On the multiplication of spherical functions of reductive spherical pairs of type A

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Let $K \subset G$ be connected reductive algebraic groups over $\mathbb{C}$.

Definition

G/K is called **spherical** *if $\mathbb{C}[G/K]$ is a multiplicity free G-module. Equivalently, (G, K) is called a* **reductive spherical pair**.

EXAMPLES.

Symmetric varieties: $K = (G^{\vartheta})^{\circ}$ for some $\vartheta \in Inv(G)$, e.g.

$$SL_n/SO_n, \quad SL_{2n}/Sp_{2n}, \quad SL_{n+1}/GL_n, \quad Sp_{2n}/GL_n,$$
$$SO_{2n}/GL_n, \quad SO_n/SO_{n-1}, \quad \mathrm{E}_6/\mathrm{F}_4, \quad \mathrm{F}_4/\mathrm{B}_4$$

Non-symmetric exampes:

$$SL_{2n+1}/Sp_{2n}, \quad SO_{2n+1}/GL_n, \quad SO_8/G_2, \quad \mathrm{G}_2/\mathrm{A}_2$$

We will assume that G is simple and simply connected.

The classification of the reductive spherical pairs with G simple goes back to Krämer (1979).

Fix $X = G/K$ spherical. Consider the decomposition of G-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)$$

with $E_X(\lambda) \simeq V(\lambda)$ irreducible of highest weight $\lambda \in \Lambda^+$

- Λ_X^+ is called the **weight monoid** of X.
- The **weight lattice** of X is the lattice Λ_X generated by Λ_X^+.

The weight monoid and weight lattice of X are well understood.

Problem

Given $\lambda, \mu \in \Lambda_X^+$, determine the decomposition of $E_X(\lambda) \cdot E_X(\mu)$: for which $\nu \in \Lambda_X^+$ does it hold $E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu)$?

Somehow, Λ_X^+ behaves as a monoid of dominant weights for a root datum (Φ_X, Λ_X) attached to X, which generalize the restricted root system of a symmetric variety.

We will be interested in the case where Φ_X is of type A.

The **root monoid** of X is

$$\mathcal{M}_X = \langle \lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Lambda_X^+, \; E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \rangle_{\mathbb{N}}$$

Theorem (Knop 1994)

$\mathcal{M}_X$ is a free monoid, and the set of free generators $\Delta_X \subset \mathcal{M}_X$ is the base of a (reduced) root system Φ_X.

In general, Λ_X is not contained in the weight lattice of Φ_X. However it is possible to define a map

$$\Phi_X \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda_X, \mathbb{Z}) \qquad \alpha \longmapsto \alpha^\vee$$

giving rise to a *based root datum*

$$\mathcal{R}_X = (\Lambda_X, \Delta_X, \Delta_X^\vee)$$

- In this way, the weight monoid Λ_X^+ gets identified with a submonoid of dominant weights for $\mathcal{R}_X$.
- If X is a symmetric variety, then $\mathcal{R}_X$ is semisimple and Φ_X is the reduced root system associated to the restricted root system of X.

Let G_X be the reductive group defined by $\mathcal{R}_X$, then

$$\Lambda_X^+ \longrightarrow Irr(G_X) \qquad \lambda \longmapsto V_X(\lambda)$$

Recall the decomposition of G-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda), \qquad\qquad E_X(\lambda) \simeq V(\lambda)$$

Conjecture

Suppose that Φ_X is of type A, *and let $\lambda, \mu, \nu \in \Lambda_X^+$. Then*

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \Longleftrightarrow V_X(\nu) \subset V_X(\lambda) \otimes V_X(\mu)$$

For symmetric varieties, in a slightly different form, the conjecture was considered by Graham-Hunziker (2009).

Theorem (Bravi-G.)

- *The previous conjecture is true if $X \neq \mathrm{F}_4/\mathrm{B}_4$, provided a conjecture of Stanley on Jack symmetric functions is true.*
- *Suppose that Φ_X is direct sum of root systems of type A_1, and assume $X \neq \mathrm{F}_4/\mathrm{B}_4$. Then the previous conjecture is true.*

Table: Symmetric pairs with restricted root system of type A

G	K	$\Phi_{G/K}$	$m_{G/K}$
SL_n, $n \geqslant 2$	SO_n	A_{n-1}	1
$\mathrm{SL}_n \times \mathrm{SL}_n$, $n \geqslant 2$	$\mathrm{diag}(\mathrm{SL}_n)$	A_{n-1}	2
SL_{2n}, $n \geqslant 2$	Sp_{2n}	A_{n-1}	4
Spin_n, $n \geqslant 5$	Spin_{n-1}	A_1	$n-2$
E_6	F_4	A_2	8

Table: The other reductive spherical pairs with root system of type A

G	K	$\Phi_{G/K}$	Sym
SL_n, $n \geqslant 3$	SL_{n-1}	A_1	
SL_n, $n \geqslant 3$	GL_{n-1}	A_1	*
SL_{2n+1}, $n \geqslant 2$	Sp_{2n}	$A_n \times A_{n-1}$	
SL_{2n+1}, $n \geqslant 2$	$GL_1 \times Sp_{2n}$	$A_n \times A_{n-1}$	
Sp_{2n}, $n \geqslant 2$	$GL_1 \times Sp_{2n-2}$	$A_1 \times A_1$	
Sp_{2n}, $n \geqslant 3$	$Sp_2 \times Sp_{2n-2}$	A_1	*
$Spin_7$	G_2	A_1	
$Spin_9$	$Spin_7$	$A_1 \times A_1$	
$Spin_8$	G_2	$A_1 \times A_1 \times A_1$	
F_4	$Spin_9$	A_1	*
G_2	SL_3	A_1	

- Several of the previous example correspond to exotic homogeneous actions on spheres:

$$SL_n/SL_{n-1} \simeq SO_{2n}/SO_{2n-1}$$

$$Spin_9/Spin_7 \simeq SO_{16}/SO_{15}$$

$$Spin_7/G_2 \simeq SO_8/SO_7$$

$$G_2/SL_3 \simeq SO_7/SO_6$$

- Other of the previous examples are also connected to exotic actions on symmetric varieties:

$$SL_{2n+1}/Sp_{2n} \simeq SL_{2n+2}/Sp_{2n+2}$$

$$Sp_{2n}/GL_1 \times Sp_{2n-2} \simeq SL_{2n}/GL_{2n-1}$$

$$SO_8/G_2 \simeq SO_8/SO_7 \times SO_8/Spin_7$$

Such isomorphisms will be fundamental to reduce the problem to the symmetric case.

Spherical functions of a reductive spherical pair

Let $X = G/K$ multiplicity free. Then

$$\mathbb{C}[X] = \mathbb{C}[G]^K \simeq \bigoplus V(\lambda)^* \otimes V(\lambda)^K$$

Being multiplicity free amounts to the property

$$\dim V(\lambda)^K \leqslant 1 \quad \forall \lambda \in \Lambda^+$$

By the decomposition $\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)$ it follows that

$$\Lambda_X^+ = \{\lambda \in \Lambda^+ \mid \dim V(\lambda)^K = 1\} :$$

therefore $\dim E_X(\lambda)^K = \dim V(\lambda)^K = 1$ for all $\lambda \in \Lambda_X^+$.

Fix nonzero elements $\varphi_\lambda \in E_X(\lambda)^K$: then $\{\varphi_\lambda\}_{\lambda \in \Lambda_X^+}$ form a basis of

$$\bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)^K = \mathbb{C}[X]^K = \mathbb{C}[G]^{K \times K}$$

- Write $\varphi_\lambda \cdot \varphi_\mu = \sum a_{\lambda,\mu}^\nu \varphi_\nu$. Then

$$\boxed{E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff a_{\lambda,\mu}^\nu \neq 0}$$

Jack polynomials and Stanley's conjecture

Let t be a parameter. The **Jack polynomials** $J_\lambda^{(t)}$ form a basis over $\mathbb{C}(t)$ of the vector space

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} \otimes \mathbb{C}(t)$$

indexed by the partitions λ of lenght $\ell(\lambda) \leqslant n$.

- $J_\lambda^{(1)}$ is scalar multiple of the Schur polynomial s_λ.

Given partitions λ, μ, ν, write $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu$ and

$$J_\lambda^{(t)} \cdot J_\mu^{(t)} = \sum_\nu f_{\lambda,\mu}^\nu(t) J_\nu^{(t)}$$

Therefore

$$f_{\lambda,\mu}^\nu(1) \neq 0 \Longleftrightarrow c_{\lambda,\mu}^\nu \neq 0$$

Conjecture (Stanley 1989)

The structure coefficients $f_{\lambda,\mu}^\nu(t)$ are ratio of polynomials in $\mathbb{N}[t]$. In particular:

$$f_{\lambda,\mu}^\nu \neq 0 \quad \Longleftrightarrow \quad f_{\lambda,\mu}^\nu(t) \neq 0 \quad \forall t > 0 \quad \Longleftrightarrow \quad c_{\lambda,\mu}^\nu \neq 0$$

Stanley's Pieri rule: The conjecture is true if $\ell(\lambda) = 1$.

Jack polynomials and symmetric varieties of type A

Let $X = G/K$ symmetric, $K = (G^\vartheta)^\circ$. Fix in G a maximal *split torus* A, namely

$$\vartheta(a) = a^{-1} \quad \forall a \in A$$

and let $T \subset G$ be a maximal torus containing A.
Then T is ϑ-stable, and the weight lattice of X is

$$\Lambda_X = \{\chi - \vartheta(\chi) \ : \ \chi \in \mathcal{X}(T)\}.$$

- The **restricted root system** of X is
$$\widetilde{\Phi}_X = \{\alpha_{|A} : \alpha \in \Phi\} \smallsetminus \{0\} :$$
a root system in $\mathcal{X}(A)_\mathbb{R}$ with weight lattice $\mathcal{X}(A/A \cap K) \simeq \Lambda_X$.
It comes with a **multiplicity function**
$$m_{\tilde{\alpha}} = |\{\alpha \in \Phi : \alpha_{|A} = \tilde{\alpha}\}$$
- Its Weyl group is the **little Weyl group**
$$\widetilde{W}_X = N_K(A)/Z_K(A)$$
- By a theorem of Richardson, restriction gives an isomorphism
$$\mathbb{C}[X]^K \xrightarrow{\sim} \mathbb{C}[A/A \cap K]^{\widetilde{W}} \simeq \mathbb{C}[A]^{\widetilde{W}}$$

Suppose moreover that $\widetilde{\Phi}$ is of type A_{n-1}.

Then $\widetilde{W} \simeq S_n$, the weight monoid Λ_X^+ is identified with the set of partitions of length $< n$, and

$$\mathbb{C}[A] \simeq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(x_1 \cdots x_n - 1) :$$

thus we have a homomorphism $\mathbb{C}[x_1, \dots, x_n]^{S_n} \to \mathbb{C}[A]^{\widetilde{W}}$.

- If m_X is the multiplicity of the restricted roots and $\lambda \in \Lambda_X^+$, then up to scalars $\varphi_\lambda \in \mathbb{C}[X]^K$ maps to the image of the specialized Jack polynomial $J_\lambda^{(2/m_X)} \in \mathbb{C}[x_1, \dots, x_n]^{S_n}$.

Recall the decompositions

$$\varphi_\lambda \cdot \varphi_\mu = \sum a_{\lambda,\mu}^\nu \varphi_\nu \qquad J_\lambda^{(t)} \cdot J_\mu^{(t)} = \sum f_{\lambda,\mu}^\nu(t) J_\nu^{(t)}$$

Therefore

$$a_{\lambda,\mu}^\nu \neq 0 \iff f_{\lambda,\mu}^\nu(2/m_X) \neq 0$$

Theorem

If X is symmetric with $\widetilde{\Phi}_X$ of type A, then the multiplication conjecture follows from Stanley's conjecture. In particular, by Stanley's Pieri rule, the conjecture is true if $\widetilde{\Phi}_X$ of type A_1.

The general case: reduction to the symmetric case

Suppose now that $X = G/K$ is spherical with Φ_X of type A.

In all cases but F_4/B_4, the problem can be reduced to the case of some symmetric variety with reduced root system of type A, in a compatible way with the dec. of Φ_X into irreducible subsystems.

- The reduction stems from the fact that (typically) the action of G on X extends to a bigger group $\widehat{G} \supset G$ making X a symmetric $\widehat{G}$-variety Y.
- Possibly we will also need a subvariety $Z \subset X$, which is symmetric for some $H \subset G$.

Let's illustrate the reduction with an example.

Definition. Let $\mathcal{R}_1 = (\Lambda_1, \Delta_1, \Delta_1^\vee)$ and $\mathcal{R}_2 = (\Lambda_2, \Delta_2, \Delta_2^\vee)$ be based root data and let $\phi : \Lambda_1 \to \Lambda_2$ be a homomorphism. Then ϕ is an **isogeny of based root data** if

- ϕ is injective with finite cokernel,
- ϕ restricts to a bijection between Δ_1 and Δ_2,
- the dual map satisfies $\phi^\vee(\phi(\sigma)^\vee) = \sigma^\vee$, for all $\sigma \in \Delta_1$.

Example: $G = SL_{2n+1}$, $K = Sp_{2n}$, $X = G/K$

Let
$$\widehat{G} = SL_{2n+2} \qquad \widehat{K} = Sp_{2n+2} \qquad H = SL_{2n}$$

Then we have
$$\widehat{G} = G \cdot \widehat{K}, \qquad G \cap \widehat{K} = K$$

giving rise to equivariant maps
$$Z =: H/K \hookrightarrow G/K \xrightarrow{\sim} \widehat{G}/\widehat{K} := Y$$

hence
$$\mathbb{C}[Y] \xleftarrow{\sim} \mathbb{C}[X] \xrightarrow{\rho} \mathbb{C}[Z]$$

This induce maps
$$\widehat{\phi} : \Lambda_X^+ \longrightarrow \Lambda_Y^+ \qquad \bar{\phi} : \Lambda_X^+ \longrightarrow \Lambda_Z^+$$

- $\widehat{\phi}(\lambda)$ is the unique $\widehat{\lambda} \in \Lambda_Y^+$ such that $E_X(\lambda) \subset E_Y(\widehat{\lambda})$;
- $\bar{\phi}(\lambda)$ is the unique $\bar{\lambda} \in \Lambda_Z^+$ such that $\rho(E_X(\lambda)) = E_Z(\bar{\lambda})$.

Notice that both Z and Y are symmetric. Moreover
$$\Phi_X = \mathsf{A}_n \times \mathsf{A}_{n-1} \qquad \Phi_Y = \mathsf{A}_n \qquad \Phi_Z = \mathsf{A}_{n-1}$$

$(\widehat{\phi}, \bar{\phi})$ induce an isogeny of root data $\mathcal{R}_X \to \mathcal{R}_Y \oplus \mathcal{R}_Z$ compatible with the multiplication in the sense

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff \begin{array}{c} E_Y(\widehat{\nu}) \subset E_X(\widehat{\lambda}) \cdot E_X(\widehat{\mu}) \\ E_Z(\bar{\nu}) \subset E_Z(\bar{\lambda}) \cdot E_Z(\bar{\mu}) \end{array}$$

Thank you