Verma modules for infinite dimensional Lie superalgebras

Prato, 13 Gennaio 2023

Theorem 1 (Kac, 1998).

A complete list of simple linearly compact Lie superalgebras is, up to isomorphisms:

- $W(m,n) = \{X = \sum_{i=1}^{m} Pi \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j} \mid P_i, Q_j \in \Lambda(n) \otimes \mathbb{C}[[x_1, \dots, x_m]]\}$;
- $S(m,n) = \{X \in W(m,n) \mid div(X) = 0\}$;
- $H(m,n) = \{X \in W(m,n) \mid X.\omega_s = 0\}$;
- $K(m,n) = \{X \in W(m,n) \mid X.\omega_c = f\omega_c\}$;
- $HO(m,m) = \{X \in W(m,m) \mid X.\omega_{os} = 0\}$;
- $SHO(m,m) = \{X \in HO(m,m) \mid div(X) = 0\}$;
- $KO(m,m+1) = \{X \in W(m,m+1) \mid X.\omega_{oc} = f\omega_{oc}\}$;
- $SKO(m,m+1;\beta) = \{X \in KO(m,m+1) \mid div_\beta(X) = 0\}$;
- $SH\tilde{O}(m,m)$;
- $SK\tilde{O}(m,m+1;1+2/m)$;
- $E(1,6)$
- $E(3,6)$
- $E(3,8)$
- $E(4,4)$
- $E(5,10)$

Main examples

$$\underline{\mathfrak{g} = E(5,10)}$$

$$\mathfrak{g}_{\bar{0}} = S_5$$
$$\mathfrak{g}_{\bar{1}} = \Omega^2_{cl}(5)$$

with bracket:

$$[X,\omega] = L_X(\omega) = d(i_X\omega)$$

$$[\omega_1,\omega_2] = \omega_1 \wedge \omega_2$$

Main examples

$$\underline{\mathfrak{g} = E(5,10)}$$

$$\mathfrak{g}_{\bar{0}} = S_5$$
$$\mathfrak{g}_{\bar{1}} = \Omega^2_{cl}(5)$$

with bracket:

$$[X,\omega] = L_X(\omega) = d(i_X\omega)$$

$$[\omega_1,\omega_2] = \omega_1 \wedge \omega_2$$

$$\underline{\mathfrak{g} = E(4,4)}$$

$$\mathfrak{g}_{\bar{0}} = W_4;$$
$$\mathfrak{g}_{\bar{1}} = (\Omega^1(4))^{-1/2}$$

with bracket:

$$[X,\omega] = L_X(\omega) - \frac{1}{2} div(X)\omega$$

$$[\omega_1,\omega_2] = d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_5 \cong sl_5$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_5 \cong sl_5$$

$$\mathfrak{g}_{-1} = span\{dx_i \wedge dx_j\} \cong \Lambda^2(\mathbb{C}^5)$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_5 \cong sl_5$$

$$\mathfrak{g}_{-1} = span\{dx_i \wedge dx_j\} \cong \Lambda^2(\mathbb{C}^5)$$

$$\mathfrak{g}_{-2} = span\{\frac{\partial}{\partial x_i}\} \cong (\mathbb{C}^5)^*$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_5 \cong sl_5$$

$$\mathfrak{g}_{-1} = span\{dx_i \wedge dx_j\} \cong \Lambda^2(\mathbb{C}^5)$$

$$\mathfrak{g}_{-2} = span\{\frac{\partial}{\partial x_i}\} \cong (\mathbb{C}^5)^*$$

$$\mathfrak{g}_1 = span\{x_i dx_j \wedge dx_k\} \cap \Omega^2_{cl} \cong F(1,1,0,0)$$

$\mathbb{Z}$-gradings

$$E(5,10) = \prod_{j \geq -2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_k] \subseteq \mathfrak{g}_{i+k}$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_5 \cong sl_5$$

$$\mathfrak{g}_{-1} = span\{dx_i \wedge dx_j\} \cong \Lambda^2(\mathbb{C}^5)$$

$$\mathfrak{g}_{-2} = span\{\frac{\partial}{\partial x_i}\} \cong (\mathbb{C}^5)^*$$

$$\mathfrak{g}_1 = span\{x_i dx_j \wedge dx_k\} \cap \Omega^2_{cl} \cong F(1,1,0,0)$$

$$\mathfrak{g}_i = (\mathfrak{g}_1)^i$$

What are the relevant properties of this grading?

$$\mathfrak{g} = \prod_{j \geq -d} \mathfrak{g}_j$$

for some $d \in \mathbb{N}$, and $\dim(\mathfrak{g}_i) < \infty$;

- it is *consistent* i.e. $\mathfrak{g}_{\bar{0}} = \prod_{j \in \mathbb{Z}} \mathfrak{g}_{2j}$, $\mathfrak{g}_{\bar{1}} = \prod_{j \in \mathbb{Z}} \mathfrak{g}_{2j+1}$;

- $\mathfrak{g}_0$ is reductive (it is in fact simple in the case of $E(5,10)$);

- $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module;

- $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$;

- $\mathfrak{g}_i = (\mathfrak{g}_1)^i$.

$E(4,4)$ has no such grading

Theorem 2 (C. - Kac, 2005).

$E(4,4)$ has, up to isomorphisms, only one irreducible grading (which is not consistent)

Why are we interested in such a grading?

Let $F = F(\lambda)$ be an irreducible finite-dimensional $\mathfrak{g}_0$-module and extend the action of $\mathfrak{g}_0$ to $\mathfrak{g}_{\geq 0}$ by letting $\mathfrak{g}_{>0}$ act trivially on F. Then we can define the (finite) Verma-module

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_{\geq 0})} F \cong \mathcal{U}(\mathfrak{g}_-) \otimes_{\mathbb{C}} F.$$

Why are we interested in such a grading?

Let $F = F(\lambda)$ be an irreducible finite-dimensional $\mathfrak{g}_0$-module and extend the action of $\mathfrak{g}_0$ to $\mathfrak{g}_{\geq 0}$ by letting $\mathfrak{g}_{>0}$ act trivially on F. Then we can define the (finite) Verma-module

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_{\geq 0})} F \cong \mathcal{U}(\mathfrak{g}_-) \otimes_{\mathbb{C}} F.$$

Question 1. What are the conditions on λ in order to have an irreducible $M(\lambda)$?

Why are we interested in such a grading?

Let $F = F(\lambda)$ be an irreducible finite-dimensional $\mathfrak{g}_0$-module and extend the action of $\mathfrak{g}_0$ to $\mathfrak{g}_{\geq 0}$ by letting $\mathfrak{g}_{>0}$ act trivially on F. Then we can define the (finite) Verma-module

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_{\geq 0})} F \cong \mathcal{U}(\mathfrak{g}_-) \otimes_{\mathbb{C}} F.$$

Question 1. What are the conditions on λ in order to have an irreducible $M(\lambda)$?

Theorem 3 (C., Caselli, Kac, 2021).

If $\mathfrak{g} = E(5, 10)$ the Verma module $M(\lambda)$ is not irreducible if and only if λ is one of the following weights:

$$(m, n, 0, 0), (m, 0, 0, n) (0, 0, m, n).$$

Tools: singular vectors and morphisms

Definition 4.

A vector $w \in M(F)$ is singular if it satisfies the following properties:

a) $x_i \frac{\partial}{\partial x_{i+1}}.w = 0 \, \forall i$;

b) $\mathfrak{g}_{>0}.w = 0$;

c) $w \notin F$.

Tools: singular vectors and morphisms

Definition 4.

A vector $w \in M(F)$ is singular if it satisfies the following properties:

a) $x_i \frac{\partial}{\partial x_{i+1}}.w = 0 \, \forall i$;

b) $\mathfrak{g}_{>0}.w = 0$;

c) $w \notin F$.

Example 1.

In the Verma module $M(F)$ with $F = F(m, n, 0, 0)$ ($m, n \in \mathbb{Z}_+$), the vector

$$w = dx_1 \wedge dx_2 \otimes v,$$

where v is a highest weight vector in F, is singular.

Example 2.

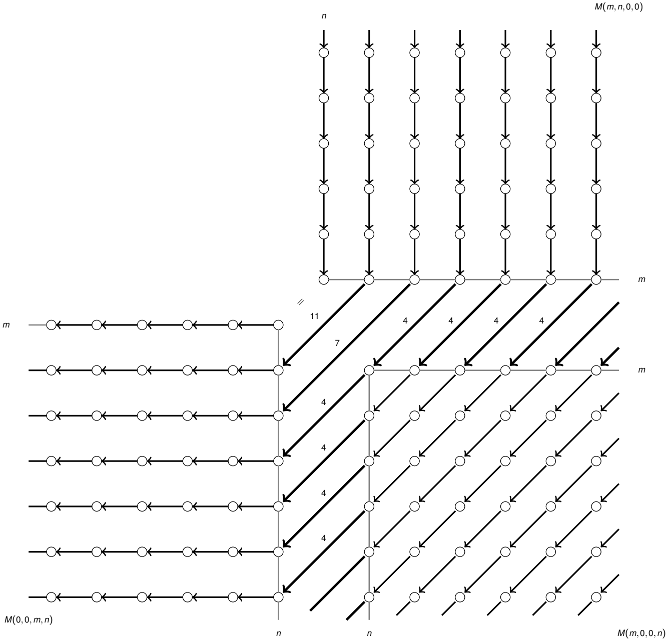
$$
\begin{aligned}
w[4_E] = {} & d_{12} d_{13} d_{14} d_{15} f_1^3 f_5^n + d_{12} d_{14} d_{15} d_{23} f_1^2 f_2 f_5^n + d_{13} d_{14} d_{15} d_{23} f_1^2 f_3 f_5^n - d_{12} d_{13} d_{15} d_{24} f_1^2 f_2 f_5^n \\
& + d_{13} d_{14} d_{15} d_{24} f_1^2 f_4 f_5^n + d_{12} d_{15} d_{23} d_{24} f_1 f_2^2 f_5^n + d_{13} d_{15} d_{23} d_{24} f_1 f_2 f_3 f_5^n + d_{14} d_{15} d_{23} d_{24} f_1 f_2 f_4 f_5^n \\
& + d_{12} d_{13} d_{14} d_{25} f_1^2 f_2 f_5^n + d_{13} d_{14} d_{15} d_{25} f_1^2 f_5^{n+1} - d_{12} d_{14} d_{23} d_{25} f_1 f_2^2 f_5^n - d_{13} d_{14} d_{23} d_{25} f_1 f_2 f_3 f_5^n \\
& + d_{12} d_{23} d_{24} d_{25} f_2^3 f_5^n + d_{13} d_{23} d_{24} d_{25} f_2^2 f_3 f_5^n + d_{14} d_{23} d_{24} d_{25} f_2^2 f_4 f_5^n + d_{15} d_{23} d_{24} d_{25} f_2^2 f_5^{n+1} \\
& - d_{12} d_{13} d_{15} d_{34} f_1^2 f_3 f_5^n - d_{12} d_{14} d_{15} d_{34} f_1^2 f_4 f_5^n + d_{12} d_{15} d_{23} d_{34} f_1 f_2 f_3 f_5^n + d_{13} d_{15} d_{23} d_{34} f_1 f_3^2 f_5^n \\
& + d_{14} d_{15} d_{23} d_{34} f_1 f_3 f_4 f_5^n + d_{12} d_{15} d_{24} d_{34} f_1 f_2 f_4 f_5^n + d_{13} d_{15} d_{24} d_{34} f_1 f_3 f_4 f_5^n + d_{14} d_{15} d_{24} d_{34} f_1 f_4^2 f_5^n \\
& - d_{12} d_{13} d_{25} d_{34} f_1 f_2 f_3 f_5^n - d_{12} d_{14} d_{25} d_{34} f_1 f_2 f_4 f_5^n + d_{13} d_{15} d_{25} d_{34} f_1 f_3 f_5^{n+1} + d_{14} d_{15} d_{25} d_{34} f_1 f_4 f_5^{n+1} \\
& - d_{12} d_{23} d_{25} d_{34} f_2^2 f_3 f_5^n - d_{13} d_{23} d_{25} d_{34} f_2 f_3^2 f_5^n - d_{14} d_{23} d_{25} d_{34} f_2 f_3 f_4 f_5^n - d_{15} d_{23} d_{25} d_{34} f_2 f_3 f_5^{n+1} \\
& - d_{12} d_{24} d_{25} d_{34} f_2^2 f_4 f_5^n - d_{13} d_{24} d_{25} d_{34} f_2 f_3 f_4 f_5^n - d_{14} d_{24} d_{25} d_{34} f_2 f_4^2 f_5^n - d_{15} d_{24} d_{25} d_{34} f_2 f_4 f_5^{n+1} \\
& + d_{12} d_{13} d_{14} d_{35} f_1^2 f_3 f_5^n - d_{12} d_{14} d_{15} d_{35} f_1^2 f_5^{n+1} - d_{12} d_{14} d_{23} d_{35} f_1 f_2 f_3 f_5^n - d_{13} d_{14} d_{23} d_{35} f_1 f_3^2 f_5^n \\
& + d_{12} d_{13} d_{24} d_{35} f_1 f_2 f_3 f_5^n - d_{13} d_{14} d_{24} d_{35} f_1 f_3 f_4 f_5^n + d_{12} d_{15} d_{24} d_{35} f_1 f_2 f_5^{n+1} \\
& + d_{14} d_{15} d_{24} d_{35} f_1 f_4 f_5^{n+1} + d_{12} d_{23} d_{24} d_{35} f_2^2 f_3 f_5^n + d_{13} d_{23} d_{24} d_{35} f_2 f_3^2 f_5^n + d_{14} d_{23} d_{24} d_{35} f_2 f_3 f_4 f_5^n \\
& + d_{15} d_{23} d_{24} d_{35} f_2 f_3 f_5^{n+1} - d_{12} d_{14} d_{25} d_{35} f_1 f_2 f_5^{n+1} - d_{13} d_{14} d_{25} d_{35} f_1 f_3 f_5^{n+1} + d_{14} d_{15} d_{25} d_{35} f_1 f_5^{n+2} \\
& - d_{12} d_{24} d_{25} d_{35} f_2^2 f_5^{n+1} - d_{13} d_{24} d_{25} d_{35} f_2 f_3 f_5^{n+1} - d_{14} d_{24} d_{25} d_{35} f_2 f_4 f_5^{n+1} - d_{15} d_{24} d_{25} d_{35} f_2 f_5^{n+2} \\
& + d_{12} d_{13} d_{34} d_{35} f_1 f_3^2 f_5^n + d_{12} d_{14} d_{34} d_{35} f_1 f_3 f_4 f_5^n + d_{12} d_{15} d_{34} d_{35} f_1 f_3 f_5^{n+1} + d_{12} d_{23} d_{34} d_{35} f_2 f_3^2 f_5^n \\
& + d_{13} d_{23} d_{34} d_{35} f_3^3 f_5^n + d_{14} d_{23} d_{34} d_{35} f_3^2 f_4 f_5^n + d_{15} d_{23} d_{34} d_{35} f_3^2 f_5^{n+1} + d_{12} d_{24} d_{34} d_{35} f_2 f_3 f_4 f_5^n \\
& + d_{13} d_{24} d_{34} d_{35} f_3^2 f_4 f_5^n + d_{14} d_{24} d_{34} d_{35} f_3 f_4^2 f_5^n + d_{15} d_{24} d_{34} d_{35} f_3 f_4 f_5^{n+1} + d_{12} d_{25} d_{34} d_{35} f_2 f_3 f_5^{n+1} \\
& + d_{13} d_{25} d_{34} d_{35} f_3^2 f_5^{n+1} + d_{14} d_{25} d_{34} d_{35} f_3 f_4 f_5^{n+1} + d_{15} d_{25} d_{34} d_{35} f_3 f_5^{n+2} + d_{12} d_{13} d_{14} d_{45} f_1^2 f_4 f_5^n \\
& + d_{12} d_{13} d_{15} d_{45} f_1^2 f_5^{n+1} - d_{12} d_{14} d_{23} d_{45} f_1 f_2 f_4 f_5^n - d_{13} d_{14} d_{23} d_{45} f_1 f_3 f_4 f_5^n - d_{12} d_{15} d_{23} d_{45} f_1 f_2 f_5^{n+1}
\end{aligned}
$$

Proposition 1.

The following claims are equivalent:

1. $M(\mu)$ *is degenerate (i.e. not irreducible);*
2. $M(\mu)$ *contains a singular vector* w *of weight* λ*;*
3. *there exists a morphism of* $\mathfrak{g}$*-modules*

$$\varphi : M(\lambda) \to M(\mu).$$



$M(m,n,0,0)$
n
m
m
m
11
7
4
4
4
4
4
4
4
4
4
$M(0,0,m,n)$
n
n
$M(m,0,0,n)$

Duality

Theorem 5 (C., Caselli, Kac, 2021).

Let $\mathfrak{g} = E(5,10)$. Then if there exists a morphism

$$\varphi : M(\lambda) \to M(\mu)$$

then there exists a morphism

$$\varphi^* : M(\mu^*) \to M(\lambda^*)$$

where $F(\lambda^*)$ is the dual module of $F(\lambda)$.

Duality

Theorem 5 (C., Caselli, Kac, 2021).

Let $\mathfrak{g} = E(5,10)$*. Then if there exists a morphism*

$$\varphi : M(\lambda) \to M(\mu)$$

then there exists a morphism

$$\varphi^* : M(\mu^*) \to M(\lambda^*)$$

where $F(\lambda^*)$ *is the dual module of* $F(\lambda)$*.*

Remark Theorem 5 was proved in a much more general situation but not for $\mathfrak{g} = E(4,4)$.

Troubles with $E(4,4)$.

Let us describe the irreducible grading of $E(4,4)$:

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

Troubles with $E(4,4)$.

Let us describe the irreducible grading of $E(4,4)$:

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}, x_j dx_i\}$$

Troubles with $E(4,4)$.

Let us describe the irreducible grading of $E(4,4)$:

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}, x_j dx_i\}$$

$$\mathfrak{g}_{-1} = span\{\frac{\partial}{\partial x_i}, dx_i\}$$

Troubles with $E(4,4)$.

Let us describe the irreducible grading of $E(4,4)$:

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}, x_j dx_i\}$$

$$\mathfrak{g}_{-1} = span\{\frac{\partial}{\partial x_i}, dx_i\}$$

$$\mathfrak{g}_1 = span\{x_i x_j \frac{\partial}{\partial x_k}, x_i x_j dx_k\}$$

Troubles with $E(4,4)$.

Let us describe the irreducible grading of $E(4,4)$:

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

where:

$$\mathfrak{g}_0 = span\{x_i \frac{\partial}{\partial x_j}, x_j dx_i\}$$

$$\mathfrak{g}_{-1} = span\{\frac{\partial}{\partial x_i}, dx_i\}$$

$$\mathfrak{g}_1 = span\{x_i x_j \frac{\partial}{\partial x_k}, x_i x_j dx_k\}$$

$$\mathfrak{g}_i = (\mathfrak{g}_1)^i$$

Let us concentrate on $\mathfrak{s} := \mathfrak{g}_0$: this is a finite-dimensional Lie superalgebra which has a consistent $\mathbb{Z}$-grading, namely:

$$\mathfrak{s}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_4 \cong sl_4$$

$$\mathfrak{s}_{-1} = span\{x_i dx_j - x_j dx_i\} \cong \Lambda^2(\mathbb{C}^4)$$

$$\mathfrak{s}_1 = span\{x_i dx_j + x_j dx_i\} \cong S^2(\mathbb{C}^4)$$

BUT

$$[\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] \neq 0.$$

We have:

$$\mathfrak{s}_{-2} = [\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] = \mathbb{C}C$$

where $C = \sum_i x_i \frac{\partial}{\partial x_i}$.

Let us concentrate on $\mathfrak{s} := \mathfrak{g}_0$: this is a finite-dimensional Lie superalgebra which has a consistent $\mathbb{Z}$-grading, namely:

$$\mathfrak{s}_0 = span\{x_i \frac{\partial}{\partial x_j}\} \cap S_4 \cong sl_4$$

$$\mathfrak{s}_{-1} = span\{x_i dx_j - x_j dx_i\} \cong \Lambda^2(\mathbb{C}^4)$$

$$\mathfrak{s}_1 = span\{x_i dx_j + x_j dx_i\} \cong S^2(\mathbb{C}^4)$$

BUT

$$[\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] \neq 0.$$

We have:

$$\mathfrak{s}_{-2} = [\mathfrak{s}_{-1}, \mathfrak{s}_{-1}] = \mathbb{C}C$$

where $C = \sum_i x_i \frac{\partial}{\partial x_i}$.
Hence

$$\mathfrak{g}_0 = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$$

is thus a non trivial central extension of the finite-dimensional Lie superalgebra $p(4)$.

Let us go back to

$$E(4,4) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus (\mathfrak{g}_1)^j$$

Here

$$\mathfrak{g}_{-1} \cong \mathbb{C}^{4|4}$$

and $\mathfrak{g}_1$ is irreducible as a $\mathfrak{g}_0$-module.

Need a preliminary study of the irreducible $\mathfrak{s}$-modules

Let $F(\lambda)$ be the irreducible sl_4-module with highest weight λ. We define

$$M(\lambda) = \mathcal{U}(\mathfrak{s}) \otimes_{\mathcal{U}(\mathfrak{s}_{\geq 0})} F(\lambda) \cong \mathcal{U}(\mathfrak{s}_{<0}) \otimes_{\mathbb{C}} F(\lambda)$$

and

$$\bar{M}(\lambda, c) = M(\lambda)/(C - c) \cong \mathcal{U}(\mathfrak{s}_{-1}) \otimes_{\mathbb{C}} F(\lambda).$$

Then every finite-dimensional irreducible $\mathfrak{s}$-module is a quotient of $\bar{M}(\lambda, c)$ for some $c \in \mathbb{C}$.

Need a preliminary study of the irreducible $\mathfrak{s}$-modules

Let $F(\lambda)$ be the irreducible sl_4-module with highest weight λ. We define

$$M(\lambda) = \mathcal{U}(\mathfrak{s}) \otimes_{\mathcal{U}(\mathfrak{s}_{\geq 0})} F(\lambda) \cong \mathcal{U}(\mathfrak{s}_{<0}) \otimes_{\mathbb{C}} F(\lambda)$$

and

$$\bar{M}(\lambda, c) = M(\lambda)/(C - c) \cong \mathcal{U}(\mathfrak{s}_{-1}) \otimes_{\mathbb{C}} F(\lambda).$$

Then every finite-dimensional irreducible $\mathfrak{s}$-module is a quotient of $\bar{M}(\lambda, c)$ for some $c \in \mathbb{C}$.

Example 3.

$E(4,4)_1$ is a quotient of $\bar{M}((3,0,0),1)$.

Progress on $E(4,4)$ (C., Caselli, Kac 2022)

- We can prove the duality theorem for $E(4,4)$.

- We can prove the duality theorem for $E(4,4)$.

Main argument: embedding $E(4,4)$ into $W(4,4)$.

Theorem 6.

If $\mathfrak{g} = \oplus_{i\geq -1}$ is a transitive $\mathbb{Z}$-graded Lie superalgebra with $\dim(\mathfrak{g}_{-1})_{\bar{0}} = m$, $\dim(\mathfrak{g}_{-1})_{\bar{1}} = n$, then there is an embedding

$$\mathfrak{g} \to W(m,n)$$

preserving the $\mathbb{Z}$-grading.

Progress on $E(4,4)$

The Lie superalgebra $\mathfrak{s}$ can be embedded into $E(5,10)$ as follows:

$$C \mapsto \frac{1}{2}\frac{\partial}{\partial x_5}$$

$$x_i dx_j - x_j dx_i \mapsto dx_i \wedge dx_j$$

$$x_i \frac{\partial}{\partial x_j} \mapsto x_i \frac{\partial}{\partial x_j}$$

$$x_i dx_j + x_j dx_i \mapsto x_i dx_j \wedge dx_5 + x_j dx_i \wedge dx_5$$

Using this embedding we can describe all degenerate $\mathfrak{s}$-modules.