Unitary minimal W-algebras

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Minimal W-algebras

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Kac and Wakimoto used quantum Hamiltonian reduction to build up a vertex algebra $W^k(\mathfrak{g}, x, f)$, associated to a datum $(\mathfrak{g}, x, f)$ and $k \in \mathbb{C}$. Here

- $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a basic Lie superalgebra, i.e. $\mathfrak{g}$ is simple, its even part $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}$ carries an even invariant non-degenerate supersymmetric bilinear form $(\cdot|\cdot)$,
- x is an *ad*-diagonalizable element of $\mathfrak{g}_{\bar{0}}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, $f \in \mathfrak{g}_{\bar{0}}$ is such that $[x, f] = -f$ and the eigenvalues of $ad\, x$ on the centralizer $\mathfrak{g}^f$ of f in $\mathfrak{g}$ are non-positive
- $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

W-algebras

The most important examples are provided by x and f to be part of an sl_2 triple $\{e, x, f\}$, where $[x, e] = e, [x, f] = -f, [e, f] = x$. In this case $(\mathfrak{g}, x, f)$ is called a *Dynkin datum*.
We recently proved that if ϕ is a conjugate linear involution of $\mathfrak{g}$ such that

$$\phi(x) = x, \quad \phi(f) = f \text{ and } \overline{(\phi(a)|\phi(b))} = (a|b),\ a, b \in \mathfrak{g}, \tag{1.1}$$

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then ϕ induces a conjugate linear involution of the vertex algebra $W^k(\mathfrak{g}, x, f)$.

Moreover, if ϕ is a conjugate linear involution of $W^k(\mathfrak{g}, x, f)$, this vertex algebra carries a non-zero ϕ-invariant Hermitian form $H(\cdot, \cdot)$ for all $k \neq -h^\vee$ if and only if $(\mathfrak{g}, x, f)$ is a Dynkin datum; moreover, such H is unique, up to a real constant factor, and we normalize it by the condition $H(\mathbf{1}, \mathbf{1}) = 1$.

Goal of the talk

Definition

A module M for a vertex algebra V is called *unitary* if there is a conjugate linear involution ϕ of V such that there is a positive definite ϕ–invariant Hermitian form on M. A simple vertex algebra V is called unitary if the adjoint module is.

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Goals of the talk

Denote by $W^k_{\min}(\mathfrak{g})$ the vertex algebra $W^k(\mathfrak{g}, x, f)$ when f is a minimal nilpotent element.

1. classify the pairs $(\mathfrak{g}, k)$ such that the simple quotient $W_k^{\min}(\mathfrak{g})$ of $W^k_{\min}(\mathfrak{g})$ is unitary.
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2. Investigate the irreducible unitary modules for $W^k_{\min}(\mathfrak{g})$:
    1. Classification
    2. Characters
3. Classify irreducible representations of $W_k^{\min}(\mathfrak{g})$ *in the unitary range*.

Minimal W-algebras

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- If ϕ fixes x and f, it also fixes e, hence $\phi(\mathfrak{g}^\natural) = \mathfrak{g}^\natural$.
- It is easy to see that unitarity of $W_k(\mathfrak{g}, x, f)$ implies, when k is not collapsing, that $\phi_{|[\mathfrak{g}^\natural,\mathfrak{g}^\natural]}$ is a compact involution.

Definition: minimal W-algebra

We ask that for the $ad\,x$-gradation $\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ one has

$$\mathfrak{g}_j = 0 \ \text{ if } |j| > 1, \text{ and } \mathfrak{g}_{-1} = \mathbb{C}f. \tag{2.1}$$

In this case $(\mathfrak{g}, x, f)$ is automatically a Dynkin datum. The corresponding W-algebra is called *minimal*, denoted by $W^k_{\min}(\mathfrak{g})$.

Restrictions on $\mathfrak{g}$ for unitarity

Proposition (KMP, CCM2022)

If $W_k^{\min}(\mathfrak{g})$ is unitary and k is not a collapsing level, then the parity of $\mathfrak{g}$ is compatible with the ad x-gradation, i.e.

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-1}}_{even} \oplus \underbrace{\mathfrak{g}_{-1/2}}_{odd} \oplus \underbrace{\mathfrak{g}_0}_{even} \oplus \underbrace{\mathfrak{g}_{1/2}}_{odd} \oplus \underbrace{\mathfrak{g}_1}_{even} \qquad (\clubsuit)$$

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Corollary

The complete list of the $\mathfrak{g}$ for which $(\clubsuit)$ holds is

$$sl(2|m) \text{ for } m \geq 3, \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0,$$
$$osp(4|m) \text{ for } m > 2 \text{ even}, \quad D(2,1;a) \text{ for } a \in \mathbb{C}, \quad F(4), \quad G(3).$$

In these cases

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}^\natural \oplus \mathfrak{s}, \quad \mathfrak{s} \cong sl_2.$$

Choice of ϕ

- It turns out that a conjugate linear involution of the minimal W-algebra $W^k_{\min}(\mathfrak{g})$ at non-collapsing level k is necessarily induced by a conjugate linear involution ϕ of $\mathfrak{g}$.
- Moreover, if $W^k_{\min}(\mathfrak{g})$ admits a unitary highest weight module, then $\mathfrak{g}^\natural$ has to be semisimple and the involution ϕ of $\mathfrak{g}$ must be *almost compact*, according to the following definition.

Definition

We say that a conjugate linear involution ϕ on $\mathfrak{g}$ is almost compact if

(i) ϕ fixes e, x, f;

(ii) ϕ is a compact conjugate linear involution of $\mathfrak{g}^\natural$.

So, in order to study unitarity of highest weight modules, it is not restrictive to assume that the conjugate linear involution of $W^k_{\min}(\mathfrak{g})$ is induced by an almost compact conjugate linear involution of $\mathfrak{g}$.

Almost compact involutions

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Example

$\mathfrak{g} = spo(2|m)$. Then $\mathfrak{g}_{\bar{0}} = sl_2 \oplus so_m$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}^2 \otimes \mathbb{C}^m$ as $\mathfrak{g}_{\bar{0}}$-module. We set

$$\mathfrak{g}_{\bar{0}}^{ac} = sl_2(\mathbb{R}) \oplus so_m(\mathbb{R}), \quad \mathfrak{g}_{\bar{1}}^{ac} = \mathbb{R}^2 \otimes \mathbb{R}^m.$$

Explicitly, let B be a non-degenerate $\mathbb{R}$-valued bilinear form of the superspace $\mathbb{R}^{2|m}$ with matrix $\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & I_m \end{array}\right)$. Then

$$\mathfrak{g}^{ac} = \{A \in sl(2|m;\mathbb{R}) \mid B(Au,v) + (-1)^{p(A)p(u)} B(u,Av) = 0\}.$$

Superconformal algebras

Recall that the most important superconformal algebras in conformal field theory are minimal W-algebras or are obtained from them by a simple modification:

(a) $W_k^{\min}(spo(2|N))$ is the Virasoro vertex algebra for $N = 0$, the Neveu-Schwarz vertex algebra for $N = 1$, the $N = 2$ vertex algebra for $N = 2$, and becomes the $N = 3$ vertex algebra after tensoring with one fermion; it is the Bershadsky-Knizhnik algebra for $N > 3$;

(b) $W_k^{\min}(psl(2|2))$ is the $N = 4$ vertex algebra;

(c) $W_k^{\min}(D(2, 1; a))$ tensored with four fermions and one boson is the big $N = 4$ vertex algebra.

Superconformal algebras

The unitary Virasoro ($N = 0$), Neveu-Schwarz ($N = 1$) and $N = 2$ simple vertex algebras were classified in the mid 80s. The above three cases cover all minimal W-algebras such that $\mathfrak{g}_0$ is abelian. Up to isomorphism, these vertex algebras depend only on the central charge $c(k)$,

$$c(k) = 1 - \frac{6}{p(p+1)} \quad \text{for Virasoro vertex algebra,} \tag{2.2}$$

$$c(k) = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \quad \text{for Neveu-Schwarz vertex algebra,} \tag{2.3}$$

$$c(k) = 3\left(1 - \frac{2}{p}\right) \quad \text{for } N = 2 \text{ vertex algebra.} \tag{2.4}$$

where $k = \frac{1}{p} - 1$. Next theorem is a result of many papers from the 80s

Theorem

The complete list of unitary $N = 0, 1,$ and 2 vertex algebras is as follows: either $c(k)$ is given as above for $p \in \mathbb{Z}_{\geq 2}$, or $c(k) \geq 1, \frac{3}{2}$ or 3 respectively.

Recap on vertex algebras

A vertex algebra structure on a vector space V is encoded in the state-field correspondence

$$a \mapsto Y(a, z) = a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

where $a_{(n)} \in End(V)$ and $a_{(n)} v = 0$ for $n >> 0$ (depending on v); hence one has infinitely many bilinear products on V: $(a, b) \mapsto a_{(n)} b$.

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By the so-called locality axiom,

$$[a(z), b(w)] = \sum_{j \in \mathbb{Z}_+} (a_{(j)}b)(w) \frac{\partial^j \delta_w(z-w)}{j!}$$

Recap on vertex algebras

Note that this relation gives rise to a Lie algebra structure on the coefficients:

$$[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}$$

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The singular part of the OPE expansion is encoded in the λ-bracket

$$[a_\lambda b] = \sum_{n\geq 0} a_{(n)}b\frac{\lambda^n}{n!}$$

whose properties may be axiomatized: $[\ _\lambda\ ] : V \otimes V \to \mathbb{C}[\lambda] \otimes V$

1. $[Ta_\lambda b] = -\lambda[a_\lambda b],\ T[a_\lambda b] = [Ta_\lambda b] + [a_\lambda Tb]$ (sesquilinearity)
2. $[b_\lambda a] = -p(a, b)[a_{-\lambda-T}b]$ (skew-symmetry)
3. $[a_\lambda[b_\mu c]] - p(a, b)[b_\lambda[a_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c]$ (Jacobi identity)

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The crucial point is that starting with λ-brackets and $(a,b) \mapsto a_{(-1)}b =: ab :$, with suitable constraints, one can recover all the n^{th}-products.

Example: *Vir*

The Virasoro algebra is defined as $Vir = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}L_i \oplus \mathbb{C}C$ with Lie algebra structure

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{(m^3-m)}{12}C, \qquad C \text{ central}$$

From a v.a. point of view, one considers the formal distribution

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

with OPE expansion

$$L(z)L(w) \sim \frac{\partial L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{C/2}{(z-w)^4}$$

Alternatively, the Virasoro vertex algebra can be viewed as the universal enveloping vertex algebra of the Lie conformal algebra $\mathbb{C}[\partial]L \oplus \mathbb{C}C$ with λ-bracket

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{\lambda^3}{12}C$$

Vertex operator algebras

Definition

A vertex operator algebra V is a vertex algebra endowed with a distinguished vector $L \in V$, called a Virasoro vector, satisfying the following conditions: if $Y(L,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c\,I,$$

$$L_{-1} = T,$$

L_0 is diagonalizable and its eigenspace decomposition has the form

$$V = \bigoplus_{n\in\mathbb{Z}_+} V_n,$$

where

$$\dim V_n < \infty \text{ for all } n \text{ and } V_0 = \mathbb{C}\mathbf{1}.$$

Universal minimal W-algebras: structure

Recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x$.

Theorem (KW)

(a) The vertex algebra $W^k_{\min}(\mathfrak{g})$ is strongly and freely generated by elements $J^{\{a\}}$, where a runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where v runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector ω.

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Rough explanation: a vector space basis $W^k_{\min}(\mathfrak{g})$ is given by monomials in (iterated) normal orders of the above generators and their derivatives.

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(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and $3/2$, respectively, with respect to ω.

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(b) The elements $J^{\{a\}}$*,* $G^{\{v\}}$ *are primary of conformal weight* 1 *and* $3/2$*, respectively, with respect to* ω*.*
This means

$$[\omega_\lambda J^{\{a\}}] = (L_{-1}+\lambda)J^{\{a\}}, \quad [\omega_\lambda G^{\{v\}}] = (L_{-1}+\tfrac{3}{2}\lambda)G^{\{v\}}$$

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(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and $3/2$, respectively, with respect to ω.
(c) The following λ-brackets hold:

$$[J^{\{a\}}{}_\lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda\left((k+h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right),\ a,b\in\mathfrak{g}^\natural,$$

where κ_0 is the Killing form of $\mathfrak{g}_0$, and

$$[J^{\{a\}}{}_\lambda G^{\{u\}}] = G^{\{[a,u]\}},\ a\in\mathfrak{g}^\natural,\ u\in\mathfrak{g}_{-1/2}.$$

(d) There are explicit formulas yielding $[G^{\{u\}}{}_\lambda G^{\{v\}}]$ for $u,v\in\mathfrak{g}_{-1/2}$.

Presentation of $W^k_{\min}(\mathfrak{g})$

Proposition (AKMPP, J-alg)

Let $u, v \in \mathfrak{g}_{-1/2}$. Then

$$G^{\{u\}}{}_{(2)}G^{\{v\}} = 4(e_\theta|[u,v])p(k)\mathbf{1}.$$

Moreover, the linear polynomial $k_i(k)$, $i \in I$, defined by $k_i(k) = k + \frac{1}{2}(h^\vee - h^\vee_{0,i})$, divides $p(k)$ and

$$G^{\{u\}}{}_{(1)}G^{\{v\}} = 4\sum_{i\in I}\frac{p(k)}{k_i(k)}J^{\{([[e_\theta,u],v])^\natural_i\}}$$

where $(a)^\natural_i$ denotes the orthogonal projection of $a \in \mathfrak{g}_0$ onto $\mathfrak{g}^\natural_i$ and $p(k)$ is an explicit monic quadratic polynomial.

Collapsing levels

Definition

We say that a level k is *collapsing* if

$$W_k^{\min}(\mathfrak{g}) = V_{k'}(\mathfrak{g}^\natural).$$

Here $V_s(\mathfrak{a})$ denotes the simple affine vertex algebra at level s attached to the Lie superalgebra $\mathfrak{a}$, i.e. the irreducible module $L(s\Lambda_0)$, where Λ_0 is the 0-th fundamental weight of the affine Lie superalgebra $\widehat{\mathfrak{a}}$.

Theorem (AKMPP)

Let $\mathfrak{g}^\natural = \bigoplus_{i\in I} \mathfrak{g}_i^\natural$. Then k is a collapsing level, if and only if $p(k) = 0$, and

$$W_k^{\min}(\mathfrak{g}) = \bigotimes_{i\in I : k_i \neq 0} V_{k_i}(\mathfrak{g}_i^\natural). \tag{4.1}$$

Unitarity: basic definitions

Let $W = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}_+} W(n)$, be a conformal vertex algebra. Set

$$(-1)^{L_0}a = e^{\pi\sqrt{-1}\Delta_a}a, \quad \sigma^{1/2}(a) = e^{\frac{\pi}{2}\sqrt{-1}p(a)}a. \tag{5.1}$$

If ϕ is a conjugate linear involution of W, set

$$g = ((-1)^{L_0}\sigma^{1/2})^{-1}\phi. \tag{5.2}$$

Definition

Assume that $\phi(L) = L$. Let g be as in (5.2). A Hermitian form $(\cdot\,,\cdot)$ on W is said to be ϕ*–invariant* if, for all $a \in W$,

$$(v, Y(a,z)u) = (Y(e^{zL_1}z^{-2L_0}ga, z^{-1})v, u), \quad u, v \in W. \tag{5.3}$$

Unitarity

$$(v, Y(a,z)u) = (Y(\underbrace{e^{zL_1} z^{-2L_0} g}_{A(z)} a, z^{-1})v, u)$$

- The operator $A(z) = e^{zL_1} z^{-2L_0} g$ with $g = (-1)^{L_0}$ was introduced by Borcherds and used in Frenkel-Lepowsky-Meurman for the construction of the dual to the V–modules.
  $g = (-1)^{L_0}\phi$ was used in Dong-Lin to define unitary structures on VOAs and this notion was generalized by Ai-Lin to vertex algebras with $\frac{1}{2}\mathbb{Z}_+$–grading compatible with parity, in which case $g = (-1)^{L_0 + 2L_0^2}\phi$. The most general definition is given in [KMP-CCM].
- The existence conditions for a ϕ–invariant Hermitian form on a vertex operator algebra W are discussed in [KMP-CCM]. For a conformal vertex algebra these conditions reduce to

$$L_1 W_1 = \{0\}.$$

Representations

W-algebras have a highest weight module theory. Let $L^W(\nu, \ell_0)$, the simple highest weight $W^k_{\min}(\mathfrak{g})$–module with highest weight (ν, ℓ_0), where

- ν is a (real) weight of $\mathfrak{g}^\natural$;
- $\ell_0 \in \mathbb{R}$ is the minimal eigenvalue of L_0.

Fact

We prove that $L^W(\nu, \ell_0)$ admits a ϕ–invariant nondegenerate Hermitian form (unique up to normalization).

Conditions for unitarity: rough statement

Basic remark

Unitarity of $L^W(\nu, \ell_0)$ implies that

- the levels $M_i(k)$ of the affine Lie algebras $\widehat{\mathfrak{g}}_i^\natural$ in $W^k_{\min}(\mathfrak{g})$ (they are explicit linear polynomials in k, related to $p(k)$), where $\mathfrak{g}_i^\natural$ are the simple components of $\mathfrak{g}^\natural$, are non-negative integers;
- ν is a dominant integral weight of level $M_i(k)$;
- a certain inequality $(*)$ holds. Moreover, $(*)$ must be an equality when ν is an "extremal" weight.

The key point to prove the above statements is an explicit norm computation:

$$\|G^{\{v\}}_{-1/2} v_{\nu,\ell_0}\|^2 = (-2(k+h^\vee)\ell_0 + (\nu|\nu+2\rho^\natural) - 2(k+1)(\xi|\nu) + 2(\xi|\nu)^2)\langle\phi(v), v\rangle.$$

Conditions for unitarity: rough statement

Next steps

1. Using the generalization of the *Fairlie construction*, we prove a partial converse result: if $M_i(k) + \chi_i \in \mathbb{Z}_+$, where χ_i are negative integers and ν is a dominant integral weight for $\mathfrak{g}^\natural$ which is not extremal, then the irreducible $W^k_{\min}(\mathfrak{g})$-module $L^W(\nu, \ell_0)$ is unitary for $\ell_0 >> 0$.
2. We prove that actually the necessary conditions are sufficient for non extremal weights, as follows. Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$. We introduce a highest weight module $\overline{M}(\widehat{\nu}_h)$ over $\widehat{\mathfrak{g}}$ $(h \in \mathbb{C})$, such that
   1. $\overline{M}(\widehat{\nu}_h)$ is irreducible, except possibly for an explicit set J of values of h.
   2. If H_0 denotes the quantum Hamiltonian reduction functor, the $W^k_{\min}(\mathfrak{g})$-module $H_0(\overline{M}(\widehat{\nu}_h))$ admits a Hermitian form, depending polynomially on h.

   By a theorem of Arakawa, $H_0(\overline{M}(\widehat{\nu}_h)) = L^W(\nu, \ell(h))$ for $h \notin J$. Then, miraculously, if $h \in J$, $\ell(h)$ does not satisfy $(*)$. Moreover $L^W(\nu, \ell_0)$ is unitary for $\ell_0 >> 0$. By continuity, the determinant of the Hermitian form on $L^W(\nu, \ell_0)$ is positive if $(*)$ holds.

Main Result: preparation

• If $\mathfrak{g} = sl(2|m)$ with $m \geq 3$ or $osp(4|m)$ with $m \geq 2$ even, then none of the $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$ are unitary for a non-collapsing level k.

Main Result: preparation

• If $\mathfrak{g} = sl(2|m)$ with $m \geq 3$ or $osp(4|m)$ with $m \geq 2$ even, then none of the $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$ are unitary for a non-collapsing level k. For the remaining $\mathfrak{g}$ the Lie algebra $\mathfrak{g}^\natural$ is semisimple (actually simple, except for $\mathfrak{g} = D(2,1;a)$, when $\mathfrak{g}^\natural = sl_2 \oplus sl_2$).

Main Result: preparation

Notation

Recall that

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

- Let $\theta_i^\vee$ be the coroots of the maximal roots θ_i of the simple components $\mathfrak{g}_i^\natural$ of $\mathfrak{g}^\natural = \bigoplus_i \mathfrak{g}_i^\natural$.
- Let $2\rho^\natural$ be the sum of positive roots of $\mathfrak{g}^\natural$,
- Let ξ be a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$ (this module is irreducible, except for $\mathfrak{g} = psl(2|2)$ when it is $\mathbb{C}^2 \oplus \mathbb{C}^2$).
- Let ν be a dominant integral weight for $\mathfrak{g}^\natural$ and $\ell_0 \in \mathbb{R}$.

Main Result

Theorem

Let $L^W(\nu, \ell_0)$ be a simple highest weight $W^k_{\min}(\mathfrak{g})$-module over $\mathfrak{g} = psl(2|2), spo(2|m)$ with $m \geq 3$, $D(2,1;a), F(4)$ or $G(3)$.

1. *This module can be unitary only if the following conditions hold:*
   1. *$M_i(k)$ are non-negative integers,*
   2. *$\nu(\theta_i^\vee) \leq M_i(k)$ for all i,*
   3. $$l_0 \geq \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu) - k - 1) =: A(k,\nu), \qquad (*)$$

      and equality holds in (*) if $\nu(\theta_i^\vee) > M_i(k) + \chi_i$ for $i = 1$ or 2.
2. *This module is unitary if the following conditions hold:*
   1. *$M_i(k) + \chi_i \in \mathbb{Z}_+$ for all i,*
   2. *$\nu(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all i (i.e. ν is not extremal),*
   3. *inequality (*) holds.*

Conjecture

Conjecture *The modules $L^W(\nu, \ell_0)$ are unitary if ν is extremal and $l_0 = R.H.S.$ of $(*)$. In other words, the necessary conditions of the above Theorem are sufficient.*

We are able to prove this conjecture only for $\mathfrak{g} = psl(2|2)$ and $spo(2|3)$, obtaining thereby a complete classification of unitary simple highest weight $W^k_{\min}(\mathfrak{g})$-modules in these two cases.

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We are able to prove this conjecture only for $\mathfrak{g} = psl(2|2)$ and $spo(2|3)$, obtaining thereby a complete classification of unitary simple highest weight $W^k_{\min}(\mathfrak{g})$-modules in these two cases.
The proof is a nice application of the theory of collapsing levels, and uses the fact that these superconformal algebras are linearizable, i.e. up to adding some extra fields the modes of the fields form a Lie superalgebra, hence we can tensor its modules.

Unitary minimal W-algebras

Since $\nu = 0$ is extremal iff k is collapsing, we obtain the following complete classification of minimal simple unitary W-algebras:

Theorem

The simple minimal W-algebra $W^{\min}_{-k}(\mathfrak{g})$ with $k \neq h^\vee$ and $\mathfrak{g}_0$ non-abelian is non-trivial unitary if and only if

1. $\mathfrak{g} = sl(2|m)$, $m \geq 3$, $k = 1$ *(in this case the W-algebra is a free boson);*
2. $\mathfrak{g} = psl(2|2)$, $k \in \mathbb{N} + 1$;
3. $\mathfrak{g} = spo(2|3)$, $k \in \frac{1}{4}(\mathbb{N} + 2)$;
4. $\mathfrak{g} = spo(2|m)$, $m > 4$, $k \in \frac{1}{2}(\mathbb{N} + 1)$;
5. $\mathfrak{g} = D(2, 1; \frac{m}{n})$, $k \in \frac{mn}{m+n}\mathbb{N}$, *where $m, n \in \mathbb{N}$ are coprime,* $k \neq \frac{1}{2}$;
6. $\mathfrak{g} = F(4)$, $k \in \frac{2}{3}(\mathbb{N} + 1)$;
7. $\mathfrak{g} = G(3)$, $k \in \frac{3}{4}(\mathbb{N} + 1)$.

Character formulas

We provide character formulas for all unitary $W^k_{\min}(\mathfrak{g})$-modules $L^W(\nu, \ell_0)$, which are obtained by applying the quantum Hamiltonian reduction to the corresponding irreducible highest weight modules over the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. There are two cases.

1. *Massive (or typical)*: the inequality $(*)$ is strict, and the character formula is easy to prove.
2. *Massless (or atypical)*, when the inequality $(*)$ is equality. There is a general KW-formula for maximally atypical tame integrable $\widehat{\mathfrak{g}}$-modules, conjectured in [KW] and proved in [GK2] in many, but not all, cases. We get correspondigly a character formula for all massless $L^W(\nu, \ell_0)$, with some exceptions.

Character formulas were also given by Eguchi-Taormina and (resp. Miki) for the $N = 4$ superconformal algebra (resp. for $W^k_{\min}(spo(2|3))$, hence for the $N = 3$ superconformal algebra) in non-rigorous fashion. Their formulas for both massive and massless representations coincide with ours.

Character formulas

Theorem

Let k be in the unitary range and let $\nu \in P_k^+$. Let $L^W(\nu, \ell_0)$ be a unitary irreducible $W^k_{\min}(\mathfrak{g})$–module. Choose h so that $\ell(h) = \ell_0$ and set $\widehat{\nu}_h = k\Lambda_0 + \nu + h\theta$.
(i) If $\ell_0 > A(k, \nu)$, then

$$chL^W(\nu, \ell_0) = \sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_h). \tag{6.1}$$

(ii) If $\ell_0 = A(k, \nu)$, and $\nu = 0$ if $\mathfrak{g} \neq D(2, 1; \frac{m}{n})$, then

$$chL^W(\nu, \ell_0) = \sum_{w \in \widehat{W}^\natural} \sum_{\gamma \in \mathbb{Z}_+ \Pi_{\bar{1}}} (-1)^\gamma det(w) ch\, M^W(w.(\widehat{\nu}_h - \gamma)), \tag{6.2}$$

where $\Pi_{\bar{1}} = \{\gamma_1, \gamma_2, \ldots\}$ is the set of isotropic simple roots for $\mathfrak{g}$, and for $\gamma = n_1\gamma_1 + \cdots$, we write $(-1)^\gamma = (-1)^{n_1 + \cdots}$.

Very recent developments (joint also with D. Adamovic)

Theorem

Let k be in the unitary range. Then all irreducible highest weight $W^k_{\min}(\mathfrak{g})$–modules $L^W(\nu, \ell_0)$ with $\ell_0 \in \mathbb{C}$ when $\nu \in P^+_k$ is not extremal, and $\ell_0 = A(k, \nu)$ otherwise, descend to $W_k^{\min}(\mathfrak{g})$.

Corollary

Any unitary $W^k_{\min}(\mathfrak{g})$–module $L^W(\nu, \ell_0)$ descends to $W_k^{\min}(\mathfrak{g})$.

Theorem

The modules appearing in the above theorem afford the complete list of inequivalent irreducible representations of $W_k^{\min}(\mathfrak{g})$ in the unitary range.

Some details on Main Theorem

Main Theorem

Let $k \neq -h^\vee$. If k lies in the unitary range, then the $W^k_{\min}(\mathfrak{g})$–module $L^W(\nu, \ell_0)$ is unitary for all non extremal $\nu \in \widehat{P}^+_k$ and $\ell_0 \geq A(k, \nu)$.

Some details on Main Theorem

For $\nu \in P_k^+ = \{\nu \in P^+ \mid \nu(\theta_i^\vee) \leq M_i(k) \text{ for all } i \geq 1\}$ and $h \in \mathbb{C}$, set

$$\widehat{\nu}_h = k\Lambda_0 + \nu + h\theta \in \widehat{\mathfrak{h}}^*.$$

Let $\widehat{\mathfrak{p}}$ be the parabolic subalgebra of $\widehat{\mathfrak{g}}$ with Levi factor $\widehat{\mathfrak{h}} + \mathfrak{g}^\natural$ and the nilradical $\widehat{\mathfrak{u}}_+ = \sum_{\alpha \in \widehat{\Delta}^+ \setminus \Delta^\natural} \widehat{\mathfrak{g}}_\alpha$. Set $\widehat{\mathfrak{u}}_- = \sum_{\alpha \in \widehat{\Delta}^+ \setminus \Delta^\natural} \widehat{\mathfrak{g}}_{-\alpha}$. Let $V^\natural(\nu)$ denote the irreducible $\mathfrak{g}^\natural$–module with highest weight ν and extend the $\mathfrak{g}^\natural$ action to $\widehat{\mathfrak{p}}$ by letting $\widehat{\mathfrak{u}}_+$ act trivially; x, K, and d act by h, k, and 0 respectively. Let $M^\natural(\widehat{\nu}_h)$ be the corresponding generalized Verma module for $\widehat{\mathfrak{g}}$, i.e.

$$M^\natural(\widehat{\nu}_h) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} V^\natural(\nu).$$

Some details on Main Theorem

Lemma

For $\nu \in P_k^+$ not extremal, we have

$$N_i \equiv N_i(k,\nu) = M_i(k) + \chi_i + 1 - (\nu|\theta_i^\vee) \in \mathbb{N}.$$

Moreover, for

$$v_i(h) := x_{-\eta_i}^{N_i} x_{-\alpha_0-\alpha_1} x_{-\alpha_1} v_{\widehat{\nu}_h},$$

the subspace $\sum_i U(\widehat{\mathfrak{g}})v_i(h)$ is a proper submodule of the $\widehat{\mathfrak{g}}$-module $M^\natural(\widehat{\nu}_h)$.

Set

$$\overline{M}(\widehat{\nu}_h) = M^\natural(\widehat{\nu}_h)/(\sum_i U(\widehat{\mathfrak{g}})v_i(h)).$$

More details on Main Theorem

Proposition

Assume that $\nu \in P_k^+$ is not extremal and that

$$(\widehat{\nu}_h + \widehat{\rho}|\alpha) \neq \frac{n}{2}(\alpha|\alpha) \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in \widehat{\Delta}^+ \setminus \widehat{\Delta}^+(\mathfrak{g}^\natural).$$

Then

(i) the module $\overline{M}(\widehat{\nu}_h)$ is irreducible;

(ii) its character is

$$ch\overline{M}(\widehat{\nu}_h) = \sum_{w \in \widehat{W}^\natural} \det(w) chM(w.\widehat{\nu}_h).$$

Some details on Main Theorem

Let $H_0 : \{\text{Category } \mathcal{O}_k \text{ for } \widehat{\mathfrak{g}}\} \to \{W^k_{\min}(\mathfrak{g})\text{-modules}\}$ be the QHR functor

Theorem

If $\ell(h) > A(k,\nu)$, then $H_0(\overline{M}(\widehat{\nu}_h))$ is an irreducible $W^k_{\min}(\mathfrak{g})$-module and its character is

$$ch\, H_0(\overline{M}(\widehat{\nu}_h)) = \sum_{w \in \widehat{W}^\natural} det(w) ch\, M^W(w.\widehat{\nu}_h).$$

where M^W is a Verma module for $W^k_{\min}(\mathfrak{g})$.

Some details on Main Theorem

Consider the free field realization

$$\Psi : W^k_{\min}(\mathfrak{g}) \to \mathcal{V}^k = V^1(\mathbb{C}a) \otimes V^{\alpha_k}(\mathfrak{g}^\natural) \otimes F(\mathfrak{g}_{1/2}).$$

If $\nu \in P^+_k$ is not extremal, let $L^\natural(\nu)$ the integrable $V^{\alpha_k}(\mathfrak{g}^\natural)$–module of h.w. ν with h.w. vector v_ν. Then

$$M(y) \otimes L^\natural(\nu) \otimes F(\mathfrak{g}_{1/2})$$

is a $\mathcal{V}^k$–module, hence, by means of Ψ, a $W^k_{\min}(\mathfrak{g})$–module. Set

$$N(y,\nu) = \Psi(W^k_{\min}(\mathfrak{g})) \cdot (1 \otimes \mathbb{C}[y] \otimes v_\nu \otimes \mathbf{1}) \subset M(y) \otimes L^\natural(\nu) \otimes F(\mathfrak{g}_{1/2}).$$

Since $M(y) \otimes L^\natural(\nu) \otimes F(\mathfrak{g}_{1/2})$ is free as a $\mathbb{C}[y]$–module, $N(y,\nu)$ is also free. If $\mu \in \mathbb{C}$, set also

$$N(\mu,\nu) = (\mathbb{C}[y]/(y-\mu)) \otimes_{\mathbb{C}[y]} N(y,\nu).$$

By construction $N(\mu,\nu)$ is clearly a highest weight module for $W^k_{\min}(\mathfrak{g})$, whose highest weight is (ν, ℓ_0) where ℓ_0 is given by an explicit formula.

Some details on Main Theorem

Since we are looking for unitary representations, we assume that $\ell_0 \in \mathbb{R}$.

Lemma

If $\ell_0 > A(k,\nu)$ then $N(\mu,\nu)$ is an irreducible $W^k_{\min}(\mathfrak{g})$–module.

Proof of the Main Theorem.

We may assume that the level is not collapsing, so that $M_i(k) + \chi_i \in \mathbb{Z}_+$. Then the Hermitian form on $L^W(\nu,\ell_0)$ is positive definite for $\ell_0 >> 0$. By the above theorem, we have $N(\mu,\nu) = L^W(\nu,\ell_0)$ if $\ell_0 = \frac{1}{2}\mu^2 - s_k\mu + \frac{(\nu|\nu+2\rho^\natural)}{2(k+h^\vee)} > A(k,\nu)$, hence $det_{\widehat{\zeta}}(\ell_0) \neq 0$ for all weights $\widehat{\zeta}$ of $N(\mu,\nu)$. It follows that the Hermitian form is positive definite for $\ell_0 > A(k,\nu)$, hence positive semidefinite for $\ell_0 = A(k,\nu)$. □