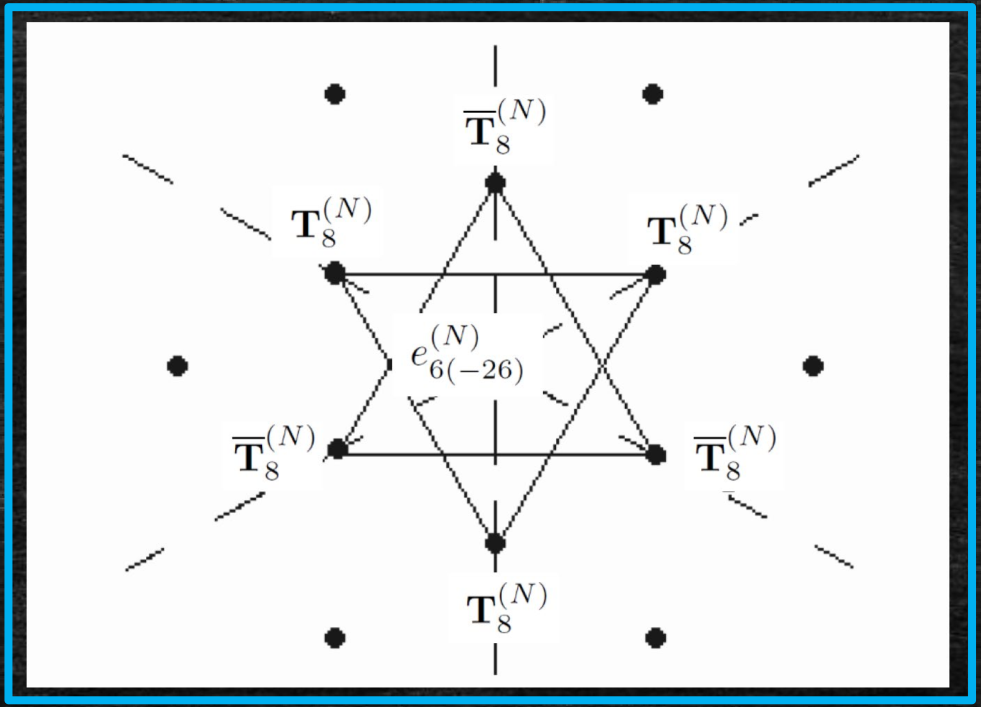

The «Magic Star» projection

Exceptional Structures and Vinberg Algebras

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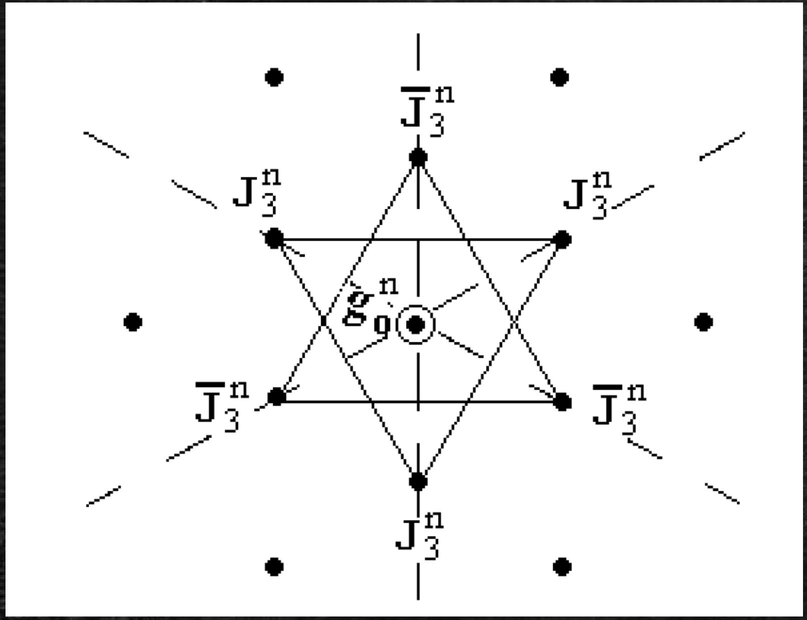
The «Magic Star» projection

- the unifying **Magic Star** (**MS**) projection/embedding for exceptional Lie algebras
- the spin-factor embedding for exceptional Lie algebras
- **Exceptional Periodicity** (EP) and the **MS** projection : beyond e8
- (rank 3) Vinberg's **HT-algebras** and their invariant structure
- EP algebras are **not** Lie algebras, but then *what are they?*
- *Further developments (work in progress…)*

Refs : Truini '12, AM,Truini '14, Truini, AM, Rios '17-'19, Truini, de Graaf, AM, '23 *to appear…*
main Ref *: Truini, Rios, AM, Contemporary Mathematics* ***721****, AMS (2019)*

the **Magic Star** (MS) projection/embedding for exceptional Lie algebras

Mukai '96, Truini '12



8D → 2D MS projection in root lattice
(not unique)

$$\mathbf{L}^{\mathbf{n}} = \mathbf{a_2} \oplus \mathbf{str}_0\left(\mathbf{J}_3^{\mathbf{n}}\right) \oplus \mathbf{3} \times \mathbf{J}_3^{\mathbf{n}} \oplus \overline{\mathbf{3}} \times \overline{\mathbf{J}_3^{\mathbf{n}}},$$

for the (compact, real form of the) exceptional sequence

n	8	4	2	1	0	$-2/3$	-1
$\mathbf{L}^{\mathbf{n}}$	$\mathbf{e}_{8(-248)}$	$\mathbf{e}_{7(-133)}$	$\mathbf{e}_{6(-78)}$	$\mathbf{f}_{4(-52)}$	$\mathbf{so}(8)$	$\mathbf{g}_{2(-14)}$	$\mathbf{su}(3)$
$\mathbf{str}_{0,c}$	$\mathbf{e}_{6(-78)}$	$\mathbf{su}(6)$	$\mathbf{su}(3) \oplus \mathbf{su}(3)$	$\mathbf{su}(3)$	$\mathbf{u}(1) \oplus \mathbf{u}(1)$	–	–

The relevant (non-compact) real form for application to (super)gravity reads

$$\mathbf{L}^{\mathbf{n}} = \mathbf{sl}(3, \mathbf{R}) \oplus \mathbf{str}_0\left(\mathbf{J}_3^{\mathbf{n}}\right) \oplus \mathbf{3} \times \mathbf{J}_3^{\mathbf{n}} \oplus \mathbf{3}' \times \mathbf{J}_3^{\mathbf{n}\prime}.$$

$$\mathbf{L}^{\mathbf{n}} = \mathbf{qconf}\left(\mathbf{J}_3^{\mathbf{n}}\right)$$

AM, Truini '14

D=(4+1)$_M$ U-duality

D=(2+1)$_M$ electric-magnetic (U-)duality symmetry

[a particular case of the so-called **super-Ehlers embedding** :
Ferrara, AM, Zumino '12, Ferrara, AM, Trigiante '12]

$$J = \sum_{I=1}^{n} q^I e_I = \begin{pmatrix} r_1 & A_1 & \overline{A}_2 \\ \overline{A}_1 & r_2 & A_3 \\ A_2 & \overline{A}_3 & r_3 \end{pmatrix} \qquad r_i \in \mathbb{R}, A_i \in \mathbb{A}$$

Jordan, Von Neumann, Wigner '34

J3 is a simple, rank-3, Euclidean Jordan algebra,
realized as the set of 3x3 Hermitian matrices over
a division (or split) normed (Hurwitz) algebra.
[for g2, this is simply R; for su(3), it is trivially the empty set]

1902-1980

From the symmetry of the Freudenthal-Rosenfeld-Tits Magic Square, e6 is both the str_o of J3(O) and the qconf of J3(C), one can apply the MS embedding **twice**, and get the simple characterization of e8 : Truini '12

$$\begin{aligned}\mathbf{e_8} &= \mathbf{a_2^c} + (3, \mathbf{J_3^8}) + (\bar{3}, \overline{\mathbf{J}}\mathbf{_3^8}) + Der_o(\mathbf{J_3^8}, \overline{\mathbf{J}}\mathbf{_3^8}) \\ &= \mathbf{a_2^c} + (3, \mathbf{J_3^8}) + (\bar{3}, \overline{\mathbf{J}}\mathbf{_3^8}) + \mathbf{a_2^f} + (3, \mathbf{J_3^2}) + (\bar{3}, \overline{\mathbf{J}}\mathbf{_3^2}) + Der_o(\mathbf{J_3^2}, \overline{\mathbf{J}}\mathbf{_3^2}) \\ &= \mathbf{a_2^c} + (3, \mathbf{J_3^8}) + (\bar{3}, \overline{\mathbf{J}}\mathbf{_3^8}) + \mathbf{a_2^f} + (3, \mathbf{J_3^2}) + (\bar{3}, \overline{\mathbf{J}}\mathbf{_3^2}) + \mathbf{a_2^{(1)}} + \mathbf{a_2^{(2)}}\end{aligned}$$

In fact (relevant nc, real forms)

$$\mathfrak{qconf}\left(\mathfrak{J}_3^{\mathbb{O}}\right) = \mathfrak{e}_{8(-24)};\ \mathfrak{str}_0\left(\mathfrak{J}_3^{\mathbb{O}}\right) = \mathfrak{e}_{6(-26)} \sim \mathfrak{sl}(3,\mathbb{O})$$

$$\mathfrak{e}_{8(-24)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{e}_{6(-26)} \oplus \mathbf{3}\times\mathbf{27} \oplus \mathbf{3}'\times\mathbf{27}'$$

$$\mathfrak{qconf}\left(\mathfrak{J}_3^{\mathbb{O}_s}\right) = \mathfrak{e}_{8(8)};\ \mathfrak{str}_0\left(\mathfrak{J}_3^{\mathbb{O}_s}\right) = \mathfrak{e}_{6(6)} \sim \mathfrak{sl}(3,\mathbb{O}_s)$$

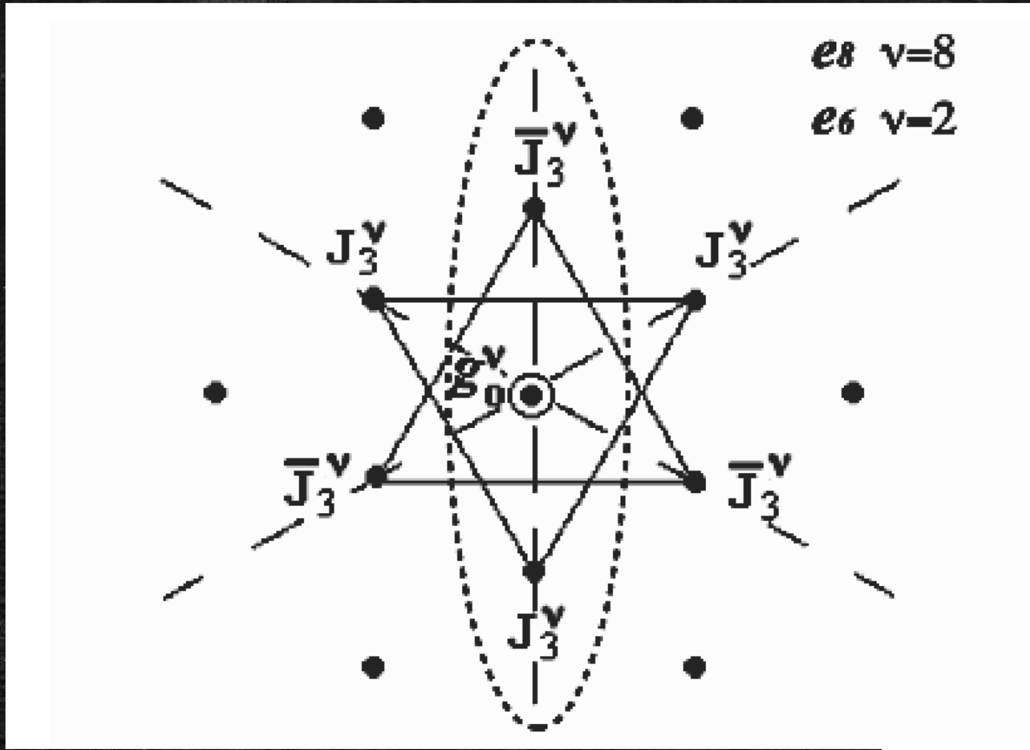
$$\mathfrak{e}_{8(8)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{e}_{6(6)} \oplus \mathbf{3}\times\mathbf{27} \oplus \mathbf{3}'\times\mathbf{27}'$$

$$\mathfrak{qconf}\left(\mathfrak{J}_3^{\mathbb{C}}\right) = \mathfrak{e}_{6(2)};\ \mathfrak{str}_0\left(\mathfrak{J}_3^{\mathbb{C}}\right) = \mathfrak{sl}(3,\mathbb{C})$$

$$\mathfrak{e}_{6(2)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathbf{3}\times(\mathbf{3},\overline{\mathbf{3}}) \oplus \mathbf{3}'\times(\overline{\mathbf{3}},\mathbf{3})$$

$$\mathfrak{qconf}\left(\mathfrak{J}_3^{\mathbb{C}_s}\right) = \mathfrak{e}_{6(6)} = \mathfrak{str}_0\left(\mathfrak{J}_3^{\mathbb{O}_s}\right);\ \mathfrak{str}_0\left(\mathfrak{J}_3^{\mathbb{C}_s}\right) = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) \sim \mathfrak{sl}(3,\mathbb{C}_s)$$

$$\mathfrak{e}_{6(6)} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})_I \oplus \mathfrak{sl}(3,\mathbb{R})_{II} \oplus \mathbf{3}\times(\mathbf{3},\mathbf{3}') \oplus \mathbf{3}'\times(\mathbf{3}',\mathbf{3})$$

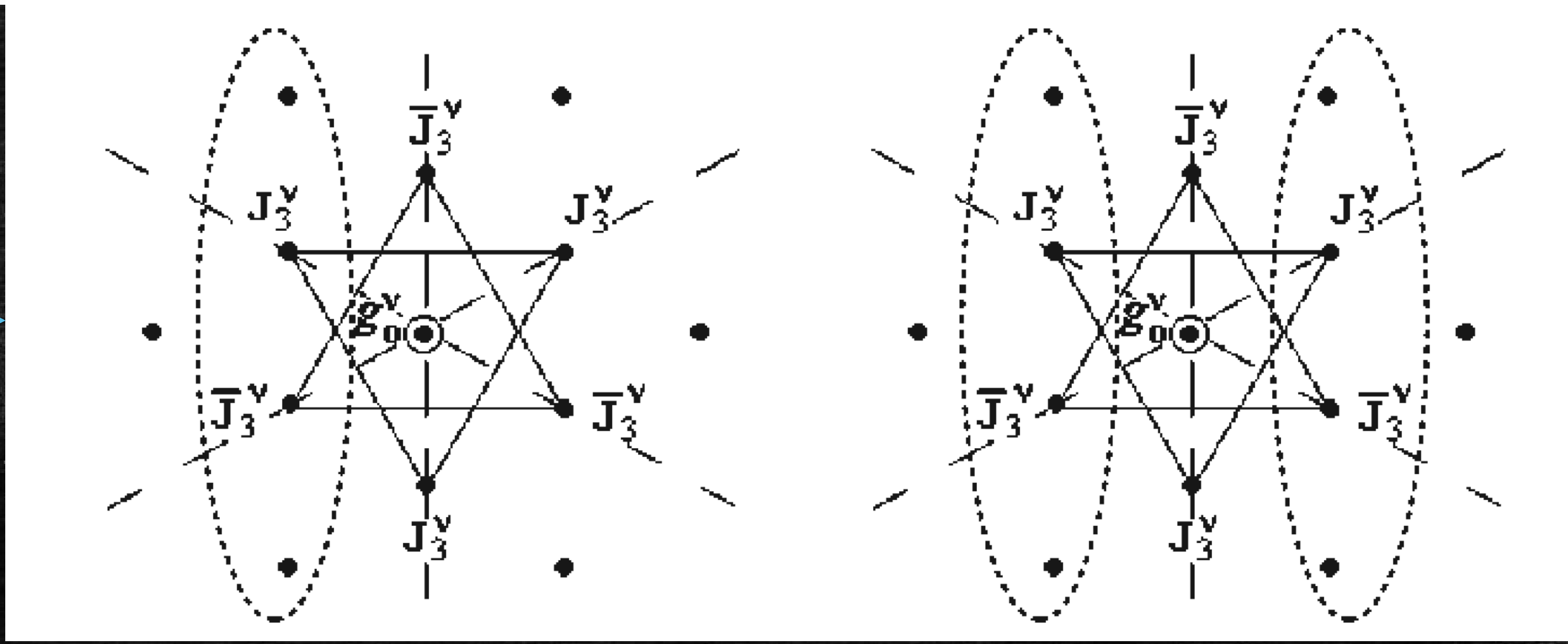


Truini, Rios, AM '17

$\mathbf{e_7}$ (resp. $\mathbf{a_5}$) as 3-graded inside $\mathbf{e_8}$ (resp. $\mathbf{e_6}$)

1905-1990

FTS (left) and KP (right) within $\mathbf{e_8}$



FTS = Freudenthal triple system
Freudenthal '59
KP = Kantor pair
Allison, Faulkner, Smirnov '17

the **spin-factor** embedding for exceptional Lie algebras

maximal Jordan algebraic embedding

$$\mathbf{J}_3^n \supset \mathbb{R} \oplus \mathbf{J}_2^n$$

$$n = \dim_{\mathbb{R}} \mathbb{A} = 1, 2, 4, 8 \;\; for \;\; \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$$

$$\mathbf{J}_2^n \approx \mathbf{\Gamma}_{1,n+1}$$

$\Gamma_{m,n}$ is a Jordan algebra with a quadratic form of signature (m, n), i.e. the Clifford algebra of O(m,n)

$$\mathbf{J}_3^n \ni J = \begin{pmatrix} r_1 & A_1 & \overline{A}_2 \\ \overline{A}_1 & r_2 & A_3 \\ A_2 & \overline{A}_3 & r_3 \end{pmatrix} \mapsto J' = \begin{pmatrix} r_1 & A_1 & 0 \\ \overline{A}_1 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \in \mathbb{R} \oplus \mathbf{J}_2^n$$

$$r_1 := x_+ + x_-, \;\; r_2 := x_+ - x_-$$

2D lightcone coords.

In the case n=8 (octonions **O**) , this implies various embeddings for the relevant symmetry algebras :

[analogous treatment can be given for n=1(**R**), n=2 (**C**) and n=4 (**H**), but it yields exceptional Lie algebras only in some cases]

$$\mathfrak{Der} : f_{4(-52)} = so(9) \oplus \mathbf{16}$$

$$\mathfrak{str}_0 : e_{6(-26)} = \begin{cases} \mathbf{16}'_{-\alpha} \oplus (so(1,9) \oplus so(1,1))_0 \oplus \mathbf{16}_{\alpha} \\ \qquad or \\ \mathbf{16}_{-\alpha} \oplus (so(1,9) \oplus so(1,1))_0 \oplus \mathbf{16}'_{\alpha} \end{cases}$$

3-grading
[with spinor polarization (s.p.)] Minchenko '06

$$\mathfrak{conf} : e_{7(-25)} = \begin{cases} (so(2,10) \oplus sl(2,\mathbb{R})) \oplus \left(\mathbf{32}^{(\prime)}, \mathbf{2}\right) = \\ \mathbf{1}_{-2\alpha} \oplus \mathbf{32}^{(\prime)}_{-\alpha} \oplus (so(2,10) \oplus so(1,1))_0 \oplus \mathbf{32}^{(\prime)}_{\alpha} \oplus \mathbf{1}_{2\alpha} \end{cases}$$

5-grading
(contact type)
[with s.p.]

5-grading (ext. Poincaré type) [with s.p.]

$$\mathfrak{qconf} : e_{8(-24)} = so(4,12) \oplus \mathbf{128}^{(\prime)} = \begin{cases} \mathbf{14}_{-2\alpha} \oplus \mathbf{64}'_{-\alpha} \oplus (so(3,11) \oplus so(1,1))_0 \oplus \mathbf{64}_{\alpha} \oplus \mathbf{14}_{2\alpha} \\ \qquad or \\ \mathbf{14}_{-2\alpha} \oplus \mathbf{64}_{-\alpha} \oplus (so(3,11) \oplus so(1,1))_0 \oplus \mathbf{64}'_{\alpha} \oplus \mathbf{14}_{2\alpha} \end{cases}$$

Cantarini, Ricciardo, Santi '17

Exceptional Periodicity (**EP**) and the **Magic Star** (**MS**) projection : **beyond e8**

Truini, Rios, Marrani '17

By exploiting **Bott periodicity** for spinor bundles, the following definitions of **(exceptional) EP algebras** can be put forward :

$N \in \mathbb{N}$ (EP level)

$$f_{4(-52)}^{(N)} \quad : \quad = so(9+8N) \oplus \psi_{\mathfrak{b}_{4+4N}}$$

$$\psi_{\mathfrak{b}_{4+4N}} \quad \equiv \quad \mathbf{2}^{4+4N} \text{ real spinor of } so(9+8N) \equiv \mathfrak{b}_{4+4N}$$

3-grading [with s.p.]

$$e_{6(-26)}^{(N)} \quad : \quad = \left\{ \begin{array}{c} \psi'_{\mathfrak{d}_{5+4N},-\alpha} \oplus (so(1,9+8N) \oplus so(1,1))_0 \oplus \psi_{\mathfrak{d}_{5+4N},\alpha} \\ or \\ \psi_{\mathfrak{d}_{5+4N},-\alpha} \oplus (so(1,9) \oplus so(1,1))_0 \oplus \psi'_{\mathfrak{d}_{5+4N},\alpha} \end{array} \right.$$

Majorana Weyl (MW)

$$\psi_{\mathfrak{d}_{5+4N}} \quad \equiv \quad 2^{4+4N} \; real\ chiral\ (semi)spinor\ of\ so(1,9+8N) \equiv (nc, real\ form\ of)\ \mathfrak{d}_{5+4N}$$

$$e_{7(-25)}^{(N)} \quad : \quad = \left\{ \begin{array}{l} (so(2,10+8N) \oplus sl(2,\mathbb{R})) \oplus \left(\psi^{(\prime)}_{\mathfrak{d}_{6+4N}}, \mathbf{2}\right) = \\ \mathbf{1}_{-2\alpha} \oplus \psi^{(\prime)}_{\mathfrak{d}_{6+4N},-\alpha} \oplus (so(2,10+8N) \oplus so(1,1))_0 \oplus \psi^{(\prime)}_{\mathfrak{d}_{6+4N},\alpha} \oplus \mathbf{1}_{2\alpha} \end{array} \right.$$

MW

$$\psi_{\mathfrak{d}_{6+4N}} \quad \equiv \quad 2^{5+4N} \; real\ chiral\ (semi)spinor\ of\ so(2,10+8N) \equiv (nc, real\ form\ of)\ \mathfrak{d}_{6+4N}$$

5-grading (contact type) [with s.p.]

[g2 does **not** enjoy an analogous EP-generalization; we will **not** be considering it henceforth]

[Nota Bene : analogous EP-generalizations can be considered for *all other nc, real forms*] in particular, e8(-24) is EP-generalized as follows :

$$
\begin{aligned}
e_{8(-24)}^{(N)} &= so(4,12+8N) \oplus \psi_{\mathfrak{d}_{8+4N}}^{(\prime)} \\
&= \begin{cases} (\mathbf{14+8N})_{-2\alpha} \oplus \psi'_{\mathfrak{d}_{7+4N,-\alpha}} \oplus (so(3,11+8N) \oplus so(1,1))_0 \oplus \psi_{\mathfrak{d}_{7+4N,\alpha}} \oplus (\mathbf{14+8N})_{2\alpha} \\ or \\ (\mathbf{14+8N})_{-2\alpha} \oplus \psi_{\mathfrak{d}_{7+4N,-\alpha}} \oplus (so(3,11+8N) \oplus so(1,1))_0 \oplus \psi'_{\mathfrak{d}_{7+4N,\alpha}} \oplus (\mathbf{14+8N})_{2\alpha} \end{cases} \\
\psi_{\mathfrak{d}_{8+4N}} &\equiv 2^{7+4N}\ real\ chiral\ (\overset{\text{MW}}{semi})spinor\ of\ so(4,12+8N) \equiv (nc, real\ form\ of)\ \mathfrak{d}_{8+4N} \\
\psi_{\mathfrak{d}_{7+4N}} &\equiv 2^{6+4N}\ real\ chiral\ (\overset{\text{MW}}{semi})spinor\ of\ so(3,11+8N) \equiv (nc, real\ form\ of)\ \mathfrak{d}_{7+4N}
\end{aligned}
$$

5-grading (extended Poincaré type) [with s.p.]

So far, this is just a bunch of definitions, exploiting Bott periodicity.

However, we anticipate that EP algebras are **not** non-reductive Lie algebras! (see further below)

[explicit construction in terms of roots and lattices can be found in Truini, Marrani, Rios '17, '18, using **Kac's asymmetry function**]

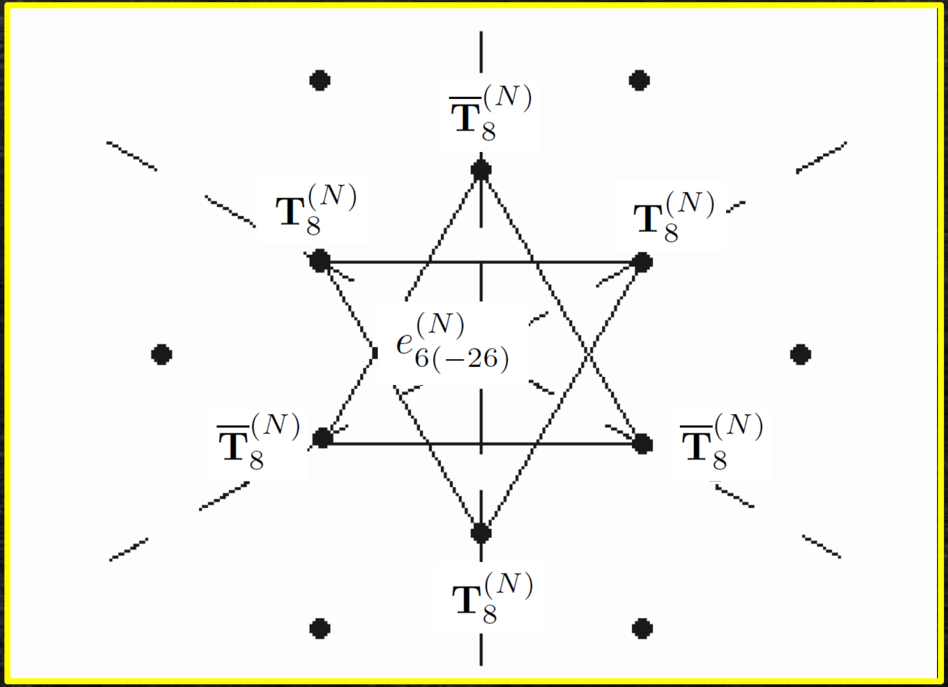
Consistently, for **N=0** EP algebras yield finite-dimensional **exceptional Lie algebras**.

key result : Truini, AM, Rios ‘17

There exists a (*non-unique*) a2 – projection/embedding of EP algebras, such that the **MS structure persists**, with **generalizations of rank-3 simple Jordan algebras** appearing on the six tips of the MS!

For simplicity’s sake, let us consider the EP-generalization e8(-24)^(N) of e8(-24) :

Essentially due to the **symmetry** between the fourth row and the fourth column of the **FRT Magic Square**, the cases of (the EP-generalization of) f4,e6,e7,e8 correspond to n=1(**R**), 2 (**C**), 4(**H**) and 8(**O**), resp.

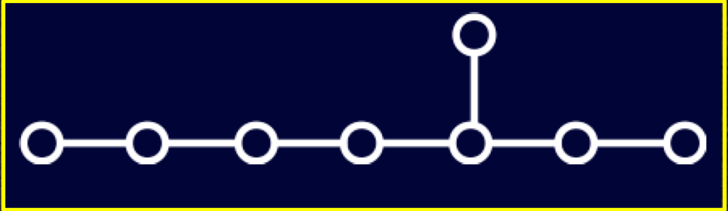


(8+4N)D → 2D MS projection of EP algebra e8(-24)^(N) in «*generalized*» root lattice (Truini, AM, Rios ‘17)

At the level of **Dynkin diagrams** and **Cartan matrices** (and their generalization) :

$$C_{e_8} = \begin{pmatrix} \mathbf{2} & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & -1 \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & -2 & -1 & \\ & & & & & -1 & 2 & \\ & & -1 & & & & & 2 \end{pmatrix}$$

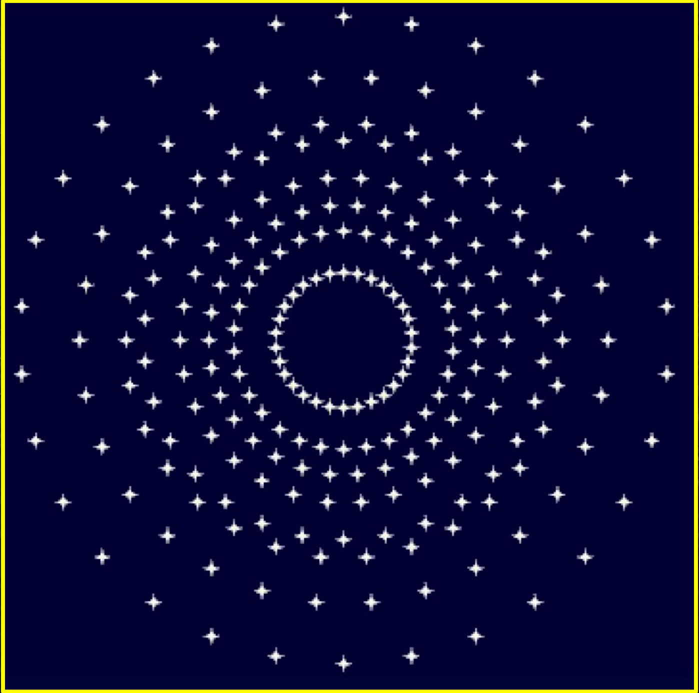
e8



e8 = Dynkin diagram of d7 + Weyl spinor of d8
= roots of d8 + MW spinor of d8

$\mathbf{e_8}$ 240 roots:

$\pm k_i \pm k_j$	$1 \leq i < j \leq 8$	112 *roots*
$\frac{1}{2}(\pm k_1 \pm k_2 \pm k_3 \pm k_4 \pm k_5 \pm k_6 \pm k_7 \pm k_8)$	even # of +	128 *roots*



Let's compare the **Cartan matrix** of e12 = e8^++++ with the **Gram matrix** of e8^(1)

$$C_{e_8^{++++}} = \begin{pmatrix} \mathbf{2} & -1 & & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & -1 \\ & & -1 & 2 & -1 & & & & & & & \\ & & & -1 & 2 & -1 & & & & & & \\ & & & & -1 & 2 & -1 & & & & & \\ & & & & & -1 & 2 & -1 & & & & \\ & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & -1 & 2 & -1 & & \\ & & & & & & & & -1 & 2 & -1 & \\ & & & & & & & & & -1 & 2 & \\ & & -1 & & & & & & & & & 2 \end{pmatrix}$$

the corresponding lattice is a root lattice, invariant under Weyl reflections. The corresponding algebra is Lie, but infinite-dimensional (generalized Kac-Moody (GKM) algebra)

$$C_{e_8^{(1)}} = \begin{pmatrix} \mathbf{3} & -1 & & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & -1 \\ & & -1 & 2 & -1 & & & & & & & \\ & & & -1 & 2 & -1 & & & & & & \\ & & & & -1 & 2 & -1 & & & & & \\ & & & & & -1 & 2 & -1 & & & & \\ & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & -1 & 2 & -1 & & \\ & & & & & & & & -1 & 2 & -1 & \\ & & & & & & & & & -1 & 2 & \\ & & -1 & & & & & & & & & \end{pmatrix}$$

the corresponding lattice is not a root lattice, it is not inv.under Weyl reflections The corresponding algebra is not Lie, but finite-dimensional (called EP algebra)

the difference is in the **norm** of the MW spinor of d12

Vinberg's **HT**-algebras

What does appear on the tips of such an **EP-generalized MS** ?
A 3x3 Hermitian generalized matrix, with structure

$$\mathbf{T}_8^{(N)} := \begin{pmatrix} r_1 & \mathbf{V}_{\mathfrak{d}_{4+4N}} & \psi_{\mathfrak{d}_{4+4N}} \\ \overline{\mathbf{V}}_{\mathfrak{d}_{4+4N}} & r_2 & \psi'_{\mathfrak{d}_{4+4N}} \\ \overline{\psi}_{\mathfrak{d}_{4+4N}} & \overline{\psi'}_{\mathfrak{d}_{4+4N}} & r_3 \end{pmatrix}$$

$$\mathbf{V}_{\mathfrak{d}_{4+4N}} \equiv 8+8N \ vector \ of \ so(8+8N) \equiv (real \ form \ of) \ \mathfrak{d}_{4+4N}$$

$$\psi_{\mathfrak{d}_{4+4N}} \equiv 2^{3+4N} \ real \ chiral \ (MW) spinor \ of \ so(8+8N) \equiv (real \ form \ of) \ \mathfrak{d}_{4+4N}$$

T8^(N) belongs to the Hermitian part of the class of the so-called **special rank-3 T-algebras**,
introduced by Vinberg as a **generalization** of rank-3 Jordan algebras in the study of homogeneous convex cones [Vinberg '60].
Thus, we will refer to T8^(N) as to a **rank-3 Hermitian T- (HT) algebra**

EP algebras are not Lie algebras

EP algebras, as presented in previous slides, **are not Lie algebras**.
In fact, e.g. for e8(-24)^(N), one can show that the spinor non-reductive part does **not** satisfy the Jacobi identity
[Truini, AM, Rios, '18-'19, and forthcoming papers...]

This is strictly related to the **non-Abelian** nature of the spinor part, which thus **cannot** be regarded as a translational extension of the reductive, simple, pseudo-othogonal part of the algebra **[see next two slides]**

Concerning physical applications :
1] the **failure of Jacobi in the spinor sector** might be related to **dark matter/dark energy** degrees of freedom;
2] the non-Abelian nature of the spinor part is crucial in order to have **non-trivial interactions among bosons and fermions** in an algebra which is not a superalgebra (or a Z_2-graded algebra) [AM, Truini '15]

Thus, in general EP algebras are not simply spinor-translational extensions of simple, pseudo-orthogonal Lie algebras.
At each level of the EP, i.e. for each fixed N**, the dimension of the algebra is finite**, and it enjoys a MS projection/embedding, which relates it to a certain **rank-3 Vinberg's HT-algebra of special type** [Vinberg '60]

The approach of EP-generalization of finite-dimensional exceptional Lie algebras is therefore very different form the usual infinite-dimensional extension through affine, extended, very extended (Kac-Moody) Lie algebras (possibly, with further Borcherds generalizations) :
In fact, **at each level of EP the algebra is finite-dimensional**.
*The «price» to pay is the **failure of Jacobi** (non-Lie nature).*

As of today, the determination of the real nature of EP algebras is still under study :
maybe, a «weaker Jacobi» identity holds for such algebras?

Some Further Developments (work in progress….)

- symmetry algebras of **(H)T-algebras** and related rings of invariants, with related physical meaning;

- ring of invariants of spinor irreprs. (some of them are examples of Vinberg's theta groups; for instance, d6 on **32**(') is «of type e7» [Brown '69], d7 on **64**(') has inv. rank-8, found in charting of Vogel's plane [Vogel '95,'99; Mkrtchyan '12]);

- EP and higher Rosenberg planes/Tits' buildings (higher ptojective planes on formal tensor products of division algebras) [**metasymplectic geometry?** cfr. e.g. Landsberg, Manivel '99]

- **Kantor-Koecher-Tits** procedure applied to (H)T-algebras, and comparison of the outcome to **EP algebras**…

- **HT-algebra pairs**, reduced Freudenthal triple systems over (H)T-algebras, (reduced) Kantor pairs over (H)T-algebras [Faulkner *et al.* '14] , and their symmetries;

- EP and higher-dimensional (global and local) **supersymmetry** (Rios, AM, Chester, '18, '19);

- How to take advantage of the failure of Jacobi? Model of emergence of space and time **purely from interactions** [AM, Truini '15; Truini, Rios, AM '17]


Thank
You!