SHEETS IN REDUCTIVE ALGEBRAIC GROUPS IN ARBITRARY CHARACTERISTICS

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OVERVIEW

1. sheets for $G \circlearrowleft X$

2. Motivation: Lusztig strata

3. Parametrization

(4. Enumeration)

G connected, reductive

$\bar{k} = k$

$G \circlearrowright X$

$$X_{(n)} = \{ x \in X \mid \dim G \cdot x = n \}$$

locally closed

sheet = an irreducible component of some $X_{(n)}$

$X = \mathrm{Lie}\, G$ $\quad k = \mathbb{C}$ $\qquad$ Borho-Kraft 1979

char k = good $\qquad$ Premet-Stewart 2015

$\Phi, \Delta = \{\alpha_1, \ldots, \alpha_n\}$

$\alpha_0 = \sum d_i \alpha_i$

$(\mathrm{char}\, k, d_i) = 1 \quad \forall i$

$\mathfrak{m} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{m}_j$ $\qquad$ $\mathrm{Lie}\, G = \mathfrak{m}_0$

$k = \mathbb{C}$ $\qquad$ $X = \mathfrak{m}_1$

$n = 2$ $\qquad$ Bulois-Hivert 2016

$n > 2$ $\qquad$ C Esposito Santi 2021

$X = G$ $\qquad$ char k = good $\qquad$ C. Esposito 2012

bad char $\qquad$ Simion 2019

$\tilde{G}$ not connected $\qquad$ any char

$G = \tilde{G}^{\circ}$ $\qquad$ $X = Gg$ $\qquad$ M. Costa Cesari 2022

Focus on

$G \supseteq G$ k good

look for a paramétrization

combinatorial

understand dependence on p

MOTIVATION: Lusztig strata

$\mathrm{char}\, k = p$ a prime or 0

G^p

$$F_p : G^p \longrightarrow \mathrm{Irr}\, W$$

$$su \longmapsto \mathrm{TR.\,IND}_{W_s}^{W} \mathrm{Spr}\left((C_G(s)^\circ \cdot u) \right)$$

s, ss
u unip
$su = us$

$s \in T$

$$W_s = N_{C_G(s)^\circ}(T) / T \leq N_G(T)/T = W$$

Fibers are called strata

Thm (Lusztig 2015)

1. $\forall \rho \in \mathrm{Irr}\, W \quad F_p^{-1}(\rho) \subseteq G_{(m)}$ for some m

2. strata are G-stable

3. $su \in \Sigma$

   $\Sigma \supseteq G \cdot \left(\{ r \in G \mid C_G(s)^0 = C_G(r)^0 \} u \right)$

4. $\mathrm{Im}\, F_p$ does NOT depend on p

$$\{\text{strata of } G^p\} \xleftrightarrow{1-1} F_p(G^p) \longleftrightarrow \{\text{strata of } G^{p'}\}$$

5. $\forall$ stratum $\exists$ at most one unip. class in it

6. $\forall \rho \in F_p(G^p) \quad \exists\, p'$ st. $F_{p'}^{-1}(\rho)$ contains a unip. element

Thm (C, 2016, 2020)

Strata are locally closed, and the irr. components are the sheets of G.

Thm (Lusztig 2022)

$$\exists \quad \text{Unip Char Sheaves}(G^{p}) \twoheadrightarrow \{\text{strata}\}$$

$$G \cdot \left(\{ r \in G \mid C_G(s)^\circ = C_G(r)^\circ \} u \right) = \tilde{J}(su)$$

$G = Sp_8(k)$ char $k \neq 2$

$s = \mathrm{diag}(1, 1, \ell, -1, -1, \ell^{-1}, 1, 1)$

$\ell \in k^*$ $\ell \neq \pm 1$

$r = \mathrm{diag}(\epsilon, \epsilon, t, -\epsilon, -\epsilon, t^{-1}, \epsilon, \epsilon)$

$\epsilon = \pm 1$ $t \in k^*$, $t \neq \pm 1$

$J(su) =$ conn. comp of $\tilde{J}(su)$ containing su

Jordan classes

<u>PROP</u> 1) Jordan classes form a poset wrt inclusion of closures

2) sheets are exactly

$J \subset G_{(m)}$ maximal

$$\boxed{\overline{J} \cap G_{(m)} \text{ is a sheet}}$$

sheets are parametrized by $\overbrace{}$ (in G adjoint)

$$\left\{ (M, u) \;\middle|\; \begin{array}{l} M \text{ is conn. centr. of some ss elt} \\ u \in M, \text{ unipotent, rigid} \end{array} \right\}$$

McNinch-Sommers, 2003

PROP (Ambrosio, C, Esposito 2022)

Fix $T \leq G$ maximal

$p = \operatorname{char} k$ or $p = 1$ if $\operatorname{char} k = 0$

$M = C_G(S)^\circ$ for some $S \subseteq T$

$$\Updownarrow$$

highest

$$\exists I \subseteq \Delta \cup \{\alpha_0\}$$

$$M = \langle T, U_{\pm\alpha}, \alpha \in I \rangle$$

$$\gcd(d_\alpha, \alpha \in \Delta \cup \{\alpha_0\} \setminus I, p) = 1$$

eg $Sp_6(k)$ $\overset{1}{\circ} \Rightarrow \overset{2}{\circ} - \overset{2}{\circ} \Leftarrow \overset{1}{\circ}$

RK For $k = \overline{\mathbb{F}_p}$ a proof could be extracted from Deriziotis thesis. Chevie contains an algorithm for computing connected centralizers in s.s. elts

EXAMPLE ($G = G_2$) level sets

dim of classes	Sheets: good char	Sheets: $p=2$	Sheets: $p=3$	# of strata
12	regular locus : irreducible			1
10				1
8				1
6				2
0				

The number of sheets varies with p

how? Is there always a bad

prime p such that

$\#$ of sheets in G^0 = $\#$ of sheets in G^p ?

YES, for G_2 & F_4.