Connected Quartic Bipartite Cayley Integral Graphs

Author: Marsha Minchenko

MONASH University Science

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Contribution

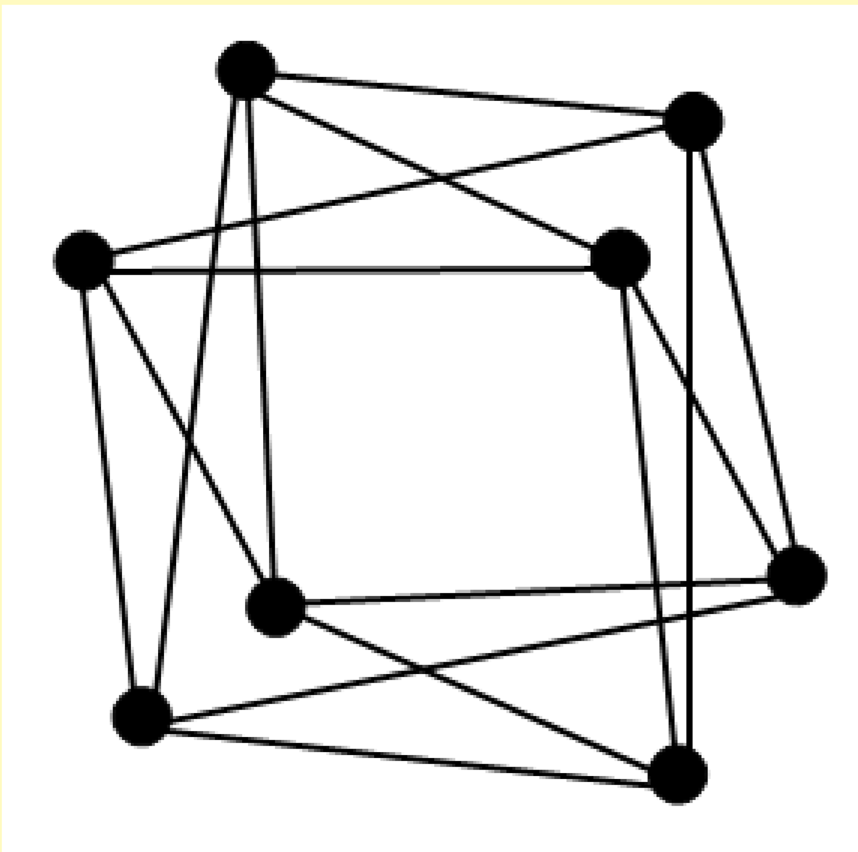
Here, for the first time, all connected quartic bipartite Cayley integral graphs are given including all non-obvious* isomorphisms.

Computations used previous results showing that all connected quartic bipartite integral graphs have one of 43 possible values for the number of vertices, all falling between 8 and 560.

Recently the quartic Cayley integral graphs on finite abelian groups were determined but here we have found the graphs considering all groups.

Explaining the Terms

An **integral graph** is a graph with integer eigenvalues with respect to the adjacency matrix of the graph.



The **spectrum** of a graph is the eigenvalues with their multiplicity.

$$\{4, 0^6, -4\}$$

The **Cayley Graph** $Cay(G, S)$ for a given group G and $S \subset G$ is the graph with vertex set G and with x connected to y if and only if $yx^{-1} \in S$.

$$Cay(C_8, \{1, 3, 5, 7\})$$

Why Quartic Bipartite?

Quartic (or 4-regular)

-The largest eigenvalue is 4 with multiplicity 1.

-(*and undirected*) Connection sets contain 0, 2, or 4 involutions.

Bipartite

-Eigenvalues are symmetric with respect to 0. The resulting spectrum:

$$\{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -1^x, -4\}$$

Sketch of the Algorithm

A. Consider all groups of order in the list of possible spectra.

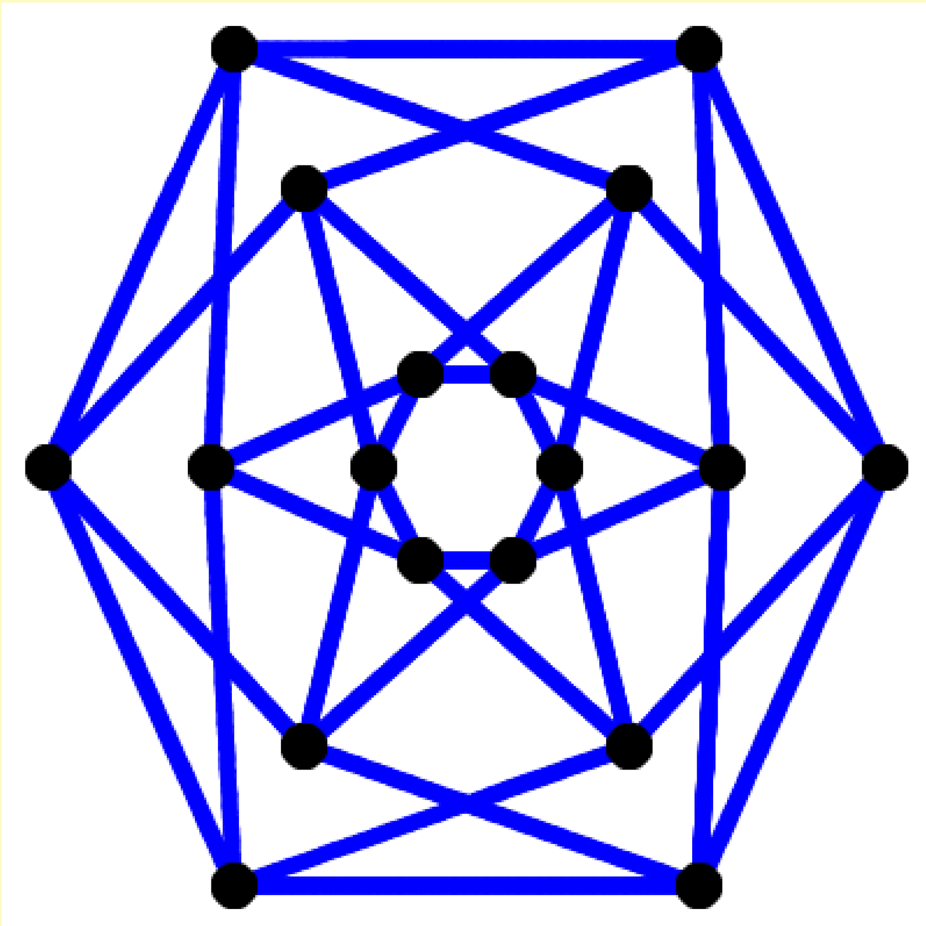
B. Form appropriate 4 element generating sets and build corresponding Cayley graphs.

C. Take only those graphs that are connected and bipartite.

D. Check if the eigenvalues are integers.

Note: GAP, Nauty, and Maple were an asset.

The Graph on 18 Vertices



Two Cayley representations for this graph:

$$Cay(C_3 \times S_3, \{(0,(12)), (0,(13)), (1,(23)), (2,(23))\})$$
$$Cay(C_6 \times C_3, \{(1,0), (5,0), (3,1), (3,2)\})$$

Results

If we let n denote the number of vertices, the following table lists the only connected quartic bipartite Cayley integral graphs up to non-obvious* isomorphism:

n	Spectrum	Group	Connection Set
8	$4, 0^6, \ldots$	C_8	0 involutions
		$C_4 \times C_2$	2 involutions
		$C_4 \times C_2$	0 involutions
		D_8	4 involutions
		D_8	2 involutions
		Q_8	0 involutions
		$C_2 \times C_2 \times C_2$	4 involutions
10	$4, 1^4, \ldots$	D_{10}	4 involutions
		C_{10}	0 involutions
12	$4, 2^2, 0^3, \ldots$	$C3 : C4$	0 involutions
		$C12$	0 involutions
		$D12$	2 involutions
12	$4, 2, 1^4, \ldots$	$D12$	4 involutions
16	$4, 2^4, 0^6, \ldots$	$C_4 \times C_4$	0 involutions
		$(C_4 \times C_2) \rtimes C_2$	2 involutions
		$(C_4 \times C_2) \rtimes C_2$	2 involutions
		$(C_4 \times C_2) \rtimes C_2$	0 involutions
		$C_4 \rtimes C_4$	0 involutions
		$C_8 \rtimes C_2$	0 involutions
		QD_{16}	2 involutions
		$C_4 \times C_2 \times C_2$	2 involutions
		$C_2 \times D_8$	4 involutions
		$C_2 \times D_8$	4 involutions
		$C_2 \times D_8$	4 involutions
		$C_2 \times D_8$	2 involutions
		$C_2 \times D_8$	2 involutions
		$C_2 \times C_2 \times C_2 \times C_2$	4 involutions
18	$4, 2^4, 1^4, \ldots$	$C_3 \times S_3$	2 involutions
		$C_3 \times S_3$	0 involutions
		$(C_3 \times C_3) \rtimes C_2$	4 involutions
		$C_6 \times C_3$	0 involutions
24	$4, 2^8, 0^6, \ldots$	$C_4 \times S_3$	2 involutions
		$(C_6 \times C_2) \rtimes C_2$	2 involutions
		$C_3 \times D_8$	0 involutions
		S_4	4 involutions
		S_4	2 involutions
		S_4	0 involutions
		$C_2 \times A_4$	0 involutions
		$C_2 \times C_2 \times S_3$	4 involutions
24	$4, 3^2, 2^2, 1^6, 0^2$	$C_4 \times S_3$	2 involutions
		$C_4 \times S_3$	2 involutions
		D_{24}	4 involutions
		D_{24}	2 involutions
		$C_2 \times (C_3 \rtimes C_4)$	0 involutions
		$(C_6 \times C_2) \rtimes C_2$	4 involutions
		$(C_6 \times C_2) \rtimes C_2$	2 involutions
		$(C_6 \times C_2) \rtimes C_2$	2 involutions
		$(C_6 \times C_2) \rtimes C_2$	0 involutions
		$C_{12} \times C_2$	0 involutions
		$C_3 \times D_8$	2 involutions
		$C_3 \times D_8$	0 involutions
		$C_2 \times C_2 \times S_3$	4 involutions
		$C_2 \times C_2 \times S_3$	4 involutions
		$C_2 \times C_2 \times S_3$	4 involutions
		$C_2 \times C_2 \times S_3$	2 involutions
		$C_2 \times C_2 \times S_3$	2 involutions
		$C_6 \times C_2 \times C_2$	2 involutions

Results Continued

n	Spectrum	Group	Connection Set
24	$4, 3^3, 1^5, 0^6, \ldots$	S_4	4 involutions
		$C_2 \times A_4$	2 involutions
30	$4, 2^{10}, 1^4, \ldots$	$C_5 \times S_3$	0 involutions
		D_{30}	4 involutions
32	$4, 2^{12}, 0^6, \ldots$	$C_8 \rtimes C_4$	0 involutions
		$(C_8 \rtimes C_2) \rtimes C_2$	2 involutions
		$C_2.((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2).(C_4 \times C_2)$	0 involutions
		$(C_4 \times C_4) \rtimes C_2$	2 involutions
		$C_4.D_8 = C_4.(C_4 \times C_2)$	0 involutions
		$(C_4 \times C_4) \rtimes C_2$	4 involutions
		$(C_8 \times C_2) \rtimes C_2$	2 involutions
		$C_2 \times QD_{16}$	2 involutions
		$(C_8 \times C_2) \rtimes C_2$	4 involutions
		$(C_2 \times D_8) \rtimes C_2$	4 involutions
		$(C_2 \times Q_8) \rtimes C_2$	2 involutions
		$(C_2 \times Q_8) \rtimes C_2$	4 involutions
36	$4, 3^4, 2^4, 1^4, 0^{10}, \ldots$	$C_3 \times (C_3 \rtimes C_4)$	0 involutions
		$(C_3 \times C_3) \rtimes C_4$	0 involutions
		$S_3 \times S_3$	4 involutions
		$S_3 \times S_3$	4 involutions
		$S_3 \times S_3$	2 involutions
		$S_3 \times S_3$	0 involutions
		$C_6 \times S_3$	2 involutions
		$C_6 \times S_3$	2 involutions
		$C_6 \times S_3$	0 involutions
		$C_2 \times ((C_3 \times C_3) \rtimes C_2)$	4 involutions
		$C_2 \times ((C_3 \times C_3) \rtimes C_2)$	2 involutions
		$C_6 \times C_6$	0 involutions
40	$4, 3^4, 2^6, 1^4, 0^{10}, \ldots$	$C_2 \times (C_5 \rtimes C_4)$	2 involutions
48	$4, 3^6, 2^4, 1^{10}, 0^6, \ldots$	$C_2 \times C_4 \times S_3$	2 involutions
		$D_8 \times S_3$	4 involutions
		$D_8 \times S_3$	4 involutions
		$D_8 \times S_3$	2 involutions
		$D_8 \times S_3$	2 involutions
		$C_2 \times ((C_6 \times C_2) \rtimes C_2)$	4 involutions
		$C_6 \times D_8$	2 involutions
		$C_2 \times S_4$	4 involutions
		$C_2 \times S_4$	4 involutions
		$C_2 \times S_4$	2 involutions
		$C_2 \times C_2 \times A_4$	2 involutions
		$C_2 \times C_2 \times C_2 \times S_3$	4 involutions
72	$4, 3^8, 2^{10}, 1^{16}, 0^2, \ldots$	$C_3 \times S_4$	0 involutions
		$(C_3 \times A_4) \rtimes C_2$	4 involutions
		$A_4 \times S_3$	0 involutions
		$C_6 \times A_4$	0 involutions
72	$4, 3^6, 2^{16}, 1^{10}, 0^6, \ldots$	$(C_3 \times (C_3 \rtimes C_4)) \rtimes C_2$	2 involutions
		$(C_6 \times S_3) \rtimes C_2$	4 involutions
		$(C_6 \times S_3) \rtimes C_2$	2 involutions
		$C_6 \times (C_3 \rtimes C_4)$	0 involutions
		$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	2 involutions
		$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	0 involutions
		$(S_3 \times S_3) \rtimes C_2$	4 involutions
		$(S_3 \times S_3) \rtimes C_2$	2 involutions
		$(S_3 \times S_3) \rtimes C_2$	2 involutions
		$(S_3 \times S_3) \rtimes C_2$	0 involutions
		$C_2 \times ((C_3 \times C_3) \rtimes C_4)$	2 involutions
		$C_2 \times ((C_3 \times C_3) \rtimes C_4)$	0 involutions
		$C_2 \times S_3 \times S_3$	4 involutions
		$C_2 \times S_3 \times S_3$	4 involutions
		$C_2 \times S_3 \times S_3$	2 involutions
		$C_2 \times C_6 \times S_3$	2 involutions
120	$4, 3^{12}, 2^{28}, 1^4, 0^{30}, \ldots$	S_5	4 involutions
		S_5	0 involutions
		$C_2 \times A_5$	2 involutions
		$S_3 \times (C_5 \rtimes C_4)$	2 involutions
		$C_5 \times S_4$	0 involutions
		$(C_5 \times A_4) \rtimes C_2$	4 involutions

*Obvious Cayley Isomorphisms

The following property was used to eliminate not all but a great number of isomorphic graphs on the same group:

$$S^\sigma = T \text{ for some } \sigma \in Aut(G)$$
$$\Rightarrow Cay(G, S) \cong Cay(G, T)$$

As can be seen, still remaining are many unexplained isomorphic Cayley graphs on the *same* or a number of *different* groups.

References

[1] D. Stevanovic, N. M.M. de Abreu, M. A.A. de Freitas, and R. Del-Vecchio. Walks and regular integral graphs. *Linear Algebra and its Applications*, 423(1):119–135, 2007.

[2] A. Abdollahi and E. Vatandoost Integral Quartic Cayley Graphs on Abelian Groups *The Electronic Journal of Combinatorics*, 18(1), P89, 2011.