On monochromatic component size for improper colourings

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Abstract

This paper concerns improper λ-colourings of graphs and focuses on the sizes of the monochromatic components (i.e., components of the subgraphs induced by the colour classes). Consider the following three simple operations, which should, heuristically, help reduce monochromatic component size: (a) assign to a vertex the colour that is least popular among its neighbours; (b) change the colours of any two adjacent differently coloured vertices, if doing so reduces the number of monochromatic edges; and (c) change the colour of a vertex, if by so doing you can reduce the size of the largest monochromatic component containing it without increasing the number of monochromatic edges. If a colouring cannot be further improved by these operations, then we regard it as locally optimal. We show that, for such a locally optimal 2-colouring of a graph of maximum degree 4, the maximum monochromatic component size is $O(2^{(2\log_2 n)^{1/2}})$. The operation set (a)–(c) appears to be one of the simplest that achieves a $o(n)$ bound on monochromatic component size. Recent work by Alon, Ding, Oporowski and Vertigan, and then Haxell, Szabó and Tardos, has shown that some algorithms can do much better, achieving a constant bound on monochromatic component size. However, the simplicity of our operation set, and of the associated local search algorithm, make the algorithm, and our locally optimal colourings, of interest in their own right.

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1 Introduction

An *assignment*, or λ*-assignment*, of a graph G is a map from $V(G)$ to some set Λ of colours such as $\{1, \ldots, \lambda\}$; i.e., it is a 'colouring' that may be improper. We will use terminology from graph colouring for such maps where the meaning is clear. For example, a *colour class* is a preimage, under the λ-assignment, of a single colour. The colour class for colour $i \in \Lambda$ is denoted by C_i.

A *chromon* of G under an assignment c is a component of a subgraph of G induced by a colour class, or in other words, a maximal connected monochromatic subgraph (sometimes called a *monochromatic component*). A k*-chromon* is a chromon with k vertices.

This paper mainly concerns 2-assignments, for a graph, which are locally optimal (in a natural sense) with respect to maximum chromon size.

A graph G is $[\lambda, C]$*-colourable* if it has a λ-assignment in which every chromon has at most C vertices. An ordinary (proper) colouring is thus a $[\lambda, 1]$-colouring, and the chromons under such a colouring are just the individual vertices.

A class of graphs Γ is $[\lambda, C]$*-colourable* if every $G \in \Gamma$ is $[\lambda, C]$-colourable. Γ is λ*-metacolourable* if there exists C such that Γ is $[\lambda, C]$-colourable. The *metachromatic number* $\boldsymbol{\chi}(\Gamma)$ is the smallest λ such that Γ is λ-metacolourable.

This is similar in spirit to the concept of fragmentability of classes of graphs that we introduced in [9]. In fact, it is to fragmentability as ordinary graph colouring is to independent sets. This paper, though, does not depend on that one.

Γ_d denotes the class of graphs of maximum degree $\leq d$.

In our main result, we give a simple local search algorithm for finding a 2-assignment of a graph of maximum degree 4 in which all chromons have size $O(2^{(2\log_2 n)^{1/2}})$. The algorithm uses three simple operations, involving changing the colour of just one or two vertices at a time, and appears to be one of the simplest algorithms that attain maximum chromon size $o(n)$. Our bound on chromon size applies to any 2-assignment that cannot be improved by applying any of our three operations. Since these operations are arguably the most natural local operations that can be done in this situation, these 2-assignments are worth studying. We also note that some such 2-assignments have maximum chromon size within a constant factor of our upper bound.

Alon et al. [4] show that (in our notation) Γ_4 is $[2, 57]$-colourable. They do not present an algorithm explicitly, though it is reasonable to expect that their approach would yield one. More recently, Haxell, Szabó and Tardos [12, §2.2] have shown that Γ_4 is $[2, 6]$-colourable, and their proof yields an efficient algorithm. Alon (private communication, 2002) also reports that an algorithm with some constant bound on chromon size can be obtained using ideas in [3].

Throughout, G is a graph and $n = |V(G)|$. An edge is *monochromatic* if its endpoints have the same colour. We use the same word for subgraphs, with the obvious meaning. If $X, Y \subseteq V(G)$, then $E(X, Y)$ denotes the set of edges with one endpoint in X and the other in Y.

2 Related work

There is considerable literature on generalisations of graph colouring (see, e.g., [19]). We briefly mention only the work we are aware of that is closest to ours. In explaining others' concepts and notation here, we generally use our own terminology.

A very general kind of colouring is introduced by Weaver and West [18] and studied further by Deuber and Zhu [8]. Let $\mathcal{P}$ be a hereditary class of graphs. Weaver and West define the $\mathcal{P}$-chromatic number $\chi_{\mathcal{P}}(G)$ of a graph G to be the smallest λ such that there is a λ-assignment for which every chromon belongs to $\mathcal{P}$. If $\mathcal{P}$ is the class of graphs of at most C vertices, then $\chi_{\mathcal{P}}(G) \leq \lambda$ if and only if G is $[\lambda, C]$-colourable. These authors study $\chi_{\mathcal{P}}$ for graphs of high girth and for cartesian and lexicographic products.

Colourings for which a forbidden induced subgraph is specified for the colour classes are investigated in [16].

In §3 we will be particularly interested in λ-assignments in which most of the chromons are paths, with the possibility of some circuits. Assignments in which *all* chromons are paths were introduced by Akiyama et al. [2]. They define the *k-path chromatic number* $\chi(G; P_k)$ of G to be the smallest λ such that there is a λ-assignment for which every chromon is a path of k vertices. So $\chi(G; P_k) \leq \lambda$ implies G is $[\lambda, k]$-colourable. They concentrate on path chromatic numbers of planar, outerplanar, regular and bounded-degree graphs. One of their results gives $\boldsymbol{\chi}(\Gamma_d) \leq \lceil (d+1)/2 \rceil$. Other work on path chromatic numbers includes study of its complexity [13].

The term "path-chromatic number" is used differently by others, such as in [1, 7, 14], where it refers to the minimum λ such that there is a λ-assignment of G for which there is no monochromatic path of k vertices, and is denoted by $P_k\,\chi(G)$. It is clear that if G is $[\lambda, C]$-colourable then $P_{C+1}\,\chi(G) \leq \lambda$.

We also mention a result of Berman and Paul, who showed in [5] that for λ-assignments of *k-trees*, the maximum chromon size is $\leq k\lceil n^{1/\lambda} \rceil$, and that such an upper bound cannot be brought down below $\lfloor n^{1/\lambda} \rfloor$.

We note that the well known concept of linear arboricity [11] involves assigning colours to *edges* so that the maximal monochromatic connected subgraphs are all paths. Restrictions on allowed path length have been considered in [6, 17].

A result of Lovász [15] yields an efficient algorithm for finding 2-colourings of graphs of maximum degree ≤ 4 with the two colour classes inducing subgraphs of maximum degree two and one respectively. This result has been used for related algorithmic problems by, e.g., Halldórsson and Lau [10]. One of the colour classes has no bound on its chromon size, but it is not surprising to find that the result has been used as a step towards bounding chromon size in both colour classes, as we now mention.

Alon et al. [4] show that: the metachromatic number of the class of planar graphs is four; Γ_d is $[\lceil (d+2)/3 \rceil, 12d^2 - 36d + 9]$-colourable; and, for any $k < 3$, every $G \in \Gamma_d$ is $[\lceil (d+2)/k \rceil, C_k]$-colourable, where the constant C_k depends on k but not on d. The latter two results use the abovementioned theorem of Lovász. Alon et al. ask whether $\boldsymbol{\chi}(\Gamma_5)$ is 2, and prove other results including some on edge-colourings.

This latter question is answered in the affirmative by Haxell et al. [12] (with $C \leq 17617$), who also prove that: Γ_4 is $[2, 6]$-colourable; $\boldsymbol{\chi}(\Gamma_8) \leq 3$; and $\boldsymbol{\chi}(\Gamma_d) \leq \lceil (d+1)/3 \rceil$ (and the same constant bound on chromon size can be used for all d). Among further results, they show that metacolourability of Γ_d can be used to prove metacolourability results about $\Gamma_{d'}$ for certain larger d'.

Alon et al. [4] also proved the lower bound $\boldsymbol{\chi}(\Gamma_d) \geq (d+3)/4$.

3 Two colours and maximum degree four

In this section, we look at using just two colours on graphs of maximum degree at most four, and consider some simple operations for which locally optimal 2-assignments have reasonably small chromons. These operations give an efficient local search algorithm whose performance we analyse. We begin with some definitions.

If x is one of the two colours at our disposal, then $\overline{x}$ is its *opposite*, i.e., the other colour.

If $v \in V(G)$, then its *cochromatic* neighbours are those with the same colour as v, and its *antichromatic* neighbours are those of different colour to v.

A vertex is *balanced*, under a particular assignment, if it has degree 4 and has two neighbours of each colour.

The number of monochromatic edges in a graph G under an assignment c is denoted by $e(G, c)$. For each k, $1 \leq k \leq n$, let $a_k = a_k(G, c)$ be the number of chromons of size k in G under c. Write $\boldsymbol{a}(G, c) = (a_n, \ldots, a_1)$.

The 2-assignments we find will be locally optimal in the sense that they cannot be improved by changing the colours of just a few vertices. The notion of "locally optimal" used is determined by specifying the colour changes that are allowed to happen. We consider briefly some different sets of allowed changes, and the maximum chromon size of the corresponding set of locally optimal colourings.

For some sets of allowed changes, we cannot get a better bound on maximum chromon size than $O(n)$. Suppose, for example, that our only allowed local change is to flip the colours of up to two vertices if, by doing so, we reduce $\boldsymbol{a}(G, c)$ under lexicographic order. Then we might get stuck with maximum chromon size $n/2$: let G be the skeleton of the antiprism based on two $n/2$-gons with n triangles between them (where n is even), with one $n/2$-gon coloured white and the other black.

This example demonstrates that, if we want to get an $o(n)$ bound on maximum chromon size by flipping the colours of at most two vertices at a time, we need to do more than just try to reduce the size of the largest chromon, or indeed $\boldsymbol{a}(G, c)$. With this in mind, it is natural to introduce operations that reduce $e(G, c)$. However, flipping just one vertex at a time, to reduce $e(G, c)$, is not enough, as the antiprism graph shows.

Note also that it is insufficient to focus just on reducing $e(G, c)$ alone, if we are only to flip one or two vertices at a time. Consider a circuit C_k, with all vertices white, together with $k/2$ black vertices, each of which is joined to four vertices of C_k in such a way that the resulting graph is 4-regular. Maximum chromon size is $2n/3$, and flipping two vertices cannot reduce $e(G, c)$.

We are thus drawn to the following set of operations:

(a) If v has more cochromatic neighbours than antichromatic ones, then flip its colour (i.e., give it the opposite colour to its present colour, so that it gets the colour least popular among its neighbours). (This reduces $e(G, c)$.)

(b) If v and w are adjacent, balanced, oppositely coloured vertices, then flip both their colours. (This reduces $e(G, c)$.)

(c) If flipping the colour of a balanced vertex v would make it belong to a smaller chromon than its current one, then do so. (This reduces $\boldsymbol{a}(G, c)$, under lexicographic order, while keeping $e(G, c)$ unchanged.)

(We require v to be balanced in (c) to avoid situations where some periodic sequence of operations can be applied forever.)

We show below that a 2-assignment for which none of these operations can be done has maximum chromon size $o(n)$. In the light of the above discussion, this set of operations appears to be one of the simplest with this property, at least among those based on flipping colours of small numbers of vertices.

It is straightforward to use these operations as the basis of an algorithm, which essentially just keeps applying operations (a), (b), (c) for as long as possible.

Algorithm 1. Finding a 2-colouring with small monochromatic components for a graph of maximum degree at most 4.

1. Input: Graph G with maximum degree ≤ 4.
2. Start with an arbitrary 2-assignment c of G.
3. do

{
    if (operation (a) can be done, somewhere in G)
        do it
    else if (operation (b) can be done, somewhere in G)
        do it
    else if (operation (c) can be done, somewhere in G)
        do it
} until none of the operations can be done any more.

4. Output: the resulting 2-assignment.

To prove our main result, we will need the following.

Lemma 1 *Let H be a graph with $\Delta(H) \leq 2\delta$. Let $V_M \subseteq V(H)$, and let M be the set of edges incident with a vertex in V_M. Let $E(H)$ be partitioned into subsets E_j, $1 \leq j \leq \mu$, and for each $e \in E(H)$ write $j(e)$ for the unique j such that $e \in E_j$. Suppose the vertices of H have positive integer weights $w(v)$, $v \in V(H)$, such that, for each $uv \in E(H) \setminus M$, $w(u) + w(v) \geq \left|E_{j(uv)}\right| + 1$. Then there exists $U \subseteq V(H) \setminus V_M$ such that*

$$\frac{|E(H-M)| - \delta + 1}{\delta} \leq |U| \leq \frac{|E(H-M)| + \delta - 1}{\delta} \tag{1}$$

and

$$\sum_{v \in U} w(v) \geq \left(|U| - \frac{\delta - 1}{\delta}\right)\left(\frac{|E(H)|}{2\mu} + \frac{1}{2}\right) + \frac{\delta - 1}{2\delta} \quad . \tag{2}$$

Proof. We begin by finding a partition $E(H-M) = S_0 \cup \cdots \cup S_{\delta-1}$ of $E(H-M)$ into δ parts, as close to equal in size as possible.

If $\delta = 1$, put $S_0 = E(H-M)$.

For the moment, suppose $\delta = 2$. Consider $H - M$. Add a matching A joining up pairs of odd-degree vertices in $H - M$, so as to make all degrees even. So all components of $H' = (H - M) \cup A$ are Eulerian. Let T be a concatenation of Eulerian trails, one from each component of H'. Form two disjoint edge sets S_0, S_1 by going around T, placing the edges of $H - M$ alternately in S_0 and S_1. Edges in A are ignored in this allocation: if the trail contains the subsequence of edges e, f, g, where $e \in S_i$ and $f \in A$, then $g \in S_{1-i}$, and certainly $f \notin S_0 \cup S_1$. It follows that S_0 and S_1 have the same size if $|E(H-M)|$ is even and differ in size by one otherwise.

Now suppose $\delta \geq 3$. It is routine to form a partition $E(H-M) = R_0 \cup \cdots \cup R_{\delta-1}$ of $E(H-M)$ into δ parts, each (when considered as a subgraph of $E(H-M)$) having maximum degree ≤ 2. Now repeatedly do the following: choose any two parts of the partition that differ in size by at least two; observe that their union has maximum degree ≤ 4, so the argument of the previous paragraph can be applied, giving two disjoint edge sets R_0', R_1', differing in size by at most one, such that $R_0' \cup R_1' = R_0 \cup R_1$. This procedure can be continued until we have a partition $E(H-M) = S_0 \cup \cdots \cup S_{\delta-1}$ in which any two parts differ in size by at most one.

Thus, for any δ, we can find a partition $E(H-M) = S_0 \cup \cdots \cup S_{\delta-1}$ such that, for all i,

$$\frac{|E(H-M)| - \delta + 1}{\delta} \le |S_i| \le \frac{|E(H-M)| + \delta - 1}{\delta}. \tag{3}$$

Suppose without loss of generality that $|E_1| \ge \cdots \ge |E_\mu|$.

Suppose that $\sum_{e \in S_i} |E_{j(e)}|$ is maximum when $i = i^*$. Then

$$\begin{aligned}
\sum_{e \in S_{i^*}} |E_{j(e)}| &\ge \frac{1}{\delta} \sum_{e \in E(H-M)} |E_{j(e)}| \\
&= \frac{1}{\delta} \sum_{j=1}^{\mu} |E_j \setminus M| \cdot |E_j| \\
&= \frac{1}{\delta} \sum_{j=1}^{\mu} (|E_j| - |M \cap E_j|) \cdot |E_j| \\
&\ge \frac{1}{\delta} \sum_{j=1}^{\mu} (|E_j| - \eta_j) \cdot |E_j|, \qquad (4)
\end{aligned}$$

where the η_j are chosen to satisfy

$$\sum_{j=1}^{\mu} \eta_j = |M| \tag{5}$$

and

$$|E_1| - \eta_1 \ge |E_2| - \eta_2 \ge \cdots \ge |E_\mu| - \eta_\mu. \tag{6}$$

The inequality (4) holds because its right hand side is minimised when the η_j satisfy (5) and (6) (due to our assumed ordering on the $|E_j|$). Applying the Cauchy-Schwartz inequality to the right-hand side of (4), we obtain

$$\begin{aligned}
\sum_{e \in S_{i^*}} |E_{j(e)}| &\ge \frac{1}{\delta\mu} \left(\sum_{j=1}^{\mu} (|E_j| - \eta_j) \right) \left(\sum_{j=1}^{\mu} |E_j| \right) && (7) \\
&= \frac{1}{\delta\mu} (|E(H)| - |M|)\, |E(H)| && \text{(by (5))} \\
&\ge \left(|S_{i^*}| - \frac{\delta - 1}{\delta} \right) \frac{|E(H)|}{\mu} \quad . && \text{(by (3))} \qquad (8)
\end{aligned}$$

Now, S_{i^*} can be thought of as the edge set of a subgraph $H_{S_{i^*}} = (V_{S_{i^*}}, S_{i^*})$ of H, where $V_{S_{i^*}}$ consists of all vertices of H incident with an edge of S_{i^*}. Clearly $\Delta(H_{S_{i^*}}) \le 2$. Hence $H_{S_{i^*}}$ is a union of disjoint cycles $C_1, \ldots, C_{s_1}$ and paths $P_1, \ldots, P_{s_2}$.

We describe how to form our set of vertices, U, from subsets of the vertex sets of these paths and cycles. For each cycle C_i, put $U_{C_i} = V(C_i)$, and observe that

$$\begin{aligned}
\sum_{v \in U_{C_i}} w(v) &= \frac{1}{2} \sum_{uv \in E(C_i)} (w(u) + w(v)) \\
&\geq \frac{1}{2} \sum_{e \in E(C_i)} \left(|E_{j(e)}| + 1\right) \\
&= \frac{1}{2} \sum_{e \in E(C_i)} |E_{j(e)}| + \frac{1}{2} |E(C_i)| . \qquad (9)
\end{aligned}$$

For each path $P_i = v_1, \ldots, v_k$, define

$$P'_i = \begin{cases} v_1, \ldots, v_{k-1}, & \text{if } w(v_1) \geq w(v_k); \\ v_2, \ldots, v_k, & \text{if } w(v_1) < w(v_k). \end{cases}$$

Put $U_{P_i} = V(P'_i)$ and observe that

$$\begin{aligned}
\sum_{v \in U_{P_i}} w(v) &\geq \frac{1}{2} \left(w(v_1) + 2 \sum_{h=2}^{k-1} w(v_h) \ + w(v_k) \right) \\
&= \frac{1}{2} \sum_{uv \in E(P_i)} (w(u) + w(v)) \\
&\geq \frac{1}{2} \sum_{e \in E(P_i)} \left(|E_{j(e)}| + 1\right) \\
&= \frac{1}{2} \sum_{e \in E(P_i)} |E_{j(e)}| + \frac{1}{2} |E(P_i)| . \qquad (10)
\end{aligned}$$

Finally put $U = \bigcup_{i=1}^{s_1} U_{C_i} \cup \bigcup_{i=1}^{s_2} U_{P_i}$. Observe firstly that $|U| = |S_{i^*}|$, so

$$\frac{|E(H-M)| - \delta + 1}{\delta} \leq |U| \leq \frac{|E(H-M)| + \delta - 1}{\delta},$$

by (3), and secondly that

$$\begin{aligned}
\sum_{v \in U} w(v) &\geq \frac{1}{2} \sum_{e \in S_{i^*}} |E_{j(e)}| + \frac{1}{2} |S_{i^*}| \qquad \text{(by (9) and (10))} \\
&\geq \frac{1}{2} \left(|S_{i^*}| - \frac{\delta - 1}{\delta} \right) \cdot \frac{|E(H)|}{\mu} + \frac{1}{2} |S_{i^*}| \qquad \text{(by (8))} \\
&= \left(|U| - \frac{\delta - 1}{\delta} \right) \cdot \left(\frac{|E(H)|}{2\mu} + \frac{1}{2} \right) + \frac{\delta - 1}{2\delta} \quad .
\end{aligned}$$

□

Note that the graphs H may have loops and parallel edges, and the degree of a vertex in H is the number of edges incident at it, with loops counting twice.

We can now prove our main result. It shows that the sizes of chromons in 2-assignments produced by Algorithm 1 — and, indeed, of any 2-assignment that is locally optimal in our sense — are bounded by a function of n whose growth rate is intermediate between $O(1)$ and $n^{1/O(1)}$.

Theorem 2 *Let G be a graph with $\Delta(G) \leq 4$. Any 2-assignment of G in which none of operations (a), (b), (c) can be applied further — in particular, any 2-assignment produced by Algorithm 1 — has maximum chromon size* $O\left(2^{(2\log_2 n)^{1/2}}\right)$.

Proof. Suppose $\Delta(G) \leq 4$.

If $v \in V(G)$ then $X(v)$ denotes the unique chromon containing v.

Let c be any 2-assignment (not necessarily one produced by Algorithm 1) for which none of operations (a), (b), (c) can be done any more. Let $\mathcal{X}$ be the set of all chromons in G under c.

We have already seen that every vertex of G has at least as many antichromatic neighbours as cochromatic ones (since operation (a) can no longer be done).

Thus, each vertex v of degree ≥ 3 is either balanced or has the same colour as a clear minority (zero or one) of its neighbours. Each chromon is thus either a path (possibly a trivial one) or a circuit, and we will refer to *path-chromons* and *circuit-chromons* with the obvious meaning. An unbalanced vertex of a chromon is said to be an *end vertex* of that chromon. Obviously, path-chromons have two end vertices, except that the trivial path of just one vertex has one end vertex, while circuit-chromons have none.

Observe next that no two differently coloured balanced vertices can be adjacent, since operation (b) can no longer be done. It follows that if v and w are adjacent vertices in different chromons, then at most one of v, w can be balanced.

Note also that an end vertex of a chromon can have at most three balanced neighbours, since $\Delta(G) \leq 4$.

If $v \in V(G)$ and X, X_1 and X_2 are chromons:

(i) $v \longrightarrow X$ indicates that v is balanced, $v \notin X$ and v is adjacent to an end vertex of X.

(ii) $X \longrightarrow v$ indicates that $v \notin X$ and some balanced $w \in X$ is adjacent to v. (It follows that v cannot be balanced.)

(iii) $X_1 \longrightarrow X_2$ indicates that $v \longrightarrow X_2$ for some balanced $v \in X_1$ (or, equivalently, $X_1 \longrightarrow w$ for some end vertex w of X_2).

The relation (iii) defines a digraph whose vertices represent chromons.

The following observation is central to the proof.

Claim 1: For each balanced vertex $v \in V(G)$,

$$|X(v)| \leq 1 + \sum_{\substack{X \in \mathcal{X}: \\ v \longrightarrow X}} |X| \tag{11}$$

(Note that the sum on the right will have just one or two terms.)

Proof. v currently belongs to a chromon of size $|X(v)|$. If its colour is flipped, it will belong to a chromon of size $1+\sum_{X\in\mathcal{X}:v\longrightarrow X}|X|$. If the inequality (11) does not hold, then operation (c) can be done. $\square$

Let $X_{\max}$ be any largest chromon, and set

$$x_0 = \begin{cases} |X_{\max}|, & \text{if } X_{\max} \text{ is a path-chromon;} \\ |X_{\max}|-2, & \text{if } X_{\max} \text{ is a circuit-chromon.} \end{cases} \tag{12}$$

Assume $x_0 > 3$.

Our aim is to find an upper bound for x_0 in terms of $|V(G)|$. We now outline roughly how we do this.

We will construct a sequence of disjoint sets $\mathcal{U}_i$ of chromons, beginning with the singleton $\mathcal{U}_0 = \{X_{\max}\}$, in which the successive $\mathcal{U}_i$ have increasing numbers of chromons. The average size of these chromons may decrease as i increases, but not at a significantly greater rate than halving. We will find lower bounds for the numbers, and average sizes, of the chromons in these sets. For $i = \log x_0$ minus a constant, we find that these bounds give a large enough lower bound on the total number of vertices in all the chromons (and hence on $|V(G)|$) that we can deduce the desired upper bound on x_0 in terms of $|V(G)|$.

At the heart of this approach is the following idea. Consider a chromon X. Each of its balanced vertices v is adjacent to two end vertices, belonging to either one or two other chromons. Claim 1 gives a lower bound on the total size of these other chromons in terms of the size of X. Doing this for all balanced $v \in X$ gives, in effect, a lower bound on the total sizes of those chromons Y such that $X \longrightarrow Y$. The process can be repeated for each of those chromons Y, and so on. Doing this sufficiently many times gives us many chromons, whose total size is bounded below in terms of $|X|$. We have glossed over several technical issues here, but make things precise below.

In outline, the construction of the $\mathcal{U}_i$ proceeds as follows. $\mathcal{U}_0 = \{X_{\max}\}$. $\mathcal{U}_1$ is a selection of those chromons Y such that $X_{\max} \longrightarrow Y$. $\mathcal{U}_2$ is a selection of those chromons Z such that: $Y \longrightarrow Z$ for some $Y \in \mathcal{U}_1$ and Z does not appear in $\mathcal{U}_0$. $\mathcal{U}_3$ is a selection of those chromons W such that: $Z \longrightarrow W$ for some $Z \in \mathcal{U}_2$ and W does not appear in $\mathcal{U}_j$, $j \leq 1$. So the construction continues. We now describe the $\mathcal{U}_i$, including the aforementioned selections, in detail.

Each $\mathcal{U}_i$ will be a subset of another set of chromons, $\mathcal{Y}_i$. These sets will be constructed inductively, beginning in the next paragraph. In addition, we will refer to the set B_i of balanced vertices of chromons in $\mathcal{U}_i$, the set D_i of end vertices of these same chromons, and the numbers $\mu_i = |\mathcal{U}_i|$, $b_i = |B_i|$ and $z_i = \sum_{X\in\mathcal{U}_i}|X|$. Finally, for all i,j with $0 \leq j < i$ write

$$\begin{aligned} M_{ji} &= E(D_j, B_i) \subseteq E(G), \\ m_{ji} &= |M_{ji}|, \\ m_{\bullet i} &= \sum_{j<i} m_{ji} \quad . \end{aligned}$$

Let B_0 be the set of balanced vertices of our largest chromon $X_{\max}$, except that if $X_{\max}$ is a circuit-chromon then we choose arbitrarily two adjacent $s, t \in X_{\max}$ and put $B_0 = X_{\max} \setminus \{s, t\}$. Thus we always have $|B_0| = |X_{\max}| - 2$. Let D_0 be the set of end vertices of $X_{\max}$, noting that $|D_0| = 0$ or 2 according as $X_{\max}$ is a circuit-chromon or a path-chromon respectively. Put $\mathcal{Y}_0 = \mathcal{U}_0 = \{X_{\max}\}$.

Now, for each $i \geq 1$, define $\mathcal{Y}_i$, $\mathcal{U}_i$ inductively as follows. Let $\mathcal{Y}_i$ be the set of chromons X such that there exists another chromon $X' \in \mathcal{U}_{i-1}$ with $X' \longrightarrow X$. To construct $\mathcal{U}_i$, we construct a graph H and use Lemma 1. The vertices of H are precisely the chromons in $\mathcal{Y}_i$; the vertex corresponding to chromon $X \in \mathcal{Y}_i$ is denoted by v_X. The edge set $E(H)$ corresponds to B_{i-1} as follows. For each $z \in B_{i-1}$, we put an edge e_z in $E(H)$ which joins all chromons X (at least one and at most two in number) such that $z \longrightarrow X$. If there exists just one such chromon X, then e_z is a loop and contributes 2 to $\deg_H v_X$. Let the chromons in $\mathcal{U}_{i-1}$ be $U_1, \ldots, U_{\mu_{i-1}}$. For each j, $1 \leq j \leq \mu_{i-1}$, put $E_j = \{e_z \mid z \in B_{i-1} \cap U_j\}$, so that the E_j partition $E(H)$. For each $z \in B_{i-1}$, let $E_{j(e_z)}$ be the unique E_j such that $e_z \in E_j$. Let $V_M = \{v_X \mid X \in \mathcal{Y}_i \cap \bigcup_{j=0}^{i-2} \mathcal{U}_j\}$, and let M be the set of edges of H incident with a vertex in V_M. Note that

$$m_{\bullet(i-1)}/2 \leq |M| \leq m_{\bullet(i-1)} \quad . \tag{13}$$

For all $v_X \in V(H)$, set $w(v_X) = |X|$. Now, if v_X and v_Y are adjacent in H via edge e_z, then

$$\begin{aligned} w(v_X) + w(v_Y) &= |X| + |Y| \\ &\geq |X(z)| - 1 \qquad \text{(by Claim 1)} \\ &= \left| E_{j(v_X v_Y)} \right| + 1. \end{aligned} \tag{14}$$

Note that the slight peculiarity of our handling of the case when $X_{\max}$ is a circuit-chromon — in (12) and the definition of B_0 — is directed at ensuring that this inequality (14) still holds when $i = 1$ and $X_{\max}$ is a circuit-chromon.

Now, if X is a circuit-chromon, then there is no vertex u such that $u \longrightarrow X$, since such a relationship depends on u being adjacent to an end vertex of X. So all chromons in $\mathcal{Y}_i$, $i \geq 1$, are path-chromons, though it is possible that $X_{\max}$ may be a circuit-chromon.

If $X \in \mathcal{Y}_i$ then each endpoint of X has at most three balanced neighbours. It follows that $\Delta(H) \leq 6$.

The hypotheses of Lemma 1 are satisfied (with $\delta = 3$), and we deduce the existence of a set $U \subseteq V(H) \setminus V_M$ with the properties guaranteed by that Lemma. Translating back into G and putting $\mathcal{U}_i = \{X \mid v_X \in U\}$, we find that (in the notation introduced earlier, and using (13)):

$$\frac{b_{i-1} - m_{\bullet(i-1)} - 2}{3} \leq \mu_i \leq \frac{b_{i-1} - m_{\bullet(i-1)}/2 + 2}{3}, \tag{15}$$

$$z_i \geq \left(\mu_i - \frac{2}{3}\right)\left(\frac{b_{i-1}}{2\mu_{i-1}} + \frac{1}{2}\right) + \frac{1}{3}. \tag{16}$$

We also have the following easy upper bound on z_i:

$$z_i \leq b_i + 2\mu_i \quad . \tag{17}$$

This is proved by observing that each of the μ_i chromons in $\mathcal{U}_i$ has at most two end vertices, so of the total number z_i of vertices in these chromons, at most $2\mu_i$ are end vertices. The rest are balanced. The inequality follows, with equality if and only if no chromon in $\mathcal{U}_i$ is trivial.

We now define some quantities that turn out to be useful lower bounds on the average sizes of chromons in $\mathcal{U}_i$. Set

$$x_i = x_0 2^{-i} - 3, \tag{18}$$

and note that

$$x_{i-1} = 2x_i + 3. \tag{19}$$

We now prove a series of further claims which will lead us to the desired result. Many of them will (at least until further notice) be subject to a technical condition given in the following definition.

We say k is *normal* if $k = 0$ or $\mu_j \geq x_j + 1/2$ for all j such that $1 \leq j \leq k$.

Claim 2: If i is normal, then

$$z_i \geq x_i \mu_i \quad .$$

Proof. We prove it by induction on i. The claim is immediate if $i = 0$.

Suppose that $i > 0$. The claim is immediate if $\mu_i = 0$, or $x_i \leq 0$, or (since $z_i \geq \mu_i$) if $0 < x_i \leq 1$. Otherwise,

$$\begin{aligned}
\frac{z_i}{\mu_i} &> \left(1 - \frac{2}{3\mu_i}\right)\left(\frac{b_{i-1}}{2\mu_{i-1}} + \frac{1}{2}\right) + \frac{1}{3} && \text{(by (16))} \\
&\geq \left(1 - \frac{2}{3\mu_i}\right)\left(\frac{z_{i-1}}{2\mu_{i-1}} - \frac{1}{2}\right) + \frac{1}{3} && \text{(by (17))} \\
&\geq \left(1 - \frac{2}{3\mu_i}\right)\frac{x_{i-1} - 1}{2} + \frac{1}{3} && \text{(by inductive hypothesis)} \\
&= \left(1 - \frac{2}{3\mu_i}\right)(x_i + 1) + \frac{1}{3} && \text{(by (19))} \\
&\geq x_i + \frac{2}{3} - \frac{1}{3(x_i + 1/2)} && \text{(since } i \text{ is normal)} \\
&> x_i \ ,
\end{aligned}$$

since now $x_i > 1$. □

Claim 3: If i is normal, then

$$b_i \geq \mu_i(x_i - 2) \ . \tag{20}$$

Proof. Use (17) and Claim 2. □

Claim 4: For any $k \geq 1$,
(i) If $X_{\max}$ is a circuit, then $m_{0k} = 0$; if it is a path, then $m_{0k} \leq 6$.
(ii) For all $j \geq 1$, $m_{jk} \leq 5\mu_j$.

Proof. (i) $X_{\max}$ is either a circuit-chromon, with no endpoints, or a path-chromon with exactly two endpoints. In the latter case, each endpoint has at most three balanced neighbours.

(ii) Now suppose $X \in \mathcal{U}_j$ and $j \geq 1$. X now has exactly two endpoints, and one of these (say w) is adjacent to some $v \in B_{j-1}$. So at most 5 of the neighbours of endpoints of X can be in $\cup_{i>j} B_i$. Since this is true for all $X \in \mathcal{U}_j$, we deduce (ii). $\square$

Claim 5: If k is normal, $k \geq 1$ and $x_k > 2$, then

$$m_{\bullet k} \leq \frac{5}{x_k - 2} \sum_{j<k} b_j \ \pm 3 \ ,$$

where the final summand is 3 if $X_{\max}$ is a path-chromon and -3 if it is a circuit-chromon.

Proof.

$$\begin{aligned}
m_{\bullet k} &= \sum_{j<k} m_{jk} \\
&= m_{0k} + \sum_{j=1}^{k-1} m_{jk} \\
&\leq (3 \pm 3) + 5 \sum_{j=1}^{k-1} \mu_j && \text{(Claim 4(i),(ii))} \\
&\leq 5 \sum_{j<k} \frac{b_j}{x_j - 2} \ \pm 3 && \text{(Claim 3)} \\
&\leq \frac{5}{x_k - 2} \sum_{j<k} b_j \ \pm 3 \ .
\end{aligned}$$

Throughout the above, the sign $\pm$ has the same interpretation as in the statement of the Claim. $\square$

Claim 6: If i is normal and $x_i > 4$, then

$$b_i > \frac{x_i - 4}{3} \sum_{j<i} b_j \ .$$

Proof. We use induction on i. It is immediate for $i = 0$, since the sum on the right is 0, and $x_0 > 2$ implies $b_0 > 0$.

Suppose $i = 1$.

$$\begin{aligned}
b_1 &\geq \mu_1(x_1 - 2) && \text{(Claim 3)} \\
&\geq \frac{x_1 - 2}{3} b_0 - \frac{2(x_1 - 2)}{3} && \text{(by (15), left side, and noting that } m_{\bullet 0} = 0) \\
&= \frac{x_1 - 3}{3} b_0 + \frac{b_0 - 2(x_1 - 2)}{3} \\
&> \frac{x_1 - 3}{3} b_0 \ > \ \frac{x_1 - 4}{3}
\end{aligned}$$

since $b_0 \ge x_0 - 2 > 2x_1 - 4$.

Now suppose that $i \ge 2$ and $b_{i-1} > ((x_{i-1} - 4)/3) \sum_{j<i-1} b_j$.

$$\begin{aligned}
b_i &\ge \mu_i(x_i - 2) && \text{(Claim 3)}\\
&\ge \frac{x_i-2}{3} b_{i-1} - \frac{x_i-2}{3} m_{\bullet(i-1)} - \frac{2(x_i-2)}{3} && \text{(by (15), left side)}\\
&\ge \frac{x_i-2}{3} b_{i-1} - \frac{5}{3} \cdot \frac{x_i-2}{x_{i-1}-2} \sum_{j<i-1} b_j \mp (x_i - 2) - \frac{2(x_i-2)}{3} && \text{(Claim 5)}\\
&> \frac{x_i-2}{3} b_{i-1} - \frac{5}{6} \sum_{j<i-1} b_j \mp (x_i - 2) - \frac{2(x_i-2)}{3} && \text{(by (19))}\\
&= \frac{x_i-4}{3} b_{i-1} + \frac{2}{3} b_{i-1} - \frac{5}{6} \sum_{j<i-1} b_j \mp (x_i - 2) - \frac{2(x_i-2)}{3}\\
&> \frac{x_i-4}{3} b_{i-1} + \frac{2}{3} \frac{x_{i-1}-4}{3} \sum_{j<i-1} b_j - \frac{5}{6} \sum_{j<i-1} b_j \mp (x_i - 2) - \frac{2(x_i-2)}{3}\\
& && \text{(by inductive hypothesis)}\\
&= \frac{x_i-4}{3} b_{i-1} + \left(\frac{4x_i}{9} - \frac{19}{18}\right) \sum_{j<i-1} b_j \mp (x_i - 2) - \frac{2(x_i-2)}{3} && \text{(by (19))}\\
&\ge \frac{x_i-4}{3} \sum_{j<i} b_j + \left(\frac{x_i}{9} + \frac{5}{18}\right) b_{i-2} \mp (x_i - 2) - \frac{2(x_i-2)}{3}.
\end{aligned}$$

But $b_{i-2} \ge \mu_{i-2}(x_{i-2} - 2) = \mu_{i-2}(4x_i + 7) \ge 4x_i + 7$ (by Claim 3, and using (19) twice). Also $x_i > 4$ implies $(x_i/9 + 5/18) > 13/18$. Hence $(x_i/9 + 5/18)b_{i-2} \mp (x_i - 2) - 2(x_i - 2)/3 > 0$, and the Claim follows. □

Claim 7: If k is normal and $x_k > 4$, then

$$m_{\bullet k} < \frac{15 b_k}{(x_k - 2)(x_k - 4)} + 3.$$

Proof. The claim is immediate if $k = 0$. Suppose then that $k \ge 1$.

$$\begin{aligned}
m_{\bullet k} &\le \frac{5}{x_k - 2} \sum_{j<k} b_j + 3 && \text{(Claim 5)}\\
&< \frac{5}{(x_k - 2)} \frac{3b_k}{(x_k - 4)} + 3 && \text{(Claim 6).}
\end{aligned}$$

□

Claim 8: If $i - 1$ is normal and $x_i > 4$, then

$$3\mu_i \ge \mu_{i-1}(x_{i-1} - 2) \left(1 - \frac{15}{(x_{i-1} - 2)(x_{i-1} - 4)} - \frac{5}{(x_{i-1} + 1/2)(x_{i-1} - 2)}\right).$$

Proof. Use (15) (left side) and Claims 7 and 3; the last term of the last factor uses Claim 3 and normality. □

Claim 9: If $0 \le i < \log_2(x_0/7)$, then i is normal.

Proof. We use induction on i. It is trivial if $i = 0$ or $x_0 \le 14$. Suppose $x_0 > 14$ and that the Claim is true for all $j < i$, where $i < \log_2(x_0/7)$. Since $i-1$ is normal, Claim 8 applies, and this together with $\mu_{i-1} \ge x_{i-1} + 1/2$ gives

$$\begin{aligned}
\mu_i &\ge (1/3)(x_{i-1}+1/2)(x_{i-1}-2)\left(1 - \frac{15}{(x_{i-1}-2)(x_{i-1}-4)} - \frac{5}{(x_{i-1}+1/2)(x_{i-1}-2)}\right) \\
&\ge (1/3)(x_{i-1}+1/2)(x_{i-1}-2) - 5\,\frac{x_{i-1}+1/2}{x_{i-1}-4} - \frac{5}{3} \\
&= (1/3)(2x_i+7/2)(2x_i+1) - 5\,\frac{2x_i+7/2}{2x_i-1} - \frac{5}{3} \qquad \text{(by (19))} \\
&= (2/3)(2x_i+7/2)(x_i+1/2) - 415/42 \\
&> x_i + 1/2,
\end{aligned}$$

since $x_i > 4$, which follows from our upper bound on i. □

It follows that, in all previous claims, if $x_0 > 7$ then any requirement of normality can be dropped. From this observation, and Claim 8, it follows that for all $i < \log_2(x_0/7)$, and provided $x_0 > 7$,

$$\begin{aligned}
\mu_i &\ge 3^{-i}\prod_{j=0}^{i-1} x_j\left(1-\frac{2}{x_j}\right)\left(1 - \frac{15}{(x_j-2)(x_j-4)} - \frac{5}{(x_j+1/2)(x_j-2)}\right) \\
&> 3^{-i}\prod_{j=0}^{i-1} x_j\left(1-\frac{2}{x_j}\right)\left(1 - \frac{20}{(x_j-2)(x_j-4)}\right) \\
&= 3^{-i}x_0^i 2^{-i(i-1)/2}\prod_{j=0}^{i-1}\left(1-\frac{3\cdot 2^j}{x_0}\right)\left(1-\frac{2}{x_j}\right)\left(1 - \frac{20}{(x_j-2)(x_j-4)}\right). \qquad (21)
\end{aligned}$$

From here it is routine to show that, for all $\varepsilon > 0$, there exists a constant $\alpha_\varepsilon > 7$ such that, if $0 \le i < \log_2(x_0/\alpha_\varepsilon)$ and $x_0 > \alpha_\varepsilon$,

$$\mu_i > x_0^i 3^{-i} 2^{-i(i-1)/2}(1-\varepsilon)^i.$$

Hence

$$\begin{aligned}
\log_2 \mu_i &> i\log_2(x_0(1-\varepsilon)/3) - i(i-1)/2 \\
&= i(\log_2 x_0 - i/2 + 1/2 + \log_2((1-\varepsilon)/3)).
\end{aligned}$$

Using $n > \mu_i$, and setting $i = \lfloor \log_2(x_0/\alpha_\varepsilon)\rfloor - 1$ and $\beta = x_0 2^{-i} > \alpha_\varepsilon$ (so $i = \log_2(x_0/\beta)$), we have

$$\begin{aligned}
\log_2 n &> \log_2(x_0/\beta)(\log_2(x_0/\beta) - (1/2)\log_2(x_0/\beta) + \log_2(\beta(1-\varepsilon)2^{1/2}/3)) \\
&> (1/2)(\log_2(x_0/\beta))^2,
\end{aligned}$$

provided $\varepsilon < 1 - 3\sqrt{2}/14$ and $x_0 > 2\alpha_\varepsilon$. It follows that

$$x_0 < \beta\, 2^{(2\log_2 n)^{1/2}}.$$

This gives us the required upper bound on maximum chromon size in terms of the number of vertices of the graph. (Note that if our lower bound $x_0 > 2\alpha_\varepsilon$ does not hold, then x_0 is bounded above by a constant.) $\square$

We now show that the worst case bound given in Theorem 2 is sharp, up to a constant factor, for the 2-assignments considered there.

Let G_1 be the 4-star $K_{1,4}$ with two of its leaves distinguished; we call these its *terminals*. The central vertex and the terminals are coloured White; the other two leaves are Black. Now, for each $k \geq 2$, form G_k recursively from G_{k-1} as follows. Take a path $v_0, v_1, \ldots, v_{3\cdot 2^{k-1}-1}$ of $3 \cdot 2^{k-1}$ vertices, and colour these vertices according to the parity of k: Black for even, White for odd. Take $3 \cdot 2^{k-2} - 1$ disjoint copies of G_{k-1}; call them $G_{k-1}^{(j)}$, $0 \leq j \leq 3 \cdot 2^{k-2} - 2$. For each such j, set $j' = 2j + 1$, join one terminal of $G_{k-1}^{(j)}$ to both $v_{j'}$ and $v_{j'+1}$, and join the other terminal to both $v_{j'+2}$ and $v_{j'+3}$ (or v_1 and v_2, if $j = 3 \cdot 2^{k-2} - 2$). Finally, the terminals of G_k are v_0 and $v_{3\cdot 2^{k-1}-1}$.

It can be shown that the 2-assignment given here cannot be improved by the local operations (a)–(c), and that the size of the largest chromon is $\Theta\left(2^{(2\log_2 n_k)^{1/2}}\right)$, where $n_k = |V(G_k)|$.

This shows that the constant factor $\sqrt{2}$ in the exponent of the upper bound of Theorem 2 cannot to be reduced by just looking at any 2-assignment that is locally optimal in our sense and improving on our analysis of maximum chromon size. The results of Alon et al. [4] and Haxell et al. [12] show however that some 2-assignments have much smaller largest chromons.

One can propose further operations that might, heuristically, improve the chromon sizes still further. For example, consider the following:

(d) Suppose u, v, w form a path of length 2 such that: u and w are nonadjacent and balanced and have the same colour, and v is differently coloured and does not have four antichromatic neighbours. Then flip the colours of u, v and w.

This reduces $e(G, c)$. A 2-assignment that is locally optimal with respect to the operations (a)–(d) has an additional property not generally found in those discussed in Theorem 2: if v is an end vertex of a chromon, then either (i) v has at most one balanced neighbour, or (ii) v has two balanced neighbours that must belong to the same chromon and are adjacent, or (iii) v has three balanced neighbours which constitute a circuit-chromon which is actually a triangle. This allows some slight tightening of the analysis in the proof of Theorem 2 (in which Lemma 1 is now used with $\delta = 2$). However, the effect is not huge: the bound still has the form $O\left(2^{(2\log_2 n)^{1/2}}\right)$, with the improvement being just a constant factor. The graphs G_k described above demonstrate, again, that this bound is best possible up to a constant factor.

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