Local-global invariants of groups, Lie algebras, and associative algebras

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The Golod–Shafarevich examples (1964) answer all these questions in the negative.
It is important to emphasize that their solution of the group-theoretic problem goes through associative algebras.

Main objects

We will discuss the **Tate–Shafarevich set** $\text{Ш}(G)$ of a group G and its analogues for Lie algebras and associative algebras.

Tate–Shafarevich set $\text{Ш}(G)$

Let a group G act on itself by conjugation. The Tate–Shafarevich set is defined with the help of (nonabelian) group cohomology corresponding to this action, by the formula

$$\text{Ш}(G) := \ker[H^1(G,G) \to \prod_{C<G \text{ cyclic}} H^1(C,G)].$$

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The definition and the name are due to Takashi Ono. The local-global flavour justifies the allusion to the object bearing the same name which appeared in the arithmetic-geometric context (related to the action of the absolute Galois group of a number field K on the group $A(\overline{K})$ of $\overline{K}$-points of an abelian K-variety A).

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Reinterpretation of $Ш(G)$

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Recall that 1-cocycles $Z^1(G, G)$ are none other than crossed homomorphisms, i.e., maps $\psi\colon G \to G$ with the property

$$\psi(st) = \psi(s)^s\psi(t) = \psi(s)s\psi(t)s^{-1}.$$

Then the correspondence $\psi(s) \mapsto f(s) = \psi(s) \cdot s$ gives a bijection between $Z^1(G, G)$ and $\mathrm{End}(G)$.

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Then the correspondence $\psi(s) \mapsto f(s) = \psi(s)\cdot s$ gives a bijection between $Z^1(G,G)$ and $\mathrm{End}(G)$. Under this correspondence, 1-coboundaries correspond to inner automorphisms. Further, a 1-cocycle whose cohomology class becomes trivial after restriction to every cyclic subgroup corresponds to a locally inner (=pointwise inner=class preserving) endomorphism, i.e., $f \in \mathrm{End}(G)$ with the property $f(g) = a^{-1}ga$ (where a depends on g).

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$$\text{Ш}(G) \cong \mathrm{Aut}_c(G)/\mathrm{Inn}(G),$$

where $\mathrm{Aut}_c(G)$ stands for the group of class-preserving automorphisms of G.

William Burnside (1852–1927)

In my survey "Local-global invariants of finite and infinite groups: Around Burnside from another side", Expo. Math., 2013 (see also an earlier survey by Manoi Yadav), one can find many classes of groups G with trivial $\text{Ш}(G)$ (they are called there Ш-rigid) as well as some interesting examples with nontrivial $\text{Ш}(G)$ (they often give rise to counter-examples to some difficult problems, such as Higman's problem on isomorphism of integral group rings).

Lie-algebraic analogue

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$$\text{Ш}(\mathfrak{g}) := \mathrm{AID}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g}),$$

where

$$\mathrm{AID}(\mathfrak{g}) := \{D \in \mathrm{Der}(\mathfrak{g}) \mid (\forall X \in \mathfrak{g})\,(\exists Z \in \mathfrak{g}) \quad D(X) = [Z, X]\}$$

(with Z depending on X) stands for the algebra of "almost inner derivations".

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Many properties of $Ш(G)$ and $Ш(\mathfrak{g})$ are known to be quite similar.

Some parallels

Say, the smallest group G with nontrivial $Ш(G)$ is of order $32 = 2^5$ (G. E. Wall, 1947), and the smallest dimension of $\mathfrak{g}$ with nontrivial $Ш(\mathfrak{g})$ is 5.

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$\text{Ш}(G) = 0$ for all finite simple groups G (Walter Feit and Gary Seitz, 1989, uses classification).
$\text{Ш}(\mathfrak{g}) = 0$ for any finite-dimensional simple Lie algebra over a field of characteristic zero (because such an algebra does not have outer derivations at all according to first Whitehead's lemma).

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It would be interesting to transfer other properties of $\text{Ш}(G)$ to $\text{Ш}(\mathfrak{g})$. Say, it is conjectured that $\text{Ш}(G)$ is always solvable; what about $\text{Ш}(\mathfrak{g})$? It is known that $\text{Ш}(G)$ may be nonabelian, the smallest example is of order 2^{15} (Sah, 1968); what about $\text{Ш}(\mathfrak{g})$?

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If this question is answered in the affirmative, we conclude that $Ш(\mathfrak{g})$ is an ideal of $\mathrm{OutDer}(\mathfrak{g})$.

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So far, this question is wide open. The next result can be viewed as a first step.

Theorem. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over $k = \mathbb{C}$. Then $\mathrm{AID}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, and hence $Ш(\mathfrak{g})$ is an ideal of $\mathrm{OutDer}(\mathfrak{g})$.*

Sketch of proof

Step 1. Note that since $\mathfrak{g}$ is nilpotent, it is algebraic, i.e. there exists an affine algebraic k-group G such that $\mathfrak{g} = \mathrm{Lie}(G)$.

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Step 3. $H := \mathrm{Aut}_{\mathrm{c}}(G)$ is a closed normal subgroup of N.

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Step 4. By Step 3, H is an affine algebraic group. Then $\mathrm{Lie}(H)$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.

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Step 4. By Step 3, H is an affine algebraic group. Then $\mathrm{Lie}(H)$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.
Step 5. We have an isomorphism $\mathrm{Lie}(H) \cong \mathrm{AID}(\mathfrak{g})$. By Step 4, this finishes the proof.

Associative algebras: derivations

Recall that a derivation $D\colon A \to M$ is a k-linear map such that

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. For a given $m \in M$, the map

$$D_m\colon A \to M, \quad m \mapsto am - ma,$$

is a derivation. Such derivations are called inner. We denote by $\mathrm{Der}(A, M)$ the set of all derivations and by $\mathrm{ad}(A, M)$ the set of all inner derivations. Clearly, they are both vector k-spaces, and $\mathrm{ad}(A, M)$ is a k-subspace of $\mathrm{Der}(A, M)$. Let $\mathrm{OutDer}(A, M) = \mathrm{Der}(A, M)/\,\mathrm{ad}(A, M)$ denote the quotient space. It is well known that $\mathrm{OutDer}(A, M)$ is isomorphic to the first Hochschild cohomology $HH^1(A, M)$.

Associative algebras: derivations

In the special case $M = A$ we abbreviate the notation $\mathrm{Der}(A, A)$, $\mathrm{ad}(A, A)$ and $\mathrm{OutDer}(A, A)$ to $\mathrm{Der}(A)$, $\mathrm{ad}(A)$ and $\mathrm{OutDer}(A)$, respectively. The first two spaces acquire a natural Lie algebra structure defined by the Lie bracket $[D, D'] = DD' - D'D$, $\mathrm{ad}(A)$ is a Lie ideal of $\mathrm{Der}(A)$, hence $\mathrm{OutDer}(A)$ also carries a Lie algebra structure. This Lie algebra is isomorphic to the Hochschild cohomology $HH^1(A)$.

Associative algebras: additive $Ш(A)$

Set

$$\mathrm{AID}(A, M) := \{D \in \mathrm{Der}(A, M) \mid (\forall a \in A)\,(\exists m \in M)\, D(a) = am - ma\}.$$

(Here m may depend on a.) We call elements of $\mathrm{AID}(A, M)$ almost inner derivations of A with coefficients in M.

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Clearly, $\mathsf{AID}(A, M)$ is a subspace of $\mathsf{Der}(A, M)$, $\mathsf{ad}(A, M)$ is a subspace of $\mathsf{AID}(A, M)$, and we define

$$\text{Ш}_{\mathrm{add}}(A, M) := \mathsf{AID}(A, M) / \mathsf{ad}(A, M).$$

It is a subspace of $\mathsf{OutDer}(A, M)$.

Associative algebras: additive $Ш(A)$

As above, in the particular case $M = A$ we shorten $\mathrm{AID}(A, M)$ and $Ш_{\mathrm{add}}(A, M)$ to $\mathrm{AID}(A)$ and $Ш_{\mathrm{add}}(A)$, respectively. One can show that $\mathrm{AID}(A)$ inherits the Lie algebra structure from $\mathrm{Der}(A)$.

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Clearly, $\mathrm{ad}(A)$ is a Lie ideal in $\mathrm{AID}(A)$, and hence $Ш_{\mathrm{add}}(A)$ also carries a natural Lie algebra structure.
We call $Ш_{\mathrm{add}}(A)$ the **additive Tate–Shafarevich algebra** of A.

Associative algebras: additive $Ш(A)$

Once a new object is introduced, the first question to ask is whether it can be nontrivial. It is not hard to construct an associative algebra A with nonzero $Ш_{\mathrm{add}}(A)$. Here is a 'generic' construction suggested by Leonid Makar-Limanov (a similar construction was communicated by Alexei Kanel-Belov).

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Take a non-commutative algebra A with an infinite set S of generators and finitary multiplication table, i.e. such that only a finite number of generators do not commute with any given generator. Let m denote a formal infinite sum of elements of A such that every generator appears only in a finite number of summands of m. Then the map $D_m \colon A \to A, \quad a \mapsto am - ma$, is well-defined and is a derivation of A. Clearly, this derivation is almost inner but not inner, so that $\text{Ш}_{\mathrm{add}}(A) \neq 0$.

Associative algebras: additive $\text{Ш}(A)$

Our further goal is to exhibit a *finitely generated* algebra A with $\text{Ш}_{\mathrm{add}}(A) \neq 0$. Towards this end, consider $A = U(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra, and $U(\mathfrak{g})$ is its universal enveloping algebra. Any $\mathfrak{g}$-bimodule M has a unique structure of a $U(\mathfrak{g})$-bimodule.

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Lemma.

(i) *For any $\mathfrak{g}$-bimodule M the vector k-spaces $\text{Ш}_{\mathrm{add}}(U(\mathfrak{g}), M)$ and $\text{Ш}(\mathfrak{g}, M)$ are isomorphic.*

(ii) *The Lie algebras $\text{Ш}_{\mathrm{add}}(U(\mathfrak{g}))$ and $\text{Ш}(\mathfrak{g}, U(\mathfrak{g}))$ are isomorphic.*

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(ii) *The Lie algebras $\text{Ш}_{\mathrm{add}}(U(\mathfrak{g}))$ and $\text{Ш}(\mathfrak{g}, U(\mathfrak{g}))$ are isomorphic.*

Corollary. *There exist finitely generated associative algebras A with $\text{Ш}_{\mathrm{add}}(A) \neq 0$.*

Associative algebras: additive $Ш(A)$

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Associative algebras: additive $Ш(A)$

The algebra $U(\mathfrak{g})$ is infinite-dimensional, so the next step is to look for *finite-dimensional* associative algebras A with $Ш_{\mathrm{add}}(A) \neq 0$. Somewhat degenerate examples arise from the following observation: a Lie algebra $\mathfrak{g}$ is associative if and only if it is two-step nilpotent. As examples of two-step nilpotent Lie algebras $\mathfrak{g}$ with $Ш(\mathfrak{g}) \neq 0$ can be produced in abundance, we obtained the needed associative algebras A for free. Note, however, that the obtained associative algebras are obviously not unital. To repair this, one can use a standard procedure of adjoining the unit to get a unital algebra $\widetilde{A} := k \oplus A$ for which we have $Ш_{\mathrm{add}}(\widetilde{A}) = Ш_{\mathrm{add}}(A) \neq 0$.

Associative algebras: additive $Ш(A)$

It is tempting to use the same examples of finite-dimensional nilpotent Lie algebras $\mathfrak{g}$ with nonzero $Ш(\mathfrak{g})$ to construct 'genuine' examples of finite-dimensional associative algebras A with nonzero $Ш_{\mathrm{add}}(A)$.

Associative algebras: additive $Ш(A)$

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Genuine examples we are looking for cannot be too small: if $\dim(A) \leq 4$, then $Ш_{\mathrm{add}}(A) = 0$.

Multiplicative $\text{Ш}(A)$

Let $G = \mathrm{Aut}_k(A)$ be the group of all k-algebra automorphisms of A. In the sequel, we shorten $\mathrm{Aut}_k(A)$ to $\mathrm{Aut}(A)$. Let $A^\times$ denote the group of invertible elements of A. Denote by $\mathrm{Inn}(A)$ the group of inner automorphisms of A. Recall that $\varphi \in \mathrm{Inn}(A)$ if there exists $a \in A^\times$ such that $\varphi(x) = axa^{-1}$. $\mathrm{Inn}(A)$ is a normal subgroup of $\mathrm{Aut}(A)$.

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Define

$$\mathrm{AIAut}(A) := \{\varphi \in \mathrm{Aut}(A) \,|\, (\forall x \in A)\, (\exists a \in A^\times)\, \varphi(x) = axa^{-1}\}.$$

(Here a may depend on x.) We call elements of $\mathrm{AIAut}(A)$ almost inner automorphisms of A.
Clearly, $\mathrm{Inn}(A)$ is a normal subgroup of $\mathrm{AIAut}(A)$.

Multiplicative $\text{Ш}(A)$

The group

$$\text{Ш}_{\mathrm{mult}}(A) := \mathsf{AIAut}(A)/\mathsf{Inn}(A)$$

is called the **multiplicative Tate–Shafarevich group** of A.
As in the additive set-up, we first make sure that there exist A with $\text{Ш}_{\mathrm{mult}}(A) \neq 0$ by providing an example (the following version is due to Be'eri Greenfeld).

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As in the additive set-up, we first make sure that there exist A with $Ш_{\mathrm{mult}}(A) \neq 0$ by providing an example (the following version is due to Be'eri Greenfeld).
Let A be the algebra of (countably) infinite matrices S over k which are eventually scalar (namely, for $i + j \gg 1, S(i,j) = \lambda\delta_{i,j}$ for some $\lambda \in k$). Consider the automorphism of A induced by conjugation by an infinite diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \dots)$ with distinct nonzero λ_i's. This is an almost inner automorphism of A which is not inner. Hence $Ш_{\mathrm{mult}}(A) \neq 0$.

Multiplicative $\text{Ш}(A)$

Both additive and multiplicative examples are reminiscent of a similar well-known construction arising in the group-theoretic set-up. Namely, let $G = \mathrm{FSym}(\Omega)$ be a finitary symmetric group (the group of all permutations of an infinite set Ω fixing all but finitely many elements of Ω). Viewing G as a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$, consider an automorphism $\varphi\colon G \to G$ induced by conjugation by some $a \in \mathrm{Sym}(\Omega) \setminus \mathrm{FSym}(\Omega)$. Clearly, φ is almost inner but not inner. Actually, in this case $\mathrm{AIAut}(G)/\mathrm{Inn}(G)$ is isomorphic to the infinite simple group $\mathrm{FSym}(\Omega)/\mathrm{Sym}(\Omega)$ (this observation is attributed to Passman), and $\text{Ш}(G)$ is even larger because there are non-surjective almost inner endomorphisms.

Multiplicative $Ш(A)$

As in the additive set-up, we are interested in exhibiting examples of finitely generated (or even finite-dimensional) algebras A with nontrivial $Ш_{\mathrm{mult}}(A)$.
In the case $A = U(\mathfrak{g})$, we did not succeed in presenting an example of $\mathfrak{g}$ with $Ш_{\mathrm{mult}}(U(\mathfrak{g})) \neq 0$.

Multiplicative Ш(A)

Consider finite-dimensional algebras A. In this case, G can be equipped with a structure of an affine algebraic k-group (not necessarily connected). Let G_A denote its identity component, it is a closed, connected, normal subgroup of finite index in G.

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Multiplicative $Ш(A)$

Recently, this structure attracted considerable attention, being an invariant of the derived equivalence class of A and being related to representation theory, in particular, to the representation type of A. It would be interesting to understand whether one can use the multiplicative and additive $Ш(A)$ in this circle of problems. First, one has to answer some basic questions. Recall that we assume A to be a finite-dimensional associative unital algebra over an algebraically field k of characteristic zero.

Multiplicative Ш(A)

Lemma. $\mathrm{AIAut}(A)$ *is a normal subgroup of* G_A.

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Clearly, Inn(A) is a closed, connected, normal subgroup of Aut(A), so that if for a certain algebra A the above question is answered in the affirmative, then $Ш_{\mathrm{mult}}(A)$ becomes a closed subgroup of Out(A), thus acquiring the structure of an affine algebraic k-group. This gives rise to the following observation.

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This gives rise to the following observation.
Lemma. *Suppose that* AIAut(A) *a closed subgroup of* Aut(A). *Then the Lie algebras* Lie$(\text{Ш}_{\text{mult}}(A))$ *and* $\text{Ш}_{\text{add}}(A)$ *are isomorphic.*

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Thus, under the assumptions of the lemma, any eventual example of an algebra A with nonzero $Ш(A)$, either additive or multiplicative, would immediately yield a required example for the other structure.

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Theorem. *Assume in addition that the algebraic k-group G_A is nilpotent. Then*

(i) $\mathrm{AIAut}(A)$ *is a closed normal subgroup of G_A with Lie algebra* $\mathrm{AID}(A)$*;*

(ii) *the Lie algebras* $\mathrm{Lie}(Ш_{\mathrm{mult}}(A))$ *and* $Ш_{\mathrm{add}}(A)$ *are isomorphic.*

Concluding parallels

Main vague parallels arise from looking at $Ш(G)$ of finite groups G. Throughout we assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra and A is a finite-dimensional associative unital algebra.

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Question.

(i) Does there exist $\mathfrak{g}$ such that the algebra $Ш(\mathfrak{g})$ is non-abelian?

(ii) Does there exist A such that the algebra $Ш_{\mathrm{add}}(A)$ is non-abelian?

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Recall that Sah disproved Burnside's statement and exhibited examples of p-groups G with non-abelian $\text{Ш}(G)$, the smallest among them is a group of order 2^{15}.

Our working hypothesis is that all these questions are answered in the affirmative.

Concluding parallels

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(i) Does there exist $\mathfrak{g}$ such that the algebra $Ш(\mathfrak{g})$ is non-solvable?

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Here we would rather expect that all Tate–Shafarevich algebras and groups appearing in these questions are solvable. Note that even in the case of finite groups G only a conditional statement is available. Sah's proof of the solvability contains a gap noticed by Murai who showed that the validity of this assertion depends on the Alperin–McKay conjecture.

Eventual generalizations

It is tempting to extend the above notions to other algebraic structures for which there exists a developed cohomology theory, with a goal to define, explore and apply analogues of Tate–Shafarevich sets to relevant problems of the corresponding research area.

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It is tempting to extend the above notions to other algebraic structures for which there exists a developed cohomology theory, with a goal to define, explore and apply analogues of Tate–Shafarevich sets to relevant problems of the corresponding research area. One has to try to equip these sets, if possible, with an additional structure (group or algebra). Also, it is very desirable to include the structure under consideration in a relevant triad, if such exists, similarly to the classical triad consisting of Lie algebras, associative algebras and groups. Here are possible instances of such an approach.

Eventual generalizations

- *Malcev algebras*

Malcev algebras arise from Lie algebras when one relaxes the Jacobi identity replacing it with a weaker condition, and keeps the anti-commutativity. One can start with derivations of such an algebra M and introduce almost inner derivations.

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Eventual generalizations

- *Leibniz algebras*

Leibniz algebras arise from Lie algebras in an opposite way, when one keeps the Jacobi identity and drops the anti-commutativity condition. Here there is a well-developed (co)homology theory (Loday, Pirashvili), and the Leibniz adjoint cohomology $HL^1(L, L)$ of a Leibniz algebra L is isomorphic to the space of outer derivations of L. One then can introduce almost inner derivations and $Ш(L)$ as in the case of Lie algebras. Adashev and Kurbanbaev (2020) provided examples of L with nonzero $Ш(L)$.

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Eventual generalizations

- *Poisson algebras*

Recall that a Poisson algebra A is equipped with structures of associative algebra and Lie algebra which are related by the Leibniz identity. The Poisson adjoint cohomology $H^1_\pi(A)$ is isomorphic to the quotient $\mathrm{Der}_\pi(A)/\mathrm{Ham}(A)$, where $\mathrm{Der}_\pi(A)$ is the Lie algebra of Poisson derivations (i.e. derivations of both associative and Lie structures) and $\mathrm{Ham}(A)$ is the ideal of Hamiltonian derivations. As in the preceding cases, we can introduce almost inner derivations and define $\text{Ш}(A)$. Here one can hope to use the Duflo isomorphism (Pevzner–Torossian) for establishing connections and analogies with other versions of Ш. We hope that this object admits a conceptual interpretation within the frame of Poisson geometry.

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THANKS FOR YOUR ATTENTION!