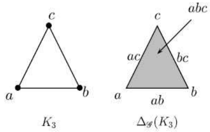

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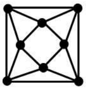
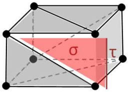
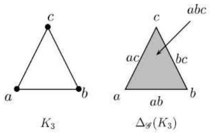






1-dismantlability

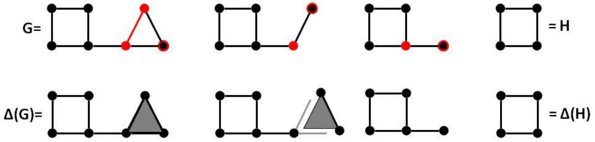
Proposition [2008]

$G \searrow^1 H \Rightarrow \Delta(G) \searrow^1 \Delta(H)$.

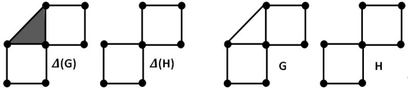
Consequence

A necessary condition for G to be 1-dismantlable is that $\Delta(G)$ is collapsible.

Example :



The converse implication is false :



1-dismantlability

Proposition [2008]

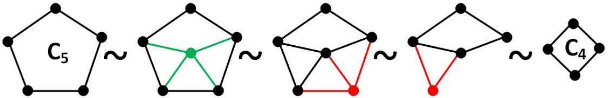
$G \searrow^1 H \Rightarrow \Delta(G) \searrow^1 \Delta(H)$.

We say that G et H have the same 1-homotopy type if there exists a sequence of graphs $G = J_1, J_2, \cdots, J_{k-1}, J_k = H$ from G to H such that $G = J_1 \xrightarrow{1} J_2 \xrightarrow{1} \ldots \xrightarrow{1} J_{k-1} \xrightarrow{1} J_k = H$ where $\xrightarrow{1}$ is the addition or the deletion of a 1-dismantlable vertex. We note $[G]_1$ the 1-homotopy type of G. In the same way we define the 1-homotopy type of $\Delta(G)$ also called simple-homotopy type in topology.

Proposition [2010]

$[G]_1 = [H]_1 \iff [\Delta(G)]_1 = [\Delta(H)]_1$.

Example : The $C_{n\geq 4}$ have the 1-homotopy type of C_4.



k-dismantlability

Even if we don't have a good characterization of the graphs that are 1-dismantlable, the link with topology of flag complexes is interesting. So, we have explored the case where we have weakened the condition of 1-dismantlability by a condition of k-dismantlablity.

Definition

- A vertex i is k-dismantlable if $N_G(i)$ is $(k-1)$-dismantlable.
- A graph G is k-dismantlable if there is an ordering $1, 2, \cdots, n$ of its vertices st. $i < n$ is k-dismantlable in $G - \{1, 2, \cdots, i-1\}$

We denote $D_k(G)$ the set of vertices of G which are k-dismantlable in G, D_k the set of the k-dismantlable graphs and $D_\infty = \bigcup_{k \geq 0} D_k$.

Proposition

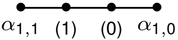
The sequence $(D_k)_{k \geq 1}$ is strictly increasing :

$$D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \ldots\ldots \subsetneq D_k \subsetneq D_{k+1} \subsetneq \ldots\ldots$$

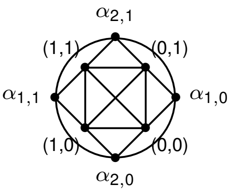
k-dismantlability

Proof :

- By induction on k, we have $D_k(G) \subseteq D_{k+1}(G)$ for all $k \geq 0$ and so a k-dismantlable ordering of G is also a $(k+1)$-dismantlable ordering.
- For the strict inclusion, we construct a sequence of graphs $(\mathfrak{Q}_n)_{n\geq 0}$, the n-cubions, with the property that $\forall n \geq 2, \mathfrak{Q}_n \in D_{n-1} \setminus D_{n-2}$.
  $V(\mathfrak{Q}_n) = \{\alpha_{i,\epsilon}, i = 1, \cdots, n \text{ and } \epsilon = 0, 1\} \cup \{x = (x_1, \cdots, x_n), x_i = 0, 1\}$
  $E(\mathfrak{Q}_n)$ defined by :
  • $\forall i \neq j, \alpha_{i,\epsilon} \sim \alpha_{j,\epsilon'}$
  • $\forall x \neq x', x \sim x'$
  • $\forall i, \alpha_{i,1} \sim (x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n), \alpha_{i,0} \sim (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n)$.



$\mathfrak{Q}_1 \in D_0$



$\mathfrak{Q}_2 \in D_1 \setminus D_0$

Properties :

- $\mathfrak{Q}_n$ has $2^n + 2n$ vertices
- $\mathfrak{Q}_n[\alpha_{1,0}, \alpha_{1,1}, \cdots, \alpha_{n,0}, \alpha_{n,1}] \cong \overline{nK_2}$
- $\mathfrak{Q}_n[x, y, \cdots] \cong K_{2^n}$

Elements of the proof :

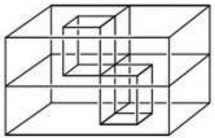
By induction, we note that

- $N_{\mathfrak{Q}_n}(\alpha_{i,\epsilon}) \cong \mathfrak{Q}_{n-1} \notin D_{n-3}$
- $N_{\mathfrak{Q}_n}(x) \searrow^0 \overline{(n-1)K_2} \notin D_\infty$

k-dismantlability

Some properties :

- The previous example of a "quasi"-wheel shows that the order of k-dismantling is important for $k \geq 1$ at least when the graph G is not k-dismantlable.
- if $D_k(G) \setminus D_{k-1}(G) \neq \emptyset$ with $k \geq 1$ then $\omega(G) \geq k+2$ where $\omega(G)$ is the number of vertices of a maximal clique of G.
  Proof : X contains at least one complete subgraph K_{k+2}. ■
- $\forall k \geq 2, \quad G \in D_k \;\Rightarrow\; [G]_{k-1} = [pt]_{k-1}$
  Proof : We proved in BFJ(2010) that if $x \in D_2(G)$ then $[G]_1 = [G - x]_1$. By iterating the process we prove that the proposition is true for $k = 2$ and by induction on k for $k \geq 2$. ■
  The converse is false : Bing's House.



k-dismantlability and transitivity

Definition

- A graph G is vertex-transitive if $\forall\, v, w \in V(G),\ \exists\, g \in \mathrm{Aut}(G), g.v = w$

Theorem

If G is a 0-dismantlable and vertex-transitive, then G is a complete graph.

Proof : Given an order $\{1, \cdots, n\}$ of 0-dismantlings, using the transitivity of G we prove by induction on i that $N_{G-\{1,\cdots,i-1\}}(i) \subset N_{G-\{1,\cdots,i-1\}}(j)$ implies that $N_G(i) \subset N_G(j)$ with $j > i$. So, $V(G) = \bigcup_i N_G(i) \subset N_G(n)$ and then $V(G) = N_G(n)$. By vertex transitivity $V(G) = N_G(i)$ for all i. ■

k-dismantlability and transitivity

Definition

- A graph G is vertex-transitive if $\forall\, v, w \in V(G),\ \exists\, g \in \mathrm{Aut}(G), g.v = w$
- A graph G is $\leq$ i-transitive if $\forall\,(\{a_1, a_2, \dots, a_i\}, \{b_1, b_2, \dots, b_i\}) \in \mathcal{C}_i(G) \times \mathcal{C}_i(G)$, $\exists\, g \in \mathrm{Aut}(G), \forall u \in \{1, \cdots, i\}, g(a_u) = b_u$ where $\mathcal{C}_i(X)$ is the set of the i-complete subgraphs of G.

conjecture

For all $k \geq 0$, if G is a k-dismantlable and $\leq k$-transitive, then G is a complete graph.

idea of the proof : the conjecture is true for $k = 1$. For $k = 2$ we prove there exist i_1 such that $V(G) = \bigcup_{1 \leq i < j \leq n} N_G(i) \cap N_G(j) \subset N_G(1) \cap N_G(i_1)$. Then $N_G(1) = V(G)$ and by vertex transitivity $V(G)$ is a complete. It is a bit long to write for $k \geq 3$ but the idea would be the same considering $V(G) = \bigcup_{1 \leq i_1 < \cdots < i_j \leq n} N_G(i_1) \cap \cdots N_G(i_j)$ ■

Remarks :

The Kneser graphs and the Johnson graphs are $\leq$ i-transitive.

k-dismantlability and evasiveness

- A graph G is *non-evasive* if for any $A \subset V(G) = \{x_1, x_2, \ldots, x_n\}$ one can guess if A is a **clique** of G in at most $n-1$ questions of the form “is x_i in A?"

Evasiveness conjecture for graphs

A non-evasive, vertex-transitive and non empty finite graph is a complete graph.

- Following a remark due to Lovász, Rivest & Vuillemin (76) pointed out that a positive answer to the evasiveness conjecture implies that a finite vertex-transitive graph with a maximal clique which intersects all other maximal cliques (Payan property) is a complete graph.
- Suppose G has the Payan property and the maximal transversal clique has cardinality equal to n then $G \in D_{n-2}$.

Question :

Is it possible to solve the particular case of evasiveness conjecture for graphs in D_n ?

Thank you !

Sorry, I cannot be with you today !