Coclass theory

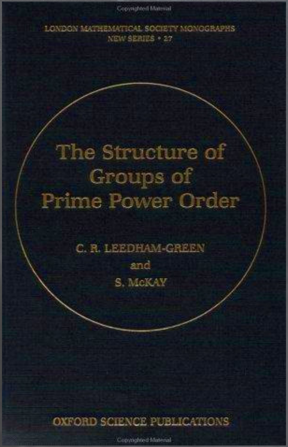
▶ Go to Automorphisms

▶ Go to End

Resources

The structure of groups of prime-power order
C. R. Leedham-Green, S. McKay
Oxford Science Publications (2002)

and some recent papers on coclass graphs
(Eick, Leedham-Green, Newman, O'Brien, D.)



Classifying p-groups by order

Recall:

order	#
1	1
2	1
4	2
8	5
16	14
32	51
64	267

order	#
128	2,328
256	56,092
512	10,494,213
1024	49,487,365,422
2048	$>$1,774,274,116,992,170

"The precise structure of p-groups is too complex for the human intellect."
(Leedham-Green & McKay 2002)

Maximal class

Maximal class

A p-group G of order p^n has **maximal class** if it has nilpotency class $n-1$.

- Groups of maximal class have been investigated in detail.
  (Wiman 1954, Blackburn 1958, Leedham-Green & McKay 1976–1984, Fernández-Alcober 1995, Vera-López et al. 1995–2008)
- The 2- and 3-groups of maximal class are classified.
  (Blackburn: Description by finitely many *parametrised presentations*.)
- The 5-groups of maximal class are investigated in detail.
  (Leedham-Green & McKay, Newman 1990, D., Eick & Feichtenschlager 2007)
- For $p \geq 7$ such a classification is open.

Coclass

Maximal class is an important special case in **coclass theory**:

Coclass

A p-group G of order p^n and nilpotency class c has **coclass** $n - c$.

Thus:

- the p-groups of maximal class are the p-groups of coclass 1,
- coclass is an isomorphism invariant.

Strategy: Investigate the p-groups of a fixed coclass.
(Leedham-Green & Newman 1980)

Leedham-Green & Newman proposed five **Coclass Conjectures A–E** on the structure of the p-groups of a fixed coclass. Their proof was a first milestone in **coclass theory** and provided a deep insight in the structure of p-groups.

Coclass

Coclass Conjectures

Theorem A: There is a function $f(p,r)$ such that every p-group of coclass r has a normal subgroup of nilpotency class 2 and index at most $f(p,r)$.

Theorem B: There is a function $g(p,r)$ such that every p-group of coclass r has derived length at most $g(p,r)$.

Theorem C: Every pro-p group of coclass r is solvable.
($=$ inverse limit of finite p-groups of coclass r.)

Theorem D: There are only finitely many isomorphism types of infinite pro-p groups of coclass r.

Theorem E: There are only finitely many isomorphism types of solvable infinite pro-p groups of coclass r.

(Leedham-Green 1994, Shalev 1994)

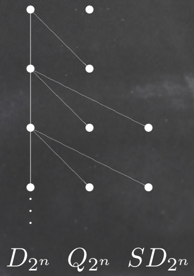
Coclass graph

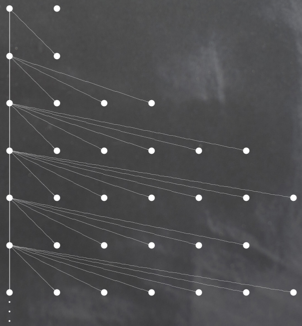
Main approach since 1999: analyse **the coclass graph $\mathcal{G}(p, r)$.**

Vertices: Isomorphism type reps of finite p-groups of coclass r.

Edges: $G \rightarrow H$ if and only if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$; then $|H| = p|G|$.

Examples: $\mathcal{G}(2,1)$ $\mathcal{G}(3,1)$





Coclass graph

The infinite paths in $\mathcal{G}(p, r)$:

- There is 1-to-1 correspondence between the **infinite pro-p groups** of coclass r (up to isom.) and the *maximal* infinite paths in $\mathcal{G}(p, r)$.

It follows from the Coclass Theorems:

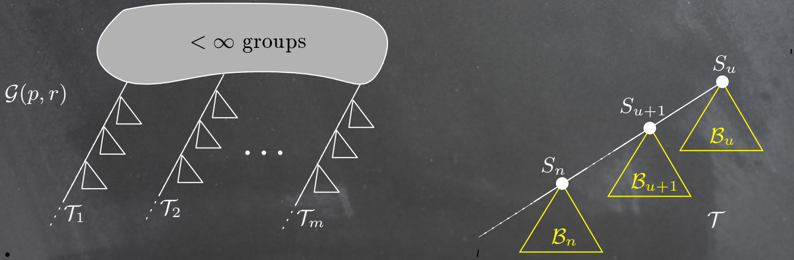
- The infinite paths are *well-understood* and finite in number!
- Only finitely many groups are not connected to an infinite path.

Number of infinite paths in $\mathcal{G}(p, r)$:

- p arbitrary and $r = 1$ (Blackburn): 1
- $p = 2$ and $r = 2, 3$ (Newman & O'Brien): 5, 54
- $p = 3$ and $r = 2, 3, 4$ (Eick): 16, ≥ 1271, ≥ 137299952383

General structure of coclass graphs

$\mathcal{G}(p,r)$ can be partitioned into a finite subgraph and finitely many infinite trees each having a unique infinite path starting at its root.
These trees are the **coclass trees** of $\mathcal{G}(p,r)$.

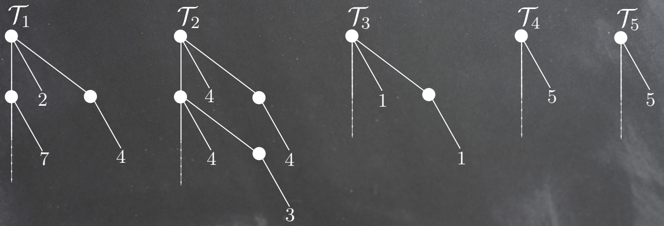


Let $\mathcal{T}$ be a coclass tree in $\mathcal{G}(p,r)$ with corresponding pro-p group S:

- The groups $S_n = S/\gamma_n(S)$ with $n \geq u$ form the **mainline** of $\mathcal{T}$.
- The finite subtrees $\mathcal{B}_n$ are the **branches** of $\mathcal{T}$.

The graph $\mathcal{G}(2,2)$

The five coclass trees of $\mathcal{G}(2,2)$:
(Newman & O'Brien 1996)

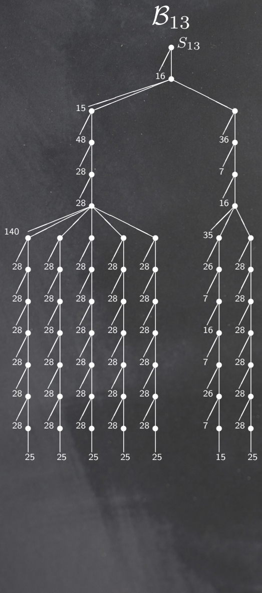


- The branches are isomorphic with periodicity 1 and 2, respectively.
- The roots have order $2^6, 2^6, 2^4, 2^4$, and 2^5, respectively.
- There are 19 groups which do not lie in any of these trees.

For arbitrary r: branches of trees in $\mathcal{G}(2,r)$ have *bounded depths*.

This does not hold for odd primes, except $(p,r) = (3,1)$.

Two branches in $\mathcal{G}(5,1)$



Based on significant computation with the p-group generation algorithm:

Central Conjecture

- $\mathcal{G}(p,r)$ can be described by a finite subgraph and *periodic patterns*.
- The p-groups of coclass r can be *classified*.
  ($\rightsquigarrow$ description by finitely many *parametrised presentations*)

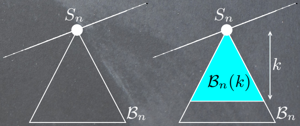
Example: the groups in $\mathcal{G}(2,1)$ of order $2^n \geq 16$

$$
\begin{aligned}
D_{2^n} &= \mathrm{Pc}\langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{-1}\rangle,\\
SD_{2^n} &= \mathrm{Pc}\langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{2^{n-2}-1}\rangle,\\
Q_{2^n} &= \mathrm{Pc}\langle a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ a^b = a^{-1}\rangle.
\end{aligned}
$$

Known results:

- The Central Conjecture is proved for $p = 2$.
  (Newman & O'Brien 1999, du Sautoy 2001, Eick & Leedham-Green 2008)
- Applications for $p = 2$: Some invariants of the groups can be described in a uniform way. (Eick 2006, 2008)
- For odd primes: Only partial results are known.

Periodicity I



$\mathcal{T}$ coclass tree with branches $\mathcal{B}_u, \mathcal{B}_{u+1}, \ldots$
The **pruned branch** $\mathcal{B}_n(k)$ is the subtree of $\mathcal{B}_n$ induced by groups of depth at most k in $\mathcal{B}_n$.

Theorem (du Sautoy 2001, Eick & Leedham-Green 2008)
There exist integers $f = f(\mathcal{T}, k)$ and $d = d(\mathcal{T})$ such that for all $n \geq f$

$$\mathcal{B}_n(k) \cong \mathcal{B}_{n+d}(k).$$

Eick & Leedham-Green determined d, an upper bound for f, and proved:

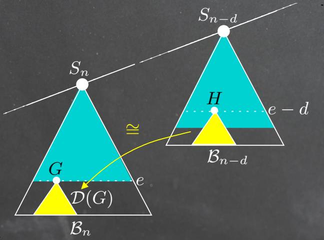
Theorem (Eick & Leedham-Green 2008)
The infinitely many groups in $\mathcal{B}_n(k)$, $n \geq u$, can be described by finitely many parametrised presentations.

These theorems prove the Central Conjecture for $p = 2$;
they are **not** sufficient to prove it for odd primes.

Periodicity II

For odd primes: Some coclass trees contain sequences of branches $\mathcal{B}_i, \mathcal{B}_{i+d}, \mathcal{B}_{i+2d}, \dots$ with strictly increasing depths.

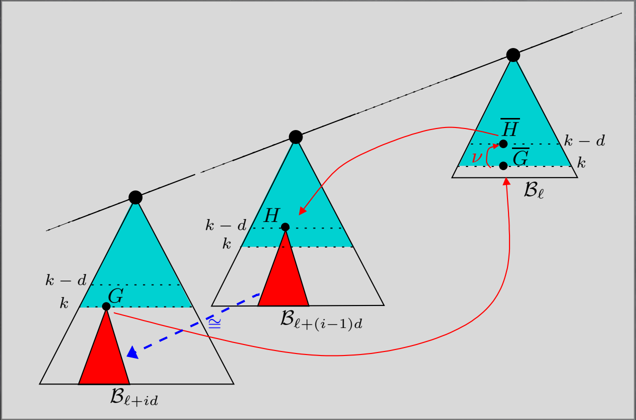
Problem: Describe the *growth* of these branches.



Conjecture (based on experiments for $\mathcal{G}(5,1)$ and $\mathcal{G}(3,2)$)
If e and n are large enough, then for every group G at depth e in $\mathcal{B}_n$ there exists a group H at depth $e-d$ in $\mathcal{B}_{n-d}$ such that $\mathcal{D}(G) \cong \mathcal{D}(H)$.

This conjecture is rather *vague* and only very little is known; some important results for $\mathcal{G}(p,1)$ exist.

Conjecture W



Conjecture W (Eick, Leedham-Green, Newman, O'Brien 2013)

Fix k and ℓ such that $\mathcal{B}_\ell(k) \cong \mathcal{B}_{\ell+jd}(k)$ for all j.
Let $\overline{K} \in \mathcal{B}_\ell$ be the group corresponding to $K \in \mathcal{B}_{\ell+jd}$.
There is a map ν from the groups at depth k in $\mathcal{B}_\ell$ to the groups at depth $k-d$ in $\mathcal{B}_\ell$ such that the picture holds... in particular, $\mathcal{D}(G) \cong \mathcal{D}(H)$

Important subtree: skeleton groups

Let $\mathcal{T}$ be a coclass tree in $\mathcal{G}(p,r)$, with associated pro-p group S.

Problem: the branches of $\mathcal{T}$ are usually pretty "thick" and "wide".

Skeleton groups (for split pro-p groups)

Let $S = P \ltimes T$ with $T \cong (\mathbb{Z}_p^d, +)$ and uniserial series $T = T_0 > T_1 > T_2 > \dots$
Let $\gamma\colon T \wedge T \twoheadrightarrow T_n$ be P-module hom and $m \geq n$ such that $\gamma(T_n \wedge T) \leq T_m$.
Let $T_{\gamma,m} = (T/T_m, \circ)$ with $(a + T_m) \circ (b + T_m) = a + b + \frac{1}{2}\gamma(a \wedge b) + T_m$;
then $C_{\gamma,m} = P \ltimes T_{\gamma,m}$ is the skeleton group defined by γ and m.

Theorem (Leedham-Green 1994)

If G is in $\mathcal{T}$, then there is $N \trianglelefteq G$ with order bounded by r and p, such that G/N is a "skeleton group"; the structure of skeleton groups is easier to understand, and the "skeleton of $\mathcal{T}$" is a significant subtree of $\mathcal{T}$.

The graph $\mathcal{G}(5,1)$

Shalev ("Problem 3", 1994): Classify the 5-groups of maximal class.

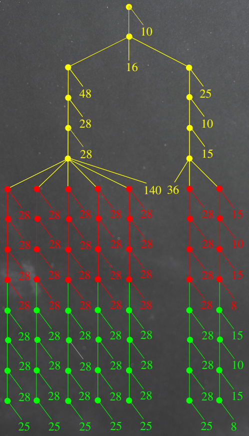
The graph $\mathcal{G}(5,1)$ has a unique coclass tree $\mathcal{T}(5)$; write $\mathcal{T}_k = \mathcal{B}_k(k-4)$.

Theorem (D. 2010)

The pruned branches $\mathcal{T}_k$ of $\mathcal{T}(5)$ can be described by a finite subgraph and the periodicities of type I & II. The groups in these pruned branches can be classified by finitely many parametrised presentations with ≤ 2 integer parameters.

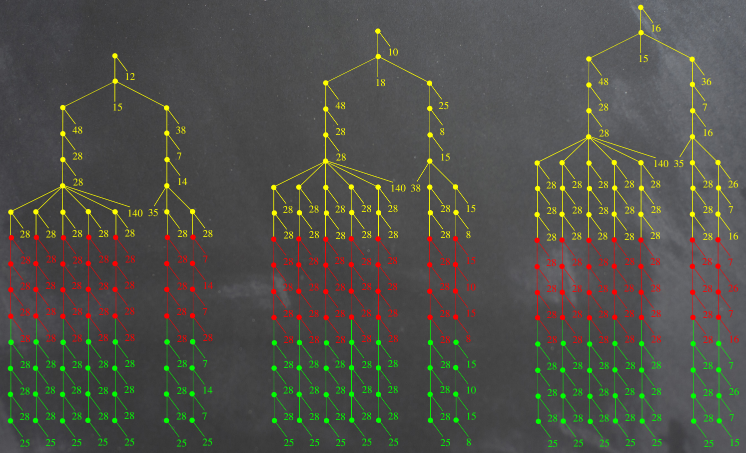
$\mathcal{G}(5,1)$: the trees $\mathcal{T}_{10+4x}$ with $x \geq 1$

Proved: $\mathcal{T}_{10+4x}$ consists of the yellow part and x copies of the red part:



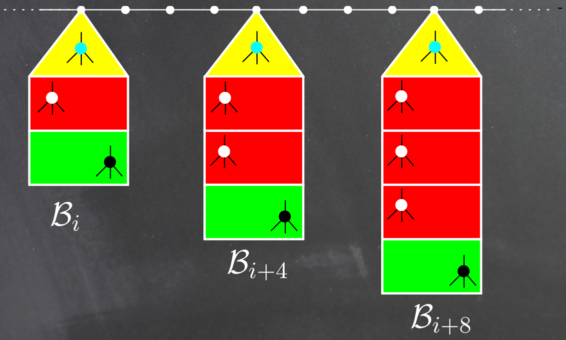
Conjecture: The difference $\mathcal{B}_{10+4x} \setminus \mathcal{T}_{10+4x}$ is the green part.

$\mathcal{G}(5,1)$: the trees $\mathcal{T}_{11+4x}$, $\mathcal{T}_{12+4x}$, and $\mathcal{T}_{13+4x}$



$\mathcal{G}(5,1)$: Periodicity classes

The origins of the periodicity classes in $\mathcal{T}_i$ with $14 \leq i \leq 17$:



"**Cyan**": 1 Parameter
"**White**": 2 Parameters
"**Black**": 1 Parameter (conjectured!)

Skip stuff

The graph $\mathcal{G}(3,2)$

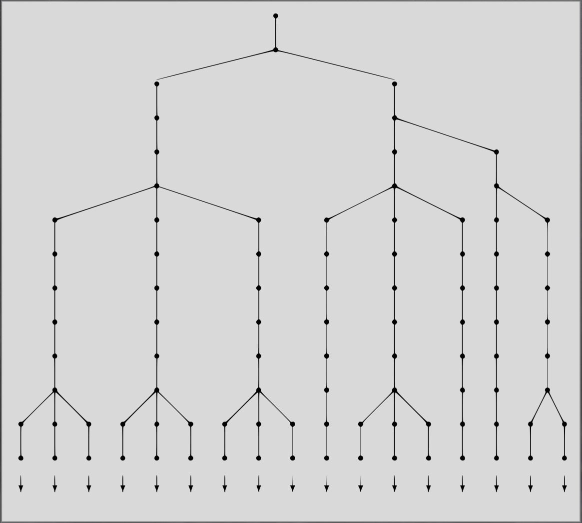
Theorem (Eick, Leedham-Green, Newman, O'Brien 2013)
Conjecture W holds for the skeletons in $\mathcal{G}(3,2)$.

Moreover:

- $\mathcal{G}(3,2)$ has 16 coclass trees, but only 4 have unbounded depths
- some coclass trees admit both, subsequences of branches of bounded depths and subsequences of branches of unbounded depths
- occurrence of "exceptional isomorphisms" between skeleton groups
- the "twigs" are described conjecturally

$\mathcal{G}(3,2)$: skeletons

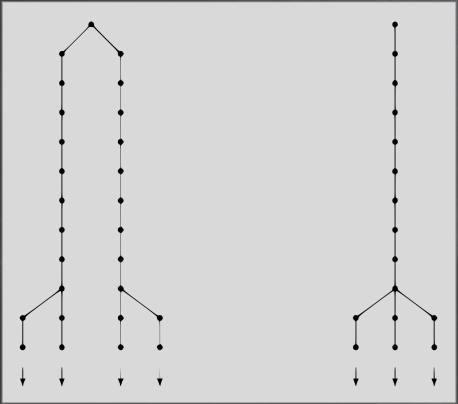
Skeletons of the split pro-3 group:



Conjectural description of twigs: usually depth 3 and up to 20,000 vertices

$\mathcal{G}(3,2)$: skeletons

Skeletons of the three non-split pro-3 groups;
skeleton only exists if class of root is congruent 0 modulo 3:



Conjectural description of twigs: up to depth 6 and 20,000 vertices

Know periodicity results

Most results and conjectures are motivated by **computer experiments**, in particular, with the p-group generation algorithm.

What is known so far:

- periodicity of type I for all graphs $\mathcal{G}(p, r)$,
- significant *local* results on periodicity of type II for the graphs $\mathcal{G}(p, 1)$,
- most of $\mathcal{G}(5, 1)$ and the skeleton structure of $\mathcal{G}(3, 2)$

Comments on periodicity of type II:

- all known results consider pruned branches
- most results consider only skeleton groups
- $\mathcal{G}(5, 1)$ and $\mathcal{G}(3, 2)$ only have branches of finite width
- D. & Eick recently considered $\mathcal{G}(p, 1)$ in more detail (2016)

There is still a lot to do – we're working on it . . . 😊

skip stuff

A new result: maximal class and 'large' aut grps

Now consider $\mathcal{G}(p,1)$ with $p \geq 7$.

Let $\mathcal{T}$ be the coclass tree with branches $\mathcal{B}_j$ and *bodies* $\mathcal{T}_j = \mathcal{B}_j(j-2p+8)$.

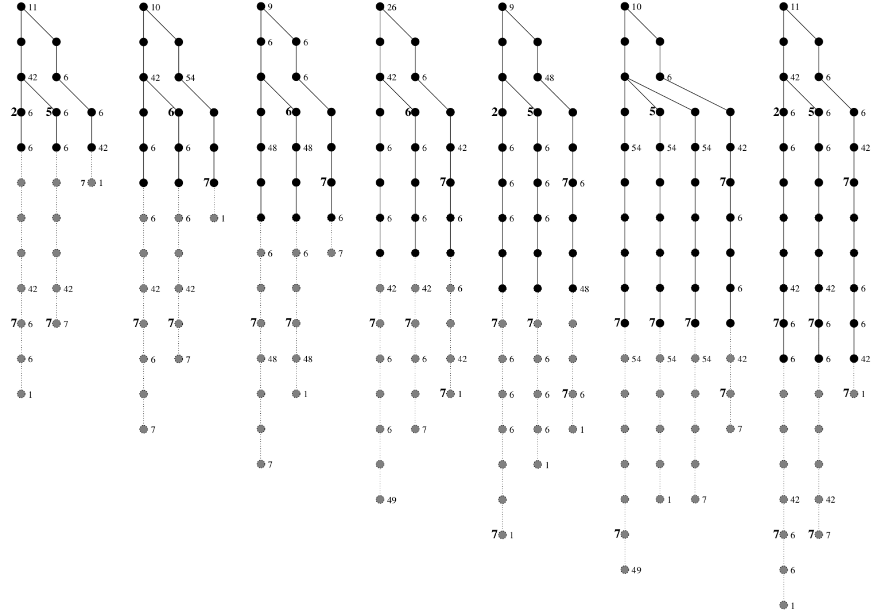
Motivated by the known periodicity results for $\mathcal{G}(p,1)$ and **promising computer experiments**, Bettina Eick and I studied the following subtrees of $\mathcal{T}$:

Definition

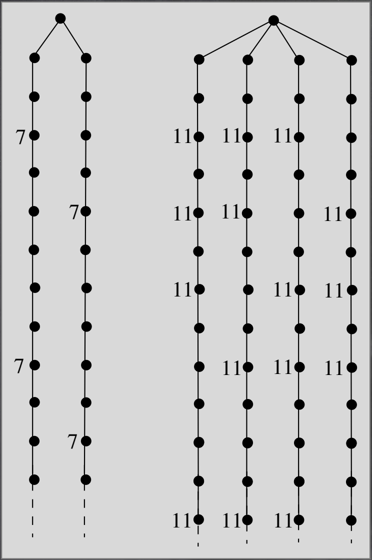
Let $\mathcal{B}_j^*$ be the subtree of $\mathcal{B}_j$ consisting of all groups whose automorphism group order is divisible by $p-1$. Let $\mathcal{S}_j^*$ be the subtree of the body $\mathcal{T}_j$ consisting of all *skeleton groups* whose automorphism group order is divisible by $p-1$.

(Note: $p-1$ is essentially the largest possible p'-part of that aut-group order.)

$\mathcal{G}(7,1)$: the trees $\mathcal{B}_j^*$ and $\mathcal{S}_j^*$ for $j = 10, \ldots, 16$



Conjectured structure of S_j^* for $p = 7, 11$



For $p = 7$:

- depth $j - 6$
- 2 groups $G_{j,1}, G_{j,2}$ at depth 1
- 7-fold ramifications at levels
  - $2 + 6\mathbb{N}$ in path of $G_{j,1}$
  - $4 + 6\mathbb{N}$ in path of $G_{j,2}$

For $p = 11$:

- depth $j - 14$
- 4 groups $G_{j,1}, \ldots, G_{j,4}$ at depth 1
- 11-fold ramifications at levels
  - $\{2, 4, 6\} + 10\mathbb{N}$ in path of $G_{j,1}$
  - $\{2, 4, 8\} + 10\mathbb{N}$ in path of $G_{j,2}$
  - $\{2, 6, 8\} + 10\mathbb{N}$ in path of $G_{j,3}$
  - $\{4, 6, 8\} + 10\mathbb{N}$ in path of $G_{j,4}$

p-groups of maximal class with 'large' aut-group

Let $d = p - 1$ and $\ell = (p-3)/2$.

Theorem (2016)

- The skeleton $\mathcal{S}_n^*$ has ℓ groups $G_{n,1}, \ldots, G_{n,\ell}$ at depth 1.
- Ramifications are always p-fold and occur exactly at depth

$$\{2, 4, \ldots, d-2\} \setminus \{d - 2i\} \; + \; d\mathbb{N}$$

in the path of $G_{n,i}$, for $i = 1, \ldots, \ell$.

The proof is heavily based on number theory and existing results for maximal class groups (19 pages, submitted 2016).

Conjectural description of twigs:
structure of twigs depends only on i, on $(e \bmod d)$, and on $(n \bmod d)$.

This is the first periodicity result supporting Conjecture W in the context of coclass trees with unbounded width.

The end . . .

- Go to Coclass
- Go to Overview

......looking back:

1. motivation
2. pc presentations
3. p-quotient algorithm
4. p-group generation
5. isomorphism test
6. automorphism groups
7. coclass theory