Non-virtually abelian discontinuous group actions vs. proper $SL(2,\mathbb{R})$-actions on homogeneous spaces

joint work with Willem A. de Graaf, Piotr Jastrzębski, and Aleksy Tralle

Maciej Bocheński

Department of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn

Lie Theory: frontiers, algorithms, and applications, Prato 2023

Definition

An action of a topological (Hausdorff) group G on a locally compact Hausdorff space M is called proper if $G_S = \{g \in G \mid gS \cap S \neq \emptyset\}$ is compact for every compact $S \subset M$.

Equivalently an action of G on M is proper if the map

$$G \times M \to M \times M, \ (g, m) \mapsto (m, gm)$$

is a proper map (that is, a pre-image of any compact set is compact).

Example

- The action of a Lie group G on itself is proper.
- The conjugation action of a non-compact Lie group on itself is not proper.

Let G be a (simple) Lie group, $H \subset G$ a closed (reductive) connected subgroup and $\Gamma \subset G$ a discrete subgroup.

Definition

The space $\Gamma \backslash G/H$ is called a Clifford-Klein form if Γ acts properly and freely on G/H. If $\Gamma \backslash G/H$ is compact then it is called a compact Clifford-Klein form.

We also say that Γ is a Clifford-Klein form for G/H and that G/H admits a compact Clifford-Klein form. If $\Gamma \backslash G/H$ is compact then we say that G/H admits a tessellation. Notice that the assumption that Γ acts freely on G/H is not very significant.

Questions:

Q1 When does a closed subgroup of G act on a homogeneous space G/H properly?

Q2 When does the homogeneous space G/H admit a compact Clifford-Klein form?

(M, J)- a smooth manifold with a geometric structure J,

$\tilde{M}$- the universal covering of M

$p : \tilde{M} \to M$, $\tilde{o} \in \tilde{M}$

$G := Aut(\tilde{M}, J)$, $H := \{g \in G \mid g\tilde{o} = \tilde{o}\}$ $\Gamma := \pi_1(M, o)$ $(o := p(\tilde{o}))$

In this case $\Gamma \subset G$ and

Theorem

Assume that G is a Lie group and acts transitively on $\tilde{M}$. Then M is diffeomorphic to $\Gamma \backslash G/H$.

$$\begin{array}{ccc} G/H & \xrightarrow{\sim} & \tilde{M} \\ \downarrow & & \downarrow \\ \Gamma\backslash G/H & \xrightarrow{\sim} & M \end{array} \qquad \begin{array}{c} gH \mapsto g\tilde{o} \\ \\ \Gamma gH \mapsto go. \end{array}$$

Q1 When does a "large" discrete subgroup of G act on a homogeneous space G/H properly?

Example

The one sheeted hyperboloid $SL(2,\mathbb{R})/SO(1,1)$ admits only finite Clifford-Klein forms.

More specifically we want to examine the connection between the following three conditions

- C1 := the space G/H admits a proper action of an infinite discrete subgroup of G,
- C2 := the space G/H admits a proper action of a non-virtually abelian discrete subgroup of G,
- C3 := the space G/H admits a proper action of a subgroup $L \subset G$ locally isomorphic to $SL(2,\mathbb{R})$.

Remark

A discrete group is non-virtually abelian if it does not contain an abelian subgroup of finite index.

Remark

Obviously $C3 \Rightarrow C2 \Rightarrow C1$.

Let

- G be a connected linear semisimple (reductive) real Lie group with the Lie algebra $\mathfrak{g}$.
- H, L be reductive subgroups of G,
- θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition w.r.t. θ.

We may assume that $\theta|_h$ and $\theta|_l$ are Cartan involutions of $\mathfrak{h}$, $\mathfrak{l}$, respectively.

Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

Definition

The real rank of a Lie algebra $\mathfrak{g}$ (denoted $\mathrm{rank}_{\mathbb{R}} \mathfrak{g}$) is the dimension of $\mathfrak{a}$.

Denote by K the maximal compact subgroup of G with the Lie algebra $\mathfrak{k}$. Let $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group of G.

Denote by $\mathfrak{a}_h, \mathfrak{a}_l$ the maximal abelian subspaces of $\mathfrak{h}$ and $\mathfrak{l}$. We may assume that $\mathfrak{a}_h, \mathfrak{a}_l \subset \mathfrak{a}$.

Theorem (Kobayashi)

The following conditions are equivalent
(i) H acts on G/L properly,
(ii) L acts on G/H properly,
(iii) $\mathfrak{a}_h \cap W\mathfrak{a}_l = \{0\}$.
Moreover, the subgroup L acts properly on G/H only if

$$\mathrm{rank}_{\mathbb{R}}(\mathfrak{l}) + \mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) \leq \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}).$$

Using the Cartan decomposition one may obtain a more general criterion. Let Σ be a system of restricted roots of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ and let Σ^+ be a subset of positive roots. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be the closed positive Weyl chamber then

$$G = Kexp(\mathfrak{a}^+)K,$$

and every $g \in G$ can be presented as a product $g = k_1 exp(X) k_2$, $k_1, k_2 \in K$ and a unique $X \in \mathfrak{a}^+$.

Definition

The map $\mu : G \to \mathfrak{a}^+$, $g \mapsto X$ is called the Cartan map.

Theorem (Benoist, Kobayashi)

Let S be a closed subgroup of G. The group S acts properly on G/H if and only if

$$\mu(H) \cap (\mu(S) + V)$$

is bounded for every compact subset $V \subset \mathfrak{a}$

Corollary (The Calabi-Markus phenomenon)

If $\mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{g})$ then only finite discrete subgroups can act properly on G/H.

G/H is C1 if and only if $\mathrm{rank}_{\mathbb{R}}\, \mathfrak{g} > \mathrm{rank}_{\mathbb{R}}\, \mathfrak{h}$.

The finite group W acts on $\mathfrak{a}$ by orthogonal transformations as a finite group generated by reflections in the hyperplanes determined by simple roots Σ. Let $w_0 \in W$ be the longest element and put

$$\mathfrak{b}^{+} := \{X \in \mathfrak{a}^{+} | -w_0(X) = X\},$$
$$\mathfrak{b} := \mathrm{Span}_{\mathbb{R}}(\mathfrak{b}^{+}).$$

Theorem ([1], Theorem 1)

G/H is C2 if and only if for every w in W, $w \cdot \mathfrak{a}_h$ does not contain $\mathfrak{b}$. In this case, one can choose a discrete subgroup of G acting properly on G/H to be free and Zariski dense in G.

Example

The homogeneous space $SL(3, \mathbb{R})/SL(2, \mathbb{R})$ is C1 but is not C2.

For many important classes of homogeneous spaces C2 and C3 are equivalent. For example, apart from irreducible symmetric spaces [9], this holds for some strongly regular homogeneous spaces ([3] (Theorem 2 and Corollary 1)b). On the other hand, there were known exactly two examples of spaces which fulfill the condition C2 but not C3.

Those two examples were obtain by finding an appropriate subspace $\mathfrak{a}_m \subset \mathfrak{a}$ such that $W\mathfrak{a}_m$ does not contain $\mathfrak{b}$ and that for any $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{g}$ there exists $g \in G$ such that

$$\mathrm{Ad}_g(\mathfrak{sl}(2,\mathbb{R})) \cap \mathfrak{a}_m \neq \{0\}.$$

In this case G/A_m, where the subgroup $A_m \subset G$ corresponds to $\mathfrak{a}_m$, is a C2 space but not a C3 space.

Thus for a space of reductive type we have the following:

$$C1 \nRightarrow C2 \nRightarrow C3.$$

Problem

Create computer algorithms and programs which verify C2 and C3 for general semisimple homogeneous spaces.

We created algorithms verifying C2 versus C3 and implemented them in the computational algebra system GAP4 (4).

Theorem

Assume that G/H is a homogeneous space given by

$$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{f}_{4(4)}) \ \ or \ \ (\mathfrak{e}_{8(8)}, \mathfrak{f}_{4(4)} \oplus \mathfrak{g}_{2(2)}),$$

(in both cases $\mathfrak{h}^c$ is maximal in $\mathfrak{g}^c$, where $\mathfrak{h}^c$, $\mathfrak{g}^c$ denote the complexifications of $\mathfrak{h}$, $\mathfrak{g}$ respectively). Then G/H is C2 but not C3:

Approach: classification of nilpotent orbits and criteria of proper actions.

Consider the class $\mathcal{GM}$ of homogeneous spaces G/H, where G is connected, linear and simple, and H is a maximal proper semisimple subgroup of G such that $\mathfrak{h}^c$ is maximal in $\mathfrak{g}^c$. For $\operatorname{rank} G \leq 8$ the corresponding pairs $(\mathfrak{g}, \mathfrak{h})$ are classified in (5).

Theorem

For any $(\mathfrak{g}, \mathfrak{h}) \in \mathcal{GM}$ such that $\operatorname{rank} \mathfrak{g} \leq 6$ conditions C2 and C3 are equivalent.

Notice that the assumption of H being maximal in G is very interesting in the context of searching for homogeneous spaces which are $C2$ but not $C3$. On one hand, one should expect that homogeneous spaces which are not C3 are given by "large" semisimple subgroups of G (to be more specific, by subgroups which have as large real rank as possible). On the other hand we are restricted by the condition C2 (so we cannot take $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = \operatorname{rank}_{\mathbb{R}} \mathfrak{h}$).

Approach

- We have used the database $\mathcal{GM}$ created by Willem de Graaf and Alessio Marrani ([5]) containing real forms of maximal semisimple subalgebras of complex simple Lie algebras of rank up to 8.
- We have filtered the database using some additional results as
  - if $\dim \mathfrak{b}_h = \dim \mathfrak{b}$, then G/H is not C2,
  - if $\dim \mathfrak{b} > \dim \mathfrak{a}_h$ than G/H is C2,
  - detailed description of the case $\dim \mathfrak{a}_h = 1$,
- we have used an algorithm (written by Willem de Graaf) which gives an embedding of $\mathfrak{h}$ into $\mathfrak{g}$ s.t. $\mathfrak{a}_h \subset \mathfrak{a}$.
- we have used the classification of nilpotent orbits and the criteria of proper actions.

Difficulties

- We had to to compute the root system of $\mathfrak{g}$ with respect to the space $\mathfrak{a}$. However, in some (but not many) cases this required eigenvalues that did not lie in the ground field that CoReLG uses. So for a few cases our procedures could not be used.
- When $\mathfrak{g}$ is a split form of type E_7 or E_8 the W-orbits of elements of $\mathfrak{a}$ corresponding to nilpotent orbits can be very large. In this case we can handle large orbits by using an algorithm due to Snow ([10], see also [6, §8.6]) for running over an orbit of the Weyl group without computing all of its elements. This algorithm has been implemented in the core system of GAP4.

Y. Benoist, *Actions propres sur les espaces homogènes réductifs*, Ann. Math. 144 (1996), 315-347.

M. Bocheński, W. de Graaf, P. Jastrzębski, A. Tralle, *Non-virtually abelian discontinuous group actions vs. proper $SL(2,\mathbb{R})$-actions on homogeneous spaces*, arXiv:2206.01069.

M. Bocheński, *Proper actions on strongly regular homogeneous spaces*, Asian J. Math. 21 (2017), 1121-1134.

The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021.

W. de Graaf, A. Marrani, *Real forms of embeddings of maximal reductive subalgebras of the complex simple Lie algebras of rank up to 8*, J. Phys. A 53 (2020), no. 15, 155203.

W. A. de Graaf *Lie algebras: theory and algorithms*. North-Holland Mathematical Library, 56. North-Holland Publishing Co., Amsterdam, 2000.

T. Kobayashi, *Proper actions on a homogeneous space of reductive type*, Math. Ann. 285 (1989), 249-263.

T. Kobayashi, T. Yoshino, *Compact Clifford-Klein forms of symmetric spaces revisited*, Pure Appl. Math. Quart 1 (2005), 603-684.

T. Okuda, *Classification of Semisimple Symmetric Spaces with Proper $SL(2,\mathbb{R})$-Actions*, J. Differential Geometry 2 (2013), 301-342.

D. M. Snow, *Weyl group orbits*. ACM Trans. Math. Software 16 (1990), no. 1, 94-108.