Computing with Finitely Presented Groups

Problem Sheet

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Question 1.1
Use Tietze transformations to show that $G = \langle g, h \mid ghghg \rangle \cong \langle a \mid \emptyset \rangle \cong (\mathbb{Z}, +)$.

Question 1.2
Use Tietze transformations to show that $G = \langle x, y, z \mid x = yzy^{-1},\ y = zxz^{-1},\ z = xyx^{-1} \rangle$ is isomorphic to $H = \langle x, y \mid xyx = yxy \rangle$, and to $K = \langle a, b \mid a^3 = b^2 \rangle$.

Question 1.3
Use von Dyck's Theorem to show that the Wicks group $W = \langle a, b \mid a^3b^4a^5b^7, a^2b^3a^7b^8 \rangle$ has C_{11} as a quotient.

Question 1.4
Do the following please; no coset enumeration is required.

a) Show that the group $A = \langle a, b \mid b^2 = a^{-1}ba, a^2 = b^{-1}ab \rangle$ has size 1.
b) Show that $B = \langle a, b \mid a^2,\ b^2a = ab^2,\ b^3 \rangle$ is cyclic of order 6.
c) Show that the group $C = \langle a, b, c \mid a^2, b^2, c^2, ba = ac, ca = ab, cb = ac \rangle$ is isomorphic to Sym_3.

Question 1.5
Let $G = \langle X \mid \mathcal{R} \rangle$ be a finitely presented group with $X = \{x_1, \ldots, x_m\}$. Let $U \leqslant G$ be a group of finite index n, and let $\{t_1, \ldots, t_n\} \subseteq G$ with $t_1 = 1$ be a right transversal of U in G. Assume we have the coset table corresponding to $[G : U] = n$, and the corresponding permutation action of G on T, such that $Ut_ig = Ut_{i^g}$ for $t_i \in T$ and $g \in G$. For $g \in G$ let $\overline{g} \in T$ be the unique element such that $Ug = U\overline{g}$. Recall that the Schreier generators are defined as

$$S = \{s_{i,j} = t_ix_j(\overline{t_ix_j})^{-1} : t_i \in T, x_j \in X, s_{i,j} \neq 1\}.$$

a) Show that $S \subseteq U$.
b) Show that if $t \in T$ and $x \in X$, then $tx^{-1}(\overline{tx^{-1}})^{-1}$ is an inverse of a Schreier generator in S.
c) Show that $U = \langle S \rangle$: take a general word in the generators X and their inverses, and then try to rewrite it as a product of Schreier generators, their inverses, and elements in T; look at the lectures for a hint.
d) Let $r \in \mathcal{R}$ be a defining relator and $t_\ell \in T$. Explain why $t_\ell r t_\ell^{-1}$ (as an element in F_X) can be written as that in the Schreier generators that we obtain when parsing the relator r in the amended coset table starting in the row with label t_ℓ.

Question 1.6
Let F be a free group of rank r and let $U \leqslant F$ be a subgroup of index n. Deduce from our discussion of the Reidemeister-Schreier method that U is a free group of rank $1 + n(r-1)$; this is the *Nielsen-Schreier Theorem.*

Question 1.7
Apply Todd-Coxeter coset enumeration to determine the size of $G = \langle a, b \mid abab^{-1}, baba^{-1} \rangle$.
Tip: Maybe start with the following definitions: $1^b = 2$, $2^a = 3$, $3^b = 4$, $4^b = 5$, $2^b = 6$, $3^a = 7$, $6^a = 8$.
Can you also determine the isomorphism type of the group?

Question 1.8
Let $n \geqslant 1$ be an integer. Show that $\langle x, y \mid x^n y^{n+1}, x^{n+1} y^{n+2} \rangle$ is the trivial group, and discuss why Todd-Coxeter coset enumeration requires the construction of at least n cosets.

Question 1.9
Consider the group $G = \langle a, b \mid a^2, b^3, (ab)^5 \rangle$ with subgroup $U = \langle a, a^b \rangle$. Coset enumeration yields the following coset table, where boldface entries specify the definitions:

nr	a	a^{-1}	b	b^{-1}
1	1	1	**2**	**3**
2	**4**	4	3	1
3	3	3	1	2
4	2	2	**5**	**6**
5	6	6	6	4
6	5	5	4	5

Find a finite presentation for U, by doing the following:

a) Write down transversal elements for each coset.
b) Write down Schreier generators and the amended coset table.
c) Write down Reidemeister relators.
d) Write down the resulting presentation for U; can you simplify it?

Question 1.10
Similar to the previous questions, consider $G = \langle a, b \mid a^2, b^3 \rangle$ with subgroup $U = \langle bab^{-1}a^{-1}, b^{-1}aba^{-1} \rangle$. Coset enumeration yields the following coset table, where boldface entries specify the definitions:

nr	a	a^{-1}	b	b^{-1}
1	**2**	2	**3**	**4**
2	1	1	**5**	**6**
3	5	5	4	1
4	6	6	1	3
5	3	3	6	2
6	4	4	2	5

Deduce that U is normal, and determine the isomorphism type of G/U; find a presentation for U.

Question 1.11
Do the following to show that the free group of rank 2 has a subgroup that is not finitely generated.

a) Let $A = \bigoplus_{z \in \mathbb{Z}} (\mathbb{Z}_2, +)$ be the group of all binary sequences $(a_z)_{z \in \mathbb{Z}}$ with only finitely many $a_i \neq 0$, with component-wise addition. Let σ be the automorphism of A that maps $a = (a_z)_{z \in \mathbb{Z}} \in A$ to $a^\sigma = (a_{z-1})_{z \in \mathbb{Z}}$, that is, σ is a "right-shift". Let $K = \langle \sigma \rangle \ltimes A$ be semidirect product where σ acts naturally on A. Do the following:
   1. Show that K can be generated by two elements.
   2. Show that the derived subgroup $K' = [K, K]$ consists exactly of all sequence b that have an even number of 1s.
   3. Deduce from 2) that K' is *not* finitely generated.

b) Let F be a free group of rank 2. Deduce from a) that F' is not finitely generated.

The following questions can be considered from Lecture 2 onwards.

Question 1.12
Let $G = \langle a, b, c \mid abcba, b^a cb \rangle$. Construct an epimorphism $G \to A$ where A is the largest abelian quotient of G.

Question 1.13
Please discuss the following.

a) Let $\mathcal{P}$ be a property of groups that is preserved by direct products and subgroups (e.g. "abelian" or "nilpotent" ...). Let G be a group and let $\mathcal{N}$ be a finite collection of normal subgroups of G such that G/U has property $\mathcal{P}$ for every $U \in \mathcal{N}$. Let $I = \bigcap_{U \in \mathcal{N}} U$, and show that G/I has property $\mathcal{P}$.

b) Let G be an fp group, and assume you can use the p-group quotient algorithm to construct finite p-group quotients of G. Discuss how a construction as in a) can be used to construct finite nilpotent quotients of G.

Question 1.14
Let $G = \mathrm{Sym}_4$.

a) Determine a polycyclic series and a polycyclic generating set for G.

b) Construct the polycyclic presentation for G which corresponds to the polycyclic generating set you have determined in a); is your presentation consistent?

Question 1.15
Consider the following polycyclic presentations: apply consistency checks to show that they are not consistent; then determine a consistent presentation for the groups.

a) $G = \mathrm{Pc}\langle a, b, c \mid a^4 = b^2 c,\ b^4,\ c^4,\ b^a = b^3,\ c^b = c^3 \rangle$.

b) $H = \mathrm{Pc}\langle u, v, w \mid u^4 = w,\ v^4 = w,\ w^4 = 1,\ w^u = w^2, \rangle$.

Question 1.16
Consider the group

$$G = \mathrm{Pc}\langle a, b, c, d, e, f \mid a^2, b^2, c^3, d^3, e^5, f^5, c^a = c^2, d^a = d^2, e^c = ef^3, f^a = e^4 f^4, f^c = e^4 f^3 \rangle.$$

Use collection to find the normal form of the element $g = feca$; maybe also collect some other random words in the generators.

Question 1.17
For an integer $n > 2$ consider the dihedral group $G = \langle r, m \mid r^{2^{n-1}},\ m^2,\ r^m = r^{2^{n-1}-1} \rangle$.

a) Find the normal form of the element $w = rmr^2m^2r^3m^3$.

b) Find a polycyclic series of G whose associated pcgs has relative orders $[2, \ldots, 2]$.

c) Find a polycyclic presentation of G, associated to the pcgs you have found in b).

Question 1.18
Compute a wpcp of the group

$$G = \langle a, b, c \mid a^9,\ b^9,\ c^9,\ [[b, a], a] = a^3,\ (aba)^9,\ (ba)^5 a = b,\ [a, c] \rangle;$$

you can use that G has order 3^3.

The following questions can be considered from Lecture 3 onwards.

Question 1.19
Let $X = \{a, b, c\}$. What are the first 20 words in X^* in the shortlex ordering defined by $b < c < a$?

Question 1.20
Consider the rewriting system $S = (X, \mathcal{R})$ where $X = \{c, d, y, z\}$ and $\mathcal{R} = \{(yz, cz), (yc, dy), (dc, cy)\}$.

a) Show that S is confluent with respect to the shortlex ordering on X^* defined by $c < d < y < z$.
b) For $n \geqslant 0$, show that $y^n z \xrightarrow{*} c^n z$.

Question 1.21
Let $X = \{a, A, b, B\}$ and consider the *wreath product ordering* on X^* where a, A, b, B have levels $1, 2, 3, 4$, respectively; this is defined as follows. Let $u, v \in X^*$. If the highest level of the letters in v is larger than the highest level of the letters in u, then $u < v$. Now suppose u and v both have letters of the same highest level r; remove all letters of level $< r$ in u and v to obtain words u' and v' (only containing letters of level r). If $u' < v'$ in shortlex ordering, then we say $u < v$. Lastly, suppose u' and v' are equal, say $u' = v' = z_1 \cdots z_t$ where each z_j has level r. Then we can write $u = x_1 z_1 x_2 z_2 \ldots z_t x_{t+1}$ and $v = y_1 z_1 y_2 z_2 \ldots z_t y_{t+1}$ where each x_i, y_i only involves letters of level $< r$. We define $u < v$ if there exists $k \leqslant t$ such that $x_i = y_i$ for $i = 1, \ldots, k$ and $x_{k+1} < y_{k+1}$ in shortlex ordering.

a) Sort the following words using the wreath product ordering defined above:

$$u_1 = a^2ba^2AB^3abaBA, \qquad u_2 = b^2abA^2, \qquad u_3 = A^{100}ba,$$
$$u_4 = A^3baBA^{10}BabaBA, \qquad u_5 = A^3baBaBabaBA.$$

b) Consider the Baumslag-Solitar group

$$B(m, n) = \langle a, b \mid b^{-1}a^m b = a^n \rangle.$$

Identify $A = a^{-1}$ and $B = b^{-1}$. Show that $B(m, n)$ has a complete rewriting system where the rules are the *inverse rules* (Aa, ε), (aA, ε), (Bb, ε), (bB, ε), and four *additional rules*

$$(a^n B, Ba^m), \quad (a^m b, ba^n), \quad (Ab, a^{m-1}bA^n), \quad (AB, a^{n-1}BA^m).$$

You can assume that the ordering is a strict well-founded ordering (see Sims, Proposition 1.7 for a proof).

Question 1.22
Show that the group

$$G = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

has a $C'(1/6)$ presentation; then use Dehn's algorithm to reduce the word $w = d^{-1}acdc^{-1}d^{-1}aba^{-2}b^{-1}cdc^{-1}$.

Question 1.23
Let $G = \langle X \mid \mathcal{R} \rangle$ be a finitely presented group and shortlex automatic structure (W, M), that is, W is a word acceptor (that accepts for each $g \in G$ a unique word in $(X \cup X^{-1})^*$ that represents g) and for each $x \in X \cup X^{-1}$ we have a multiplier automaton M_x. Assume that for every $w \in L(W)$ and $x \in X \cup X^{-1}$, we can compute $w' \in L(W)$ such that $(w', w) \in L(M_x)$. Discuss how this can be used to solve the word problem in G.

Question 1.24
Consider the group $G = \langle z \mid \emptyset \rangle \cong (\mathbb{Z}, +)$, with monoid generating set $X = \{z, z^{-1}\}$. A word acceptor W is given by the finite state automaton in Figure 1 (left), where unspecified transitions lead to a fail state. Note that the accepted language is exactly $L(W) = \{z^i : i \in \mathbb{Z}\}$, the set of all reduced words in G. We need two multiplier automaton, called M_z and $M_{z^{-1}}$. These are represented by the same finite state automaton in Figure 1 (right), but with different accept states: the accept state for M_z is z, and the one for $M_{z^{-1}}$ is z^{-1}. The transition (x, x) stands for (z, z) and (z^{-1}, z^{-1}). As before, unspecified transitions lead to a fail state.

Convince yourself that this provides an automatic structure for G; e.g., if $u, v \in L(W)$, then $(u, v)^+ \in L(M_z)$ if and only if $u =_G vz$.[1] Motivated by this, please get an automatic structure for the group $(\mathbb{Z}, +)^2$.

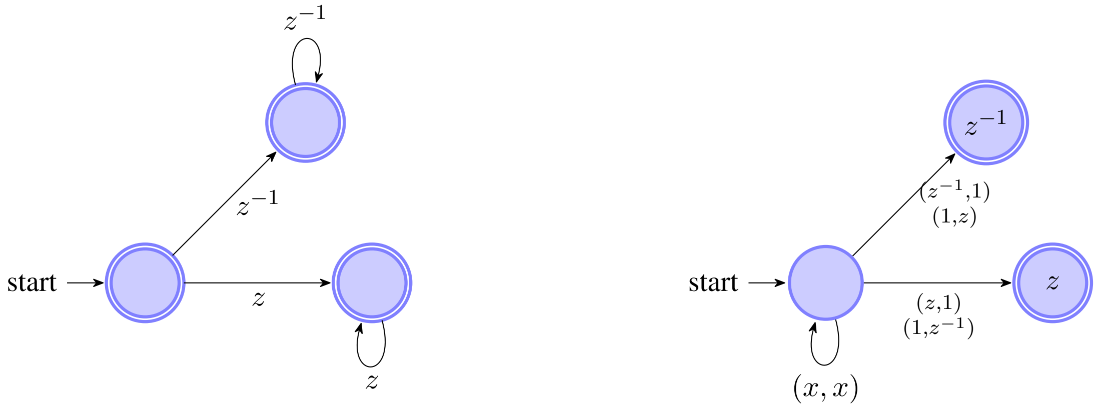

Figure 1: Word acceptor (left) and multipliers (right) for $\langle z \mid \emptyset \rangle$.

[1]The notation $(u, v)^+$ denotes *padding*: if $u = u_1 \dots u_n$ and $v = v_1 \dots v_m$, then $(u, v)^+ = (u_1, v_1) \dots (u_\ell, v_\ell)$ where $\ell = \max\{m, n\}$ and one defines $u_i = 1$ and $v_j = 1$ for all $i > n$ and $j > m$; here 1 represents the empty word in X^*