Complex Hecke algebras are real

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Coxeter-like presentation

$$\langle s_1, \ldots, s_n \mid s_i^{o(s_i)} = 1, \text{ homogeneous positive relations } \rangle$$

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Remark

$G(1,1,n) \cong A_{n-1},\ G(2,1,n) \cong B_n,\ G(2,2,n) \cong D_n,\ G(m,m,2) \cong I_2(m),$

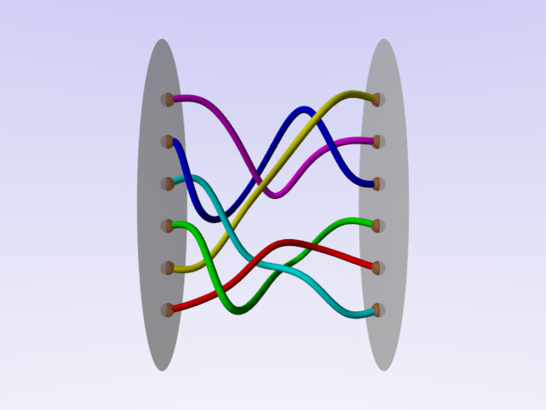
$G_{23} \cong H_3,\ G_{28} \cong F_4,\ G_{30} \cong H_4,\ G_{35} \cong E_6,\ G_{36} \cong E_7,\ G_{37} \cong E_8.$

Braid groups

Symmetric group S_n

↓

Braid group
on n strands



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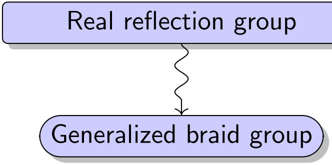

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Definition (Broué-Malle-Rouquier 1998)

The braid group $B(W)$ associated to W is the fundamental group $\pi_1(X/W, \underline{x})$.

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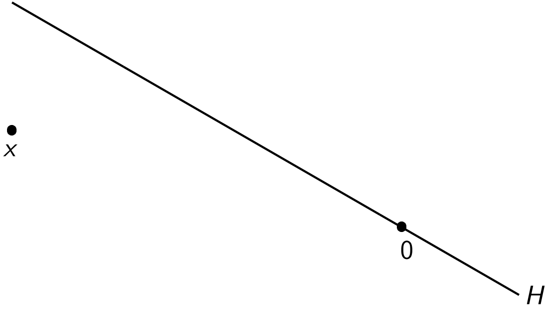

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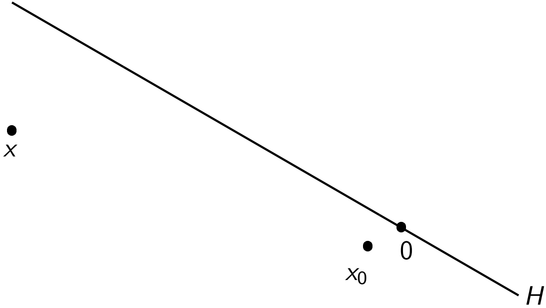

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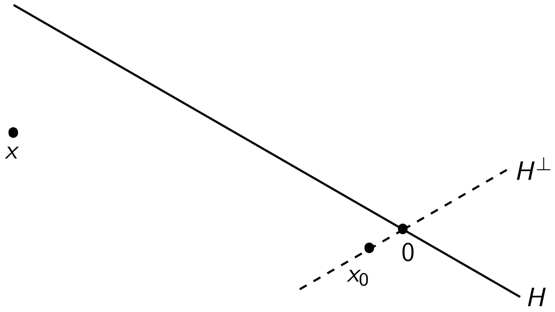

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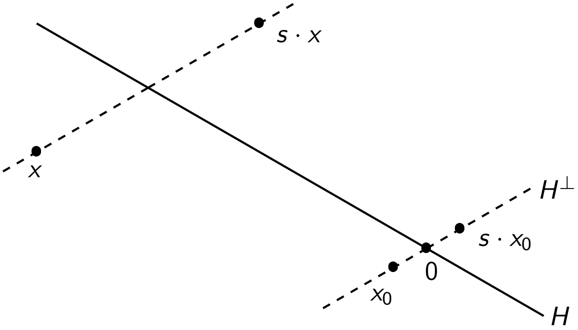

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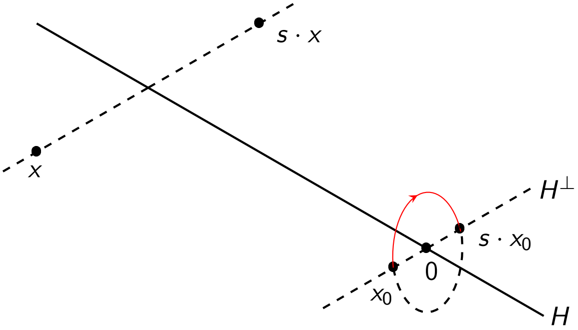

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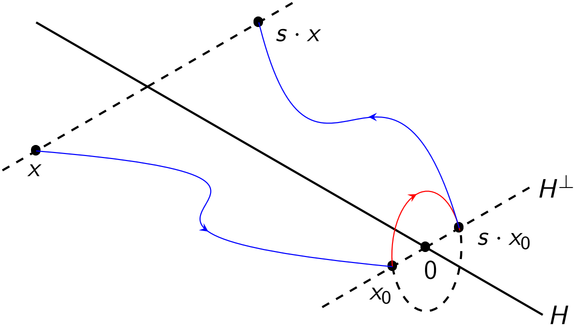

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Theorem [Bessis 1999]

The complex braid group $B(W)$ is generated by distinguished braided reflections, whose image inside W are the distinguished pseudo-reflections that generate W.

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Broué, Malle, Michel, Rouquier

Spets $\rightsquigarrow$ complex reflection group

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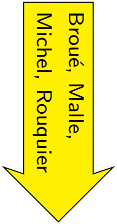

Spets $\rightsquigarrow$ complex reflection group $\rightsquigarrow$ generic Hecke algebra

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The BMR freeness conjecture

Let W be a complex reflection group and let $H(W)$ the associated Hecke algebra defined over $R(W)$.

Conjecture [Broué-Malle-Rouquier 1998]

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- Non-real case
  - ⋆ $G(de, e, n)$ [Ariki-Koike, Broué-Malle, Ariki]
    Alternative basis for $G(e, e, n)$, $G(d, 1, n)$ [Neaime]
  - ⋆ G_4 [Broué-Malle, Berceanu-Funar, C., Marin]
  - ⋆ $G_5, \dots, G_{16}$ [C.], G_{12} also by [Marin-Pfeiffer]
  - ⋆ G_{17}, G_{18}, G_{19} [Tsuchioka]
  - ⋆ $G_{20}, \dots, G_{34}$ [Marin, Marin-Pfeiffer]

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We end up with a nice basis: $1 \in \mathcal{B}$, its elements are braid group elements and it has an inductive form.

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- Non-real case
  - $\star$ $G(de, e, n)$ [Bremke-Malle, Malle-Mathas]
  - $\star$ G_4 [Malle-Michel, Marin-Wagner, Boura-C.-Chlouveraki-Karvounis]
  - $\star$ $G_5, \ldots, G_8$ [Boura-C.-Chlouveraki-Karvounis]
  - $\star$ G_{12}, G_{22}, G_{24} [Malle-Michel]
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$$t(b_k b_\ell) \overset{m \leq k}{=} t(b_m g_i b_\ell) \overset{\texttt{C++}}{=} t(b_m \sum \lambda_j b_j) = \sum \lambda_j t(b_m b_j).$$

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Example

Let $W = \langle s_1, ..., s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \ldots}_{m_{ij}-\text{factors}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}-\text{factors}} \rangle$ a real reflection group. Then:

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Then, $T_w^{\vee} = a_w^{-1} T_{w^{-1}}$.

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Let W be a complex reflection group and let $H(W)$ the associated Hecke algebra, defined over $R(W) = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$, which admits a basis $\mathcal{B} = \{b_w,\ w \in W\}$.

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Theorem [C. 2017]

There is an optimal basic set for G_4, G_8, G_{16} with respect to any θ.

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- For each $C \in Cl(W)$ we set $y_C := \sum_{w \in W} f_{w,C}\, b_w^{\vee}$.

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For each $C \in Cl(W)$ we recall that $y_C := \sum_{w \in W} f_{w,C}\, b_w^\vee$.

Theorem [C.-Pfeiffer 2021]

The set $\{y_C,\, C \in Cl(W)\}$ is a basis of the center $Z(\mathbb{C}(\mathbf{u})H(W))$.

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Remarque

Let W be a real reflection group. Then, we have the following:

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The set $\{y_C,\ C \in Cl(W)\}$ is a basis of the center $Z(\mathbb{C}(\mathbf{u})H(W))$.

Remarque

Let W be a real reflection group. Then, we have the following:

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The center of the generic Hecke algebra

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Let W be a complex reflection group and let $H(W)$ the associated Hecke algebra, which admits a basis $\mathcal{B} = \{b_w,\, w \in W\}$.

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Conjecture [C.-Pfeiffer 2021]

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Last slide

Thank you!