

SHEETS IN REDUCTIVE ALGEBRAIC GROUPS IN ARBITRARY CHARACTERISTICS

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OVERVIEW

1. sheets for $G \times X$
2. Motivation: Lusztig strata
3. Parametrization
- (4. Enumeration)

G connected, reductive
 $\bar{k} = k$

$G \curvearrowright X$

$$X_{(n)} = \{ x \in X \mid \dim G \cdot x = n \}$$

locally closed

sheet = an irreducible
component of some
 $X_{(n)}$

$$X = \text{Lie } G \quad k = \mathbb{C}$$

$$\text{char } k = \underline{\text{good}}$$

Borho-Kraft 1979
~~Premet-Stewart~~ 2015

$$\Phi, \Delta = \{\alpha_1, \dots, \alpha_n\}$$

$$\alpha_0 = \sum d_i \alpha_i$$

$$(\text{char } k, d_i) = 1 \quad \forall i$$

$$\mathfrak{m} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{m}_j$$

$$\text{Lie } G = \mathfrak{m}_0$$

$$k = \mathbb{C}$$

$$X = \mathfrak{m}_1$$

$$n=2$$

Boloiis-Hivert
 2016

$$n > 2$$

C. Esposito Sant
 2021

$$X = G$$

$$\text{char } k = \text{good}$$

C. Esposito
 2012

$$\text{bad char}$$

Simion 2019

$\tilde{G}$ not connected any char

$$G = \tilde{G}^0$$

$$X = Gg$$

M. Costa Cesari
 2022

Focus on
 $G \supseteq G$ k good

look for a parametrization
 combinatorial

understand dependence on p

MOTIVATION: Lusztig strata

$\text{char } k = p$ a prime or 0

G^F

$$F_p : G^F \longrightarrow \text{Irr } W$$

$$su \longmapsto \text{TR. IND}_{W_S}^W \text{Spr}(C_{G(s)}^\circ \cdot u)$$

S, S_S
 u unip
 $su = us$

$s \in T$

$$W_S = \frac{N_{C_{G(s)}^\circ}(T)}{T} \leq \frac{N_G(T)}{T} = W$$

Fibers are called strata

Thm (Lusztig 2015)

$$\textcircled{1} \quad \forall g \in \text{In } W \quad \overline{F}_p^{-1}(g) \subseteq G(m) \\ \text{for some } m$$

$\textcircled{2}$ strata are G -stable

$$\textcircled{3} \quad su \in \Sigma \\ \Sigma \supseteq G \cdot \left(\{ r \in G \mid C_G(s)^0 = C_G(r)^0 \} u \right)$$

$\textcircled{4}$ $\text{In } \overline{F}_p$ does NOT depend on p

$$\{ \text{strata of } G^p \} \xleftrightarrow{1-1} \overline{F}_p(G^p)$$

$$\{ \text{strata of } G^{p'} \} \xleftrightarrow{\quad} \overline{F}_p(G^p)$$

$\textcircled{5}$ $\forall$ stratum $\exists$ at most one unip. class in it

$\textcircled{6}$ $\forall p \in \overline{F}_p(G^p) \quad \exists p' \text{ st. } \overline{F}_{p'}^{-1}(p) \text{ contains a unip. element}$

Thm (C, 2016, 2020)

Strata are locally closed,
and the irr. components are the
sheets of G .

Thm (Lustig 2022)

$\exists$ Unip Char Sheaves (G^F) $\rightarrow$ $\{ \text{strata} \}$

$$G \cdot \left(\{ r \in G \mid C_G(s)^\circ = C_G(r)^\circ \} u \right) = \tilde{J}(su)$$

$$G = Sp_8(k) \quad \text{char } k \neq 2$$

$$s = \text{diag}(1, 1, l, -1, -1, l^{-1}, 1, 1)$$

$$l \in k^* \quad l \neq \pm 1$$

$$r = \text{diag}(\epsilon, \epsilon, t, -\epsilon, -\epsilon, t^{-1}, \epsilon, \epsilon)$$

$$\epsilon = \pm 1 \quad t \in k^*, \quad t \neq \pm 1$$

$$J(su) = \text{conn. comp of } \tilde{J}(su) \\ \text{containing } su$$

Jordan classes

Prop 1) Jordan classes form a poset wrt inclusion of closures

2) sheets are exactly

$J \subset G_{\text{cm}}$ maximal

$\overline{J \cap G_{\text{cm}}}$ is a sheet

sheets are parametrized by
($m \in G$ adjoint)

$q(M, u)$ M is conn. centr. of some ss elt
 $u \in M$, unipotent, rigid

McNinch-Summers, 2003

PROP (Ambrosio, C, Esposito 2022)

Fix $T \leq G$ maximal

$p = \text{char } k$ or $p=1$ if $\text{char } k = 0$

$M = C_G(S)^0$ for some $S \in T$

$\Uparrow \parallel \Downarrow$
 $\exists I \subseteq \Delta \cup \{ \alpha_0 \}$ ^{highest}

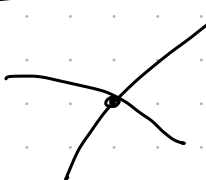
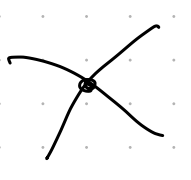
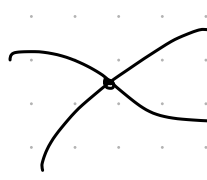
$M = \langle T, U_{\pm \alpha}, \alpha \in I \rangle$

$\gcd(d_{\alpha}, \alpha \in \Delta \cup \{ \alpha_0 \} \setminus I, p) = 1$

eg $\mathbb{Q}_6(k) \overset{1}{\circ} \overset{2}{\Rightarrow} \overset{2}{\circ} \overset{2}{\dashrightarrow} \overset{1}{\circ}$

RK For $k = \overline{\mathbb{F}_q}$ a proof could be extracted from Deriziotis thesis. Chene contains an algorithm for computing connected centralizers in ss. qlts

EXAMPLE $(G = G_2)$ level sets

dim of classes	sheets			# of strata
	good char	$p=2$	$p=3$	
12	regular locus : irreducible			1
10				1
8				1
6				2
0				1

The number of sheets varies with p
how? Is there always a bad

prime p such that

of sheets in $G^0 = \#$ of sheets in G^A ?
YES, for G_2 & F_4 .