A formula for tangent plane dimension at T-fixed points in flat linear degenerations of the flag variety.

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Plan of the Talk

- (Flat) Linear Degenerations of Flag Variety
- GKM Structures
- Linear Degenerations as Quiver Representations
- An (Effective) formula for dimension of Tangent Plane
- Smoothness Criteria

Definition (Complete Flag Variety)

$$\mathfrak{Fl}(\mathbb{C}^{n+1}) = \{(V_1, \ldots, V_n),\ V_i \subset \mathbb{C}^{n+1} \mid \dim V_i = i, V_i \subset V_{i+1}\}$$

Properties:

- Projective smooth algebraic variety
- Its dimension is equal to $\binom{n+1}{2}$
- Homogeneous Variety $SL_{n+1}\mathbb{C}/B$

Classical Flag Varieties

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A different point of view:

$$V_i \subset W_i = \mathbb{C}^{n+1}$$

$$\mathrm{id}_i : W_i \to W_{i+1} \qquad \mathrm{id}_i V_i \subset V_{i+1}$$

Consider $\mathbf{f} = (f_1, \ldots, f_{n-1}) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{n+1})^{n-1}$

Definition (f Degeneration of Flag Variety)

$$\mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1}) = \{(V_1, \ldots, V_n),\ V_i \subset \mathbb{C}^{n+1} \mid \dim V_i = i, f_i(V_i) \subset V_{i+1}\}$$

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Fix a basis $\{e_1, \ldots, e_{n+1}\}$ of $\mathbb{C}^{n+1}$. Consider $I \subset [n+1]$

$$\pi_I(e_i) = \begin{cases} 0 & \text{if } i \in I, \\ e_j & \text{for } j \notin I \end{cases}$$

Example

For $\mathbf{I} = (I_1, \ldots, I_{n-1}),\ I_j \subset [n+1]$

$$\mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1}) = \{(V_1, \ldots, V_n),\ V_j \subset \mathbb{C}^{n+1} \mid \dim V_j = j, \pi_{I_j}(V_j) \subset V_{j+1}\}$$

Definition (Flat Degenerations)

$\mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1})$ *is flat if it is and equidimensional variety of dimension* $\binom{n+1}{2}$

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$$(g_1, \ldots, g_n) \cdot (f_1, \ldots, f_{n-1}) := (g_2 f_1 g_1^{-1}, \ldots, g_n f_{n-1} g_{n-1}^{-1})$$

For every $X = \mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1})$ there exists $\mathfrak{Fl}^{\mathbf{l}}(\mathbb{C}^{n+1}) \in G \cdot X$

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Theorem (Cerulli Irelli - Fang - Feigin - Fourier - Reineke, '17)

1. *$\mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1})$ is flat if and only if $|I_j| \leq 2$ and $|I_j \cup I_{j+1}| \leq 3$*
2. *If $|I_j| \leq 1$, the variety $\mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1})$ is irreducible.*

X is a projective complex algebraic variety endowed with the action of a torus T

Definition (GKM Variety)

A T variety X is a GKM variety if:

1. *The number of T fixed points and of 1 dimensional T orbits in X is finite*
2. $H^{2i+1}(X) = 0$ *for all* $i \geq 0$.

Theorem (M.Lanini, A. Pütz, '20)

For every $\mathbf{f}$ *exists a suitable torus* T *acting on* $X = \mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1})$ *such that* X *has a structure of GKM variety.*

The Moment Graph $G(X, T)$

Definition

- *The vertices of* $G(X, T)$ *are indexed by the set of fixed points* X^T,
- *There is an edge from* x *to* y *if* x *and* y *are in the same* 1 *dimensional orbit.*

Warning! This is an unoriented version of classical moment graph.

Theorem

The cohomology of X *can be completely recovered by (the oriented version of)* $G(X, T)$.

A Remarkable Example: Feigin Degeneration

Definition (Feigin Degeneration)

$X = \mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1})$ *such that* $\mathrm{rk} f_i = n$ *for all* i *and Ker* $f_1, \dots,$ *Ker* f_{n-1} *are linealy independent.*

Remark

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Theorem (Cerulli Irelli - Feigin - Reineke)

Let X *be the Feigin degeneration. There exists an algebraic group* A *acting on* X *and a torus* $T \subset A$ *such that:*

- $X = \sqcup X_p$ *where* p *runs between elements of* X^T
- *The cell* X_p *concides with the orbit* $A \cdot p$

Consider a quiver $Q = (Q_0, Q_1)$ (a finite directed graph).

Definition (Representation M of $Q = (Q_0, Q_1)$ (over $\mathbb{C}$))

- *a family of $\mathbb{C}$ vector spaces $(M_i)_{i \in Q_0}$,*
- *a sequence of maps $(\varphi_\alpha)_{\alpha \in Q_1}$ such that $\varphi_\alpha \in \mathrm{Hom}_{\mathbb{C}}(M_{s(\alpha)}, M_{t(\alpha)})$.*

Definition (Subrepresentation)

A subrepresentation $N \subset M$ is defined by a collection of subspaces $N_i \subset M_i$ compatible with the maps φ_α.

Linear Degenerations and Quiver Representations

The (equioriented) quiver of type A_n

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

Consider $\mathbf{f} = (f_1, \ldots, f_{n-1}) \in \mathrm{End}(\mathbb{C}^{n+1})$ and set $M_i = \mathbb{C}^{n+1}$

$$M^{\mathbf{f}} : \quad \mathbb{C}^{n+1} \xrightarrow{f_1} \mathbb{C}^{n+1} \xrightarrow{f_2} \ldots \xrightarrow{f_{n-2}} \mathbb{C}^{n+1} \xrightarrow{f_{n-1}} \mathbb{C}^{n+1},$$

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Remark

$$p \in \mathfrak{Fl}^{\mathbf{f}}(\mathbb{C}^{n+1}) \quad \longleftrightarrow \quad N_p \subset M^{\mathbf{f}}, \dim(N_p)_i = i$$

Euler Formula

Problem:Determining the dimension of the tangent plane at a fixed point p

Theorem (Euler Formula)

$$\dim T_p \mathfrak{Fl}^I(\mathbb{C}^{n+1}) = \frac{n(n+1)}{2} + \sum \dim \mathrm{Ext}^1_{Rep(Q)}(U_i, U_j)$$

where U_i runs between indecomposables of p and U_j between indecomposables of M/p

Problem 1): classify fixed points
Problem 2): determinig the indecomposables of p and M/p
Problem 3): compute $\mathrm{Ext}^1_{Rep(Q)}(p, M/p)$

Coefficient Quivers

Consider M representation of Q and set $B = \sqcup B_i$ where B_i is a basis of M_i.

Definition (Coefficient quiver $Q(M, B)$)

- *it has $|B|$ vertices labelled by the elements of B,*
- *$v_k^i \to v_h^j$ iff there exists $\alpha : i \to j$ and the coefficient of v_h^j in $f_\alpha(v_k^i)$ in a B_j-expansion is non zero.*

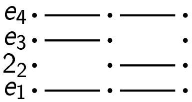

Figure: The Coefficient Quiver of Feigin Degeneration of $\mathfrak{Fl}(\mathbb{C}^4)$

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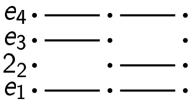

Figure: The Coefficient Quiver of Feigin Degeneration of $\mathfrak{Fl}(\mathbb{C}^4)$

Remark

For $X = \mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1})$ the connected components of $Q(M^{\mathbf{I}}, B)$ corresponds to indecomposables of $M^{\mathbf{I}}$.

Recovering the Moment Graph: Fixed Points

Consider $X = \mathfrak{Fl}^{\mathbf{l}}(\mathbb{C}^{n+1})$ and $M^{\mathbf{l}}$ the associated quiver representation

Definition (Successor closed subquiver (for short S.C.S))

Full subquiver Q' of Q such that $v = s(\alpha) \in Q'_0 \Rightarrow t(\alpha) \in Q'_0$.

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X^T is in bijection with S.C.S. of $M^{\mathbf{I}}$ such that $|Q'_0 \cap B_j| = j$ for all $j \leq n$.

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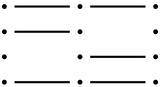
Successor closed subquivers are encoded by sequences of integers.

Definition (Admissible Sequence for $\mathfrak{Fl}^{\mathbf{l}}(\mathbb{C}^{n+1})$)

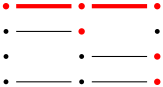
$S = (S_1, \ldots, S_n)$, $S_j \subset [n+1]$ such that $|S_j| = j$ and $S_j \subset S_{j+1} \cup I_j$.

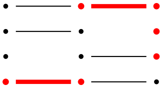
Feigin Degeneration Revised

For Feigin Degeneration we have $I_j = \{j+1\}$.



A fixed point p_S is associated to an admissible family $S = (S_1, \dots, S_n)$ such that $S_j \subset S_{j+1} \cup \{j+1\}$.





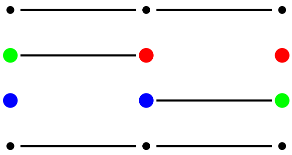
$S = (\{4\}, \{4,3\}, \{4,2,1\})$ $\qquad$ $S' = (\{1\}, \{1,4\}, \{4,3,2\})$

Smoothness for Feigin Degeneration

Definition (CIFR Condition)

$S = (S_1, \ldots, S_n)$ *has the CIFR Condition if for all for* $1 \leq h < k \leq n$

$$k \in S_h \Rightarrow h + 1 \in S_k$$



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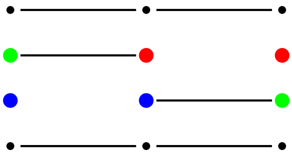


Theorem

The point p_S *is smooth if and only if* S *has the CIFR Condition.*

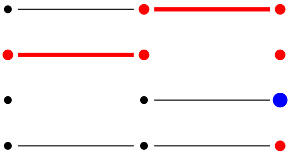
Singularity Sets

$$\text{pos}(t) = \{j \in [n+1] \mid t \in I_j\}$$

Definition

$\text{Sing}_i(Q_S)$ *is the set of pairs* $(j, h+1)$*,* $h \in \text{pos}(i)$*,* $j \notin S_{h+1}$*, such that there exists* $k \leq h$*,* $k \in \text{pos}(j)$ *satisfying*

1. $i \in S_t \; \forall t$ *such that* $k \leq t \leq h$
2. *the segment spanned by the vertices* $\{v_i^k, \ldots, v_i^h\}$ *is connected in* Q_S*.*



A Formula for Dimension of Tangent Plane

$$\mathrm{Sing}(Q_S) = \bigsqcup \mathrm{Sing}_i(Q_S)$$

Theorem

$$\dim T_{p_S}\mathfrak{Fl}^I(\mathbb{C}^{n+1}) = \frac{n(n+1)}{2} + |\mathrm{Sing}(Q_S)|.$$

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$$\dim T_{p_S}\mathfrak{Fl}^I(\mathbb{C}^{n+1}) = \frac{n(n+1)}{2} + |\mathrm{Sing}(Q_S)|.$$

1. Determine $\mathrm{pos}(i)$ for every i
2. For each $h \in \mathrm{pos}(i)$ determine the set $D(i)$ of $j \notin S_{h+1}$
3. For each $j \in D(i)$ determine $D(i)^{\leq h} = \mathrm{pos}(j) \cap \{x \leq h\}$
4. Check the two conditions for elements of $D(i)^{\leq h}$

A Complicated Example

$$I = \{\{5\}, \{6\}, \emptyset, \{3,6\}, \emptyset\}$$

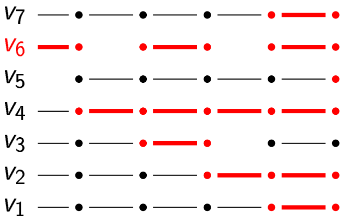

Figure: The quiver Q_S.

$$S = \{\{6\}, \{4,6\}, \{3,4,6\}, \{2,3,4,6\}, \{1,2,4,6,7\}, \{1,2,4,5,6,7\}\}$$

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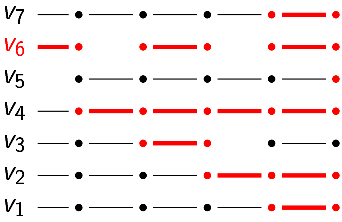

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$$|\mathrm{Sing}(Q_S)| = |\mathrm{Sing}_6(Q_S)| + |\mathrm{Sing}_3(Q_S)| = 2+1 \Rightarrow \dim T_{p_S}\mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1}) = 24$$

Recovering the Moment Graph: Mutations

Definition (Movable part of Q_S)

A movable part of a linear segment $L \subset Q_S$ is a connected subquiver $L' \subset L$ such that L' has the same starting point of L.

Definition (Mutation)

Consider $Q_S, Q_{S'}$ S.C.S of $Q(M)$, we say that there is a mutation from $Q_{S'}$ to Q_S if Q_S is obtained from $Q_{S'}$ moving exactly one movable part.

Figure: A Mutation from the 4-th line to the 3-rd

Theorem

Consider $p_S \in X^T$. The dimension of tangent plane at p_S is equal to the valence of p_S in $G(X, T)$.

$\mathrm{Mut}_S(i,j)$ is the set of mutations from i-th row of p_S to its j-th row.

Definition (Oriented Mutation (Multi)Graph $\widetilde{G}_S = (V(\widetilde{G}_S), E(\widetilde{G}_S))$)

1. *$V(\widetilde{G}_S)$ is the set $\{1, \ldots, n+1\}$,*
2. *there are exactly $|\mathrm{Mut}_S(i,j)|$ edges oriented from i to j.*

We denote by G_S the unoriented undeliyng graph.

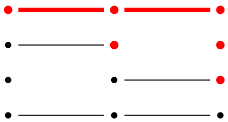

Figure: A Fixed point p_S

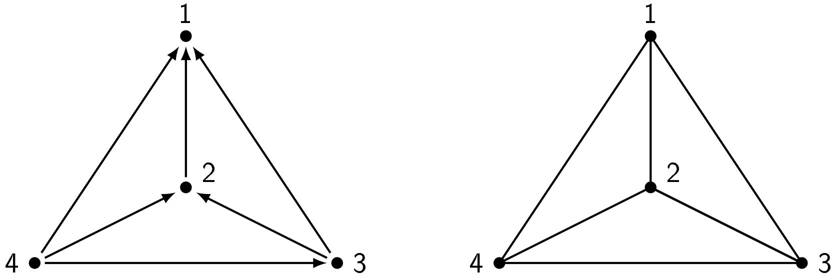

Figure: The oriented and unoriented of mutation graphs of p_S.

Smoothness Criteria

Definition (Generalized CIFR Condition)

S has the Generalized CIFR Condition if, for every $i \in [n+1]$,

$$i \in S_k,\ k \in \mathrm{pos}(j) \Rightarrow j \in S_{h+1},\ \forall h \in \mathrm{pos}(i),\ h \geq k.$$

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Theorem (Smoothness Criteria)

Let $\mathfrak{Fl}^{\mathbf{I}}(\mathbb{C}^{n+1})$ be a flat degeneration. The following conditions are equivalent:

1. *The point p_S is smooth;*
2. *The admissible sequence S has the Generalized CIFR Condition;*
3. *The Mutation graph G_S is the complete graph over $n+1$ vertices;*
4. *The oriented Mutation graph $\tilde{G}_S$ is a transitive tournament over $n+1$ vertices;*

Thank you for the attention!