Symmetry superalgebras in parabolic supergeometries

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Lie Theory: frontiers, algorithms, and applications
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Based on joint works with B. Kruglikov & D. The

Plan of the talk:

- Prelude on Lie superalgebras and supermanifolds
- Prolongations and symmetry bounds
- Realizations of G_2 as symmetry algebra
- The Lie superalgebra $G(3)$: parabolic subalgebras & Spencer cohomology
- Realizations of $G(3)$ as supersymmetry of geometric structures
- Latest developments: the mixed contact and the odd-contact $F(4)$ results

Prelude: Lie superalgebras

Def. A *Lie superalgebra* is a complex vector space of the form

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that

- $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{g}_{\bar{0}}$, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{1}}$, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subset \mathfrak{g}_{\bar{0}}$;
- for any homogeneous X, Y (i.e. with $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$)

$$[X,Y] = -(-1)^{|X||Y|}[Y,X] \qquad \left(|X| = \text{ parity of } X = \begin{cases} 0 \\ 1 \end{cases} \right)$$

- for any homogeneous X, Y, Z

$$(-1)^{|Z||X|}[X,[Y,Z]] + (-1)^{|Y||Z|}[Z,[X,Y]] + (-1)^{|X||Y|}[Y,[Z,X]] = 0$$

Simple Lie superalgebras

Finite-dimensional simple complex Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ were classified by V. Kac in 1977 and split into two families:

- *classical*, for which the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible;
- *Cartan Lie superalgebras*, analogs to simple Lie algebras of vector fields.

Classical LSA include in turn the LSA with a non-degenerate "Killing form":

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$
$\mathfrak{sl}(m\|n)$ $m, n \geqslant 1$	$\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathbb{C}$	$\left(\mathbb{C}^m \otimes (\mathbb{C}^n)^*\right) \oplus \left((\mathbb{C}^m)^* \otimes \mathbb{C}^n\right)$
$\mathfrak{osp}(m\|2n)$ $m, n \geqslant 1$	$\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$	$\mathbb{C}^m \otimes \mathbb{C}^{2n}$
$\mathfrak{osp}(4\|2;\alpha)$ $\alpha \neq 0, \pm 1, \infty$	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$F(4)$	$\mathfrak{so}(7) \oplus \mathfrak{sl}(2)$	$\mathbb{S} \otimes \mathbb{C}^2$
$G(3)$	$G_2 \oplus \mathfrak{sl}(2)$	$\mathbb{C}^7 \otimes \mathbb{C}^2$

Rem. The smallest representation of *exceptional LSA* is the adjoint representation.

Prelude: Supermanifolds

Def. A *supermnfd* of dimension $(m|n)$ is a pair $M = (M_o, \mathcal{A}_M)$, where

- M_o is an m-dimensional mnfd,
- $\mathcal{A}_M$ is a sheaf of superalgebras on M_o such that its sections admit a local analytic expansion in the "odd anticommuting coordinates":

$$f = f_0(x) + f_{\alpha_1}(x)\theta^{\alpha_1} + \cdots + f_{\alpha_1 \ldots \alpha_n}(x)\theta^{\alpha_1} \wedge \cdots \wedge \theta^{\alpha_n} \ .$$

Def. A *superdistribution* on $M = (M_o, \mathcal{A}_M)$ is a graded $\mathcal{A}_M$-subsheaf $\mathcal{D}$ of the tangent sheaf $\mathcal{T}M = \mathrm{Der}(\mathcal{A}_M)$ that is locally a direct factor.

Example. $M = \mathbb{C}^{5|2}$ with even coordinates x, u, p, q, z and odd coordinates θ, ν. The subsheaf generated by the supervector fields

$$D_x = \partial_x + p\partial_u + q\partial_p + q^2\partial_z \ , \quad \partial_q \ ,$$
$$D_\theta = \partial_\theta + q\partial_\nu + \theta\partial_p + 2\nu\partial_z \ ,$$

is a superdistribution of rank $(2|1)$.

Geometric structures

Filtered geometric structures, in particular G-structures, are defined through reductions of frame bundles. Tensors, connections, PDEs are such structures.

Example I. The *supermetric*

$$g = \frac{1}{\left(1 + k\|x\|^2\right)^2} \sum_{i=1}^{m} dx_i^2 + \sum_{\alpha=1}^{n} d\theta^\alpha d\theta^{\alpha+n}$$

on $M = \mathbb{R}^{m|2n}$ is an $\mathrm{OSp}(m|2n)$-structure and it has symmetry superalgebra

$$\mathfrak{aut}(M,g) = \begin{cases} \mathfrak{osp}(m+1|2n) & \text{if } k > 0\,, \\ \mathfrak{osp}(m|2n) \ltimes \mathbb{R}^{m|2n} & \text{if } k = 0\,, \\ \mathfrak{osp}(m,1|2n) & \text{if } k < 0\,. \end{cases}$$

Example II. The *supersymplectic form*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dx^{i+n} + \sum_{\alpha=1}^{m} d\theta^\alpha \wedge d\theta^\alpha$$

on $M = \mathbb{R}^{2n|m}$ is an $\mathrm{SpO}(2n|m)$-structure and it has an infinite-dimensional symmetry superalgebra $\mathfrak{aut}(M,\omega)$, unless M is purely odd.

Filtered geometric structures

The *weak derived flag* of a superdistribution $\mathcal{D} \subset \mathcal{T}M$ is defined as the filtration $\mathcal{D}^{-1} = \mathcal{D} \subset \mathcal{D}^{-2} \subset \cdots \subset \mathcal{D}^{-k} = \mathcal{D}^{-k+1} + [\mathcal{D}, \mathcal{D}^{-k+1}] \subset \cdots$, and then setting $\mathrm{gr}(\mathcal{T}M)_{-k} = \mathcal{D}^{-k}/\mathcal{D}^{-k+1}$ for all $k > 0$, we arrive at a sheaf

$$\mathrm{gr}(\mathcal{T}M) = \bigoplus_{k<0} \mathrm{gr}(\mathcal{T}M)_k$$

of $\mathcal{A}_M$-modules and LSA over M_o. The distribution is called *strongly regular* if $\exists$ LSA $\mathfrak{m} = \oplus_{-\mu \leqslant k < 0} \mathfrak{m}_k$ such that $\mathrm{gr}(\mathcal{T}M) \cong \mathcal{A}_M \otimes \mathfrak{m}$ locally.

Def. The *Tanaka–Weisfeiler prolongation* $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ of $\mathfrak{m}$ is maximal graded LSA s.t.:

(i) $\mathrm{pr}_{<0}(\mathfrak{m}) = \mathfrak{m}$;

(ii) $[X, \mathfrak{m}_{-1}] = 0$ for $X \in \mathrm{pr}_{\geqslant 0}(\mathfrak{m})$ implies $X = 0$.

There is a version $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ of the prolongation if $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{m})$ is also assigned, and similarly for higher-order reductions.

Tanaka–Weisfeiler prolongation and Spencer cohomology

Rem I. Although $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ can be obtained via an iterative process, one can test a candidate $\mathfrak{g}$ that extends $\mathfrak{m} \oplus \mathfrak{g}_0$ via the criteria:

- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ if and only if $H^1_{\geqslant 0}(\mathfrak{m}, \mathfrak{g}) = 0$;
- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ if and only if $H^1_{+}(\mathfrak{m}, \mathfrak{g}) = 0$.

Rem II. Kostant's version of BBW Thm efficiently computes these groups in the classical setting but in super-setting his "harmonic cohomology" is usually bigger.

Geometric prolongations

Thm[B. Kruglikov, A.S., D. The, 2022 Transform. Groups]
Let $(M, \mathcal{D})$ be strongly regular superdistribution, with symmetry superalgebra $\mathfrak{aut}(M, \mathcal{D})$, and assume that the prolongation $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ is finite-dimensional. Then:

1. $\exists$ fiber bundle $\pi : P \to M$ of $\dim P = \dim \mathfrak{g}$ with *absolute parallelism* Φ s.t. any equivalence of $(M, \mathcal{D})$ lifts to an equivalence of Φ,
2. $\dim \mathfrak{aut}(M, \mathcal{D}) \leqslant \dim(\mathfrak{g})$ in the strong sense (inequality applies to both even and odd dimensions), and $\mathrm{Aut}(M, \mathcal{D})$ is a Lie supergroup,
3. If $\dim(\mathfrak{aut}(M, \mathcal{D})) = \dim(\mathfrak{g})$ then

$$(M, \mathcal{D}) \underset{\substack{\text{locally,} \\ \text{up to deform.}}}{\cong} (G/G_{\geqslant 0}, \mathfrak{g}_{\geqslant -1}) \ ,$$

where $G_{\geqslant 0} \subset G$ is the closed Lie subsupergroup with $Lie(G_{\geqslant 0}) = \mathfrak{g}_{\geqslant 0}$.

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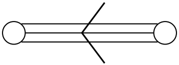
Idea: we construct a tower of bundles $M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \cdots$ consisting of partial frames (modification of Cartan–Tanaka method).

Some geometric realizations of G_2

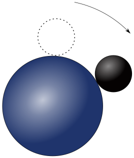


This is an abstract description via Dynkin diagrams. What about *realizations as symmetries*?

– $GL_7(\mathbb{C})$ acts with open orbit on 3-forms on $\mathbb{C}^7$ and $G_2 = \mathrm{Stab}_{GL_7(\mathbb{C})}(\phi)$ for generic $\phi \in \bigwedge^3(\mathbb{C}^7)^*$ (Engel, 1900);

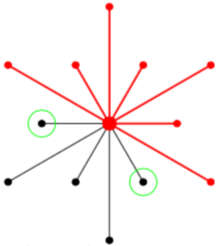
– Compact form $G_2 = \mathrm{Aut}(\mathbb{O})$ (Cartan, 1914);

– Configuration space M of a 2-sphere rolling on another w/o twisting or slipping is 5-dimensional, with the constraints given by a rank 2 distribution $\mathcal{D} \subset \mathcal{T}M$ of filtered growth $(2, 3, 5)$. If the ratio of the radii of spheres is 3, then split $G_2 = \mathrm{Aut}(M, \mathcal{D})$ (Bryant, Zelenko, Bor–Montgomery, Baez–Huerta).



$(2,3,5)$-geometry from the G_2 root diagram

G_2/P_1



Fundamental invariant of $(2,3,5)$-distributions: *binary quartic field* (Cartan 1910). Modern perspective: the quartic arises from $H^{4,2}(\mathfrak{m},\mathfrak{g}) \cong S^4(\mathbb{C}^2)$, where $\mathfrak{g} = G_2$ has $|3|$-grading $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ with negative part

$$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5 \rangle$$

$$[e_1, e_2] = e_3 \ , \qquad [e_1, e_3] = e_4 \ , \qquad [e_2, e_3] = e_5 \ ,$$

and 0-degree component $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m}) \cong \mathfrak{gl}(2)$.

Some geometric realizations of G_2

– Engel (1893): G_2 as symmetry of contact distribution $\mathcal{C}$ on 5-dim. mnfd with field of twisted cubics $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$;

– Cartan (1893, 1910): G_2 as symmetry of

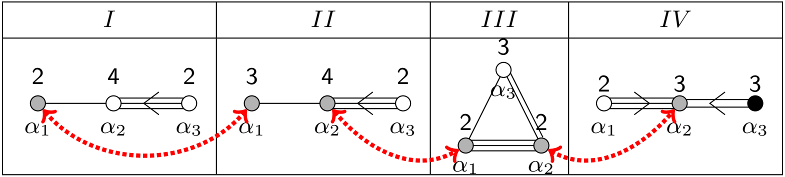
Dim	Geometric structure	Model
5	ODE with flat $(2,3,5)$-distribution	$du - u' dx$, $du' - u'' dx$, $dz - (u'')^2 dx$, *Hilbert–Cartan equation* $z' = (u'')^2$
6	Pair of PDE (with flat contact distribution)	$u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$

Today: realizations of the Lie superalgebra $G(3) = \big(G_2 \oplus \mathfrak{sp}(2)\big) \oplus \big(\mathbb{C}^7 \otimes \mathbb{C}^2\big)$.

Simple root systems of $\mathfrak{g} = G(3)$

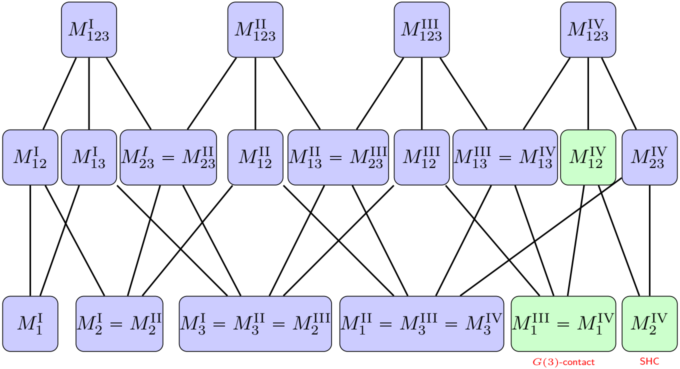
Fix Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}} = G_2 \oplus \mathfrak{sp}(2) \rightsquigarrow$ root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$. The Killing form of $G(3)$ is non-degenerate and induces non-degenerate $\langle -, - \rangle$ on $\mathfrak{h}^*$. If $\alpha \in \Delta_{\bar{0}}$ then $\langle \alpha, \alpha \rangle \neq 0$ and the *even reflection* $S_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ preserves $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$. The Weyl group $W = \langle S_\alpha \mid \alpha \in \Delta_{\bar{0}} \rangle$ is generated by even reflections. Up to W-equivalence, there are four inequivalent simple systems $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$. They are related by *odd reflections* along isotropic $\alpha \in \Pi_{\bar{1}}$ as indicated below:

$$S_\alpha(\beta) = \begin{cases} \beta + \alpha, & \langle \alpha, \beta \rangle \neq 0; \\ \beta, & \langle \alpha, \beta \rangle = 0, \ \beta \neq \alpha; \\ -\alpha, & \beta = \alpha. \end{cases}$$

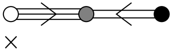
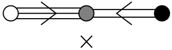


Roots: even ○, odd isotropic ● (grey) or odd nonisotropic ●.

Map of $G(3)$-supergeometries



We considered two $\mathbb{Z}$-gradings of $\mathfrak{g} = G(3)$ with (graded) growths:

	marked Dynkin diagram	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$	$\dim(\mathfrak{g}_{-3})$
$G(3)$-contact		$4\|4$	$1\|0$	
SHC		$2\|4$	$1\|2$	$2\|0$

Geometric structures associated to M_1^{IV} and M_2^{IV}

G(3)-contact super-PDE:

$$u_{xx} = \frac{1}{3}(u_{yy})^3 + 2u_{yy}u_{y\nu}u_{y\tau}, \quad u_{xy} = \frac{1}{2}(u_{yy})^2 + u_{y\nu}u_{y\tau},$$
$$u_{x\nu} = u_{yy}u_{y\nu}, \quad u_{x\tau} = u_{yy}u_{y\tau}, \quad u_{\nu\tau} = -u_{yy}.$$

where $u = u(x, y, \nu, \tau) : \mathbb{C}^{2|2} \to \mathbb{C}^{1|0}$.

Super Hilbert-Cartan equation (SHC):

$$z_x = \frac{(u_{xx})^2}{2} + u_{x\nu}u_{x\tau}, \quad z_\nu = u_{xx}u_{x\nu}, \quad z_\tau = u_{xx}u_{x\tau}, \quad u_{\nu\tau} = -u_{xx},$$

where $(u, z) = (u(x, \nu, \tau), z(x, \nu, \tau)) : \mathbb{C}^{1|2} \to \mathbb{C}^{2|0}$.

Thm[Kruglikov, S., The, 2021 Adv. Math.] These super-PDE have *symmetry superalgebra* $G(3)$. Unlike HC eqn, whose general solution depends on one function of one variable, solutions of SHC depend only on five constants.

Spencer cohomology of SHC grading

Thm[Kruglikov, S., The] Let $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ be the SHC grading of $\mathfrak{g} = G(3)$. Then $H^{d,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $d \geqslant 0$, so that $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{m})$. Moreover $H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{1}} = 0$ for all $d > 0$ while

$$H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{0}} \cong \begin{cases} 0 & \text{for all } d > 0, d \neq 2, \\ S^2\mathbb{C}^2 \boxtimes \Lambda^2\mathbb{C}^2 & \text{if } d = 2, \end{cases}$$

Rem I. As a $(\mathfrak{g}_0)_{\bar{0}}$-module, the space $C^{4,2}(\mathfrak{m}, \mathfrak{g})$ has a unique submodule $S^4\mathbb{C}^2 \boxtimes \mathbb{C}$, which is the space of Cartan's classical binary quartic invariants. Its elements are *not* closed in the complex $C^{\bullet}(\mathfrak{m}, \mathfrak{g})$.

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Rem II. This suggests the Cartan quartic of underlying generic rank 2 distribution on 5-dim. mnfd should admit a square root, hence it must be of Petrov type D (pair of double roots), N (quadruple root) or O (identically zero).

Finding models with desired symmetry

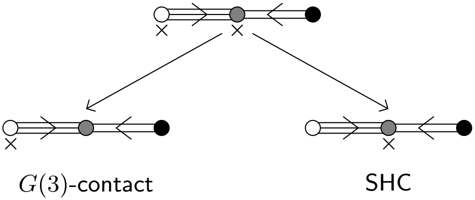
Two steps:

1. Find an explicit description of an *invariant geometric structure*. E.g. start with the $(2,3,5)$ symbol algebra and integrate structure eqns or use BCH to arrive at a local model (equivalent to the Hilbert–Cartan eqn).
   Rem. We obtained SHC eqn also in this way but it is too involved.

2. Prove that this homogeneous model has the *expected symmetry dimension*. Tanaka–Weisfeiler prolongation, via results on $H^1(\mathfrak{m}, \mathfrak{g})$, gives upper bound.
   Rem. In classical setting we have harmonic curvature as a test for flatness but this is unavailable in the super-setting.

$G(3)$-double fibration

We investigated the *$G(3)$-twistor correspondence*



Strategy: flag supermnfd $G(3)/P_1$ is contact supermnfd $(M, \mathcal{C})$ with the additional reduction of structure group $COSp(3|2) \subset CSpO(4|4)$, which we realize as $(1|2)$-twisted cubic $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$. Osculate $\mathcal{V}$ to get *PDE $\mathcal{E} \cong G(3)/P_{12}$*. Cartan superdistrib. of $\mathcal{E}$ has "Cauchy characteristic", we quotient by it to get *SHC eqn $\overline{\mathcal{E}} \cong G(3)/P_2$*.

$G(3)$-contact case

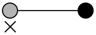
Idea: contact supermnfd + additional geometric structure.

k	$(\mathfrak{g}_k)_{\bar{0}}$	$(\mathfrak{g}_k)_{\bar{1}}$	dim
0	$\mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$	$S^2\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$7\|6$
-1	$S^3\mathbb{C}^2 \boxtimes \mathbb{C}$	$\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$4\|4$
-2	$\mathbb{C} \boxtimes \mathbb{C}$		$1\|0$

Prop. $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{osp}(3|2) \subset \mathbb{C} \oplus \mathfrak{spo}(4|4)$ is a *maximal subalgebra*.

The basis of $V := \mathfrak{g}_{-1}$ given by $\{x^3, x^2y, xy^2, y^3 | xe, xf, ye, yf\}$ allows to make explicit the invariant $CSpO$-structure on V. The topological point $[x^3] \in \mathbb{P}(V_{\bar{0}})$ has stabilizer $\mathfrak{q} \subset \mathfrak{f} := \mathfrak{osp}(3|2)$ that is a parabolic subalgebra:

$$\mathfrak{f} = \mathfrak{f}_{-1} \oplus \overbrace{\mathfrak{f}_0 \oplus \mathfrak{f}_1}^{\mathfrak{q}}$$



k	$(\mathfrak{f}_k)_{\bar{0}}$	$(\mathfrak{f}_k)_{\bar{1}}$
1	X_1	A_1, A_2
0	H_1, H_2, X_2, Y_2	A_3, A_4
-1	Y_1	A_5, A_6

The $(1|2)$-twisted cubic $\mathcal{V}$

Def. The $COSp(3|2)$-orbit $\mathcal{V} \subset \mathbb{P}(V)$ through $[x^3]$ is called $(1|2)$-*twisted cubic*.

We describe $\mathcal{V}$ locally by exponentiating the action of $\mathfrak{f}_{-1} = \mathrm{span}\{Y|A,B\} \cong \mathbb{C}^{1|2}$ through $[x^3]$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \overset{\exp(\lambda Y)}{\longmapsto} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \overset{\exp(\theta A)}{\longmapsto} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline \theta \\ 0 \\ 0 \\ -\theta\lambda \end{pmatrix} \overset{\exp(\phi B)}{\longmapsto} \begin{pmatrix} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} + \phi\theta\lambda \\ -\frac{\lambda^2}{2} + \phi\theta \\ \hline \theta \\ \phi \\ \phi\lambda \\ -\theta\lambda \end{pmatrix},$$

with λ even parameter and θ, ϕ odd. By maximality, this supervariety $\mathcal{V} \subset \mathbb{P}(V)$ characterizes the reduction of the structure group $COSp(3|2) \subset CSpO(4|4)$.

Osculations of $\mathcal{V}$

Repeatedly applying $\mathfrak{f}_{-1}$ to $[x^3]$ yields the so-called osculating sequence

$$0 \subset V^0 \subset V^1 \subset V^2 \subset V^3 = V$$

of higher order *affine tangent spaces* of $\mathcal{V}$ at $[x^3]$.

Important fact I: The affine tangent space $V^1 \subset V \cong \mathbb{C}^{4|4}$ is *Lagrangian* w.r.t. $CSpO$-structure on V (in particular $\dim V^1 = (2|2)$).

Important fact II: The associated graded v.s. $\mathrm{gr}(V) = N_0 \oplus \cdots \oplus N_3$ has natural $\mathfrak{osp}(1|2)$-equivariant $\mathbb{Z}$-graded superalgebra structure and $N_1 \otimes N_1 \to N_2 \cong N_1^*$ is a supersymmetric *cubic form* $\mathfrak{C} \in S^3 N_1^*$ on $N_1 \cong \mathbb{C}^{1|2}$. (It dualizes to the product of simple Jordan superalgebra structure on N_1 called the *Kaplansky superalgebra*.) Explicitly $\mathfrak{C} = \frac{1}{3}\lambda^3 + 2\lambda\theta\phi$.

General framework for 2nd order super-PDE

Global	Local
Contact supermfld $(M^{5\|4}, \mathcal{C}) \cong J^1(\mathbb{C}^{2\|2}, \mathbb{C}^{1\|0})$	(x^i, u, u_i), $\sigma = du - \sum_{i=1}^4 u_i dx^i$ $\mathcal{C} = \langle \sigma = 0 \rangle = \langle \partial_{x^i} + u_i \partial_u, \partial_{u_i} \rangle$
$\mathcal{C}$ has frames of conformal symplectic-orthogonal supervector fields	$d\sigma\|_{\mathcal{C}} = \left(\begin{array}{cccc\|cccc} &&&&1&&& \\ &&&&&1&& \\ &&&&&&1& \\ &&&&&&&1 \\ \hline -1&&&&&&& \\ &-1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \end{array}\right)$ $\partial_{x^i} + u_i \partial_u, \; \partial_{u_i}$ is adapted frame
Lagrangian subspace of $\mathcal{C}$ at $m \in M$	$\langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j} \rangle$
Lagrange–Grassmann bundle $(\widetilde{M}^{9\|8}, \widetilde{\mathcal{C}}) \cong J^2(\mathbb{C}^{2\|2}, \mathbb{C}^{1\|0})$	$(x^i, u, u_i, u_{ij} = \pm u_{ji})$ $\widetilde{\mathcal{C}} = \langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j}, \partial_{u_{ij}} \rangle$

A *2nd order super-PDE* is a submanifold of Lagrange–Grassmann bundle $\widetilde{M}$ and an *external symmetry* is a symmetry of $(\widetilde{M}, \widetilde{\mathcal{C}})$ that preserves the submanifold.

Key steps of the proof

- *Lagrangian lift.* At any "point" of $(M, \mathcal{C})$ we have $(1|2)$-parametric family of Lagrangian subspaces of $\mathcal{C}$: the affine tangent spaces along $\mathcal{V}$. It gives $(6|6)$-dimensional submanifold $\mathcal{E} \subset \widetilde{M}$, i.e., the $G(3)$-contact super-PDE;
- *Cubic form.* The $G(3)$-contact super-PDE can be parametrically written as

$$\begin{pmatrix} u_{00} & u_{0a} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(T^3) & \frac{3}{2}\mathfrak{C}_a(T^2) \\ \frac{3}{2}\mathfrak{C}_a(T^2) & 3\mathfrak{C}_{ab}(T) \end{pmatrix} \qquad (a, b = 1, 2, 3) \ .$$

  This extends to $G(3)$ a formula giving explicit realizations of all exceptional Lie algebras – for different cubic forms – obtained by D. The in 2018.
- *Symmetries.* External symmetries of $G(3)$-contact super-PDE are derived explicitly by a hand computation using expression of generating functions on $(M, \mathcal{C})$ via the cubic form on the Kaplanski superalgebra;

Key steps of the proof

- *Spencer cohomology.* The previous computation tells that supersymmetry dimension is $(17|14)$, i.e., the upper bound coming from Tanaka–Weisfeiler prolongation is attained. Moreover, $\exists$ grading element $\Longrightarrow$ the symmetry superalgebra is exactly $G(3)$.
- *Cauchy characteristic reduction.* On $\mathcal{E} \cong G(3)/P_{12}$ we have the Cartan superdistribution $\mathcal{H} \subset \mathcal{T}\mathcal{E}$ of rank $(3|4)$. The Cauchy characteristic space

$$\mathrm{Ch}(\mathcal{H}) = \{X \in \Gamma(\mathcal{H}) \mid \mathcal{L}_X \mathcal{H} \subset \mathcal{H}\}$$

  is a module for the space of superfunctions of $\mathcal{E}$ and it is generated by a nowhere-vanishing even supervector field. The quotient $\overline{\mathcal{E}} = \mathcal{E}/\mathrm{Ch}(\mathcal{H})$ is then $(5|6)$-dimensional and is endowed with superdistribution of rank $(2|4)$.
- *SHC-equation.* We have $\overline{\mathcal{E}} \cong G(3)/P_2$ with the Cartan superdistribution associated to SHC-eqn.

Geometric realizations of $F(4)$

F(4) mixed-contact super-PDE:

$$(u_{ij}) = \begin{pmatrix} u_{00} & u_{0a} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(T^3) & \frac{3}{2}\mathfrak{C}_a(T^2) \\ \frac{3}{2}\mathfrak{C}_a(T^2) & 3\mathfrak{C}_{ab}(T) \end{pmatrix} \qquad (a, b = 1, 2, 3, 4)$$

where $u : \mathbb{C}^{3|2} \to \mathbb{C}^{1|0}$. Here $T = (\lambda, \mu|\theta, \phi)$ and $\mathfrak{C}(T^3) = \lambda\mu^2 + 2\mu\theta\phi$.

F(4) odd-contact super-PDE:

$$u_{0ab} = u_{ab}u_{123}, \quad 1 \leqslant a < b \leqslant 3.$$

where $u : \mathbb{C}^{0|4} \to \mathbb{C}^{1|0}$. This is quite a different story!

Thm[A.S., D. The, 2022 arXiv] These super-PDE have *symmetry superalgebra* $F(4)$.

Thanks!