Lie algebras over subfields.

Lie Theory: frontiers, algorithms, and applications.

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0 Overview

1 Overview

1 Main question in this talk

In this talk, we will consider Lie algebras $\mathfrak{g}$ over different fields F. As the fields may vary, we emphasize the field we are working on by writing $\mathfrak{g}^F$. Feel free to only think about the easy example $\mathbb{R} \subset \mathbb{C}$ if that makes you more comfortable.

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Construction (Field extension)

Given a field extension $F \subset E$, we have the Lie algebra $\mathfrak{g}^E = \mathfrak{g}^F \otimes_F E$ by extending the scalars.

Example

It is a well-known fact that the complexification of the real Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{u}(n)$ are both equal to $\mathfrak{gl}(n, \mathbb{C})$.

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Question

How can we construct from a given Lie algebra $\mathfrak{g}^E$ new examples of Lie algebras over subfields $F \subset E$?

We will recall two different methods for making Lie algebras over the smaller field, which are very distinct in nature.

1 First method: forms of Lie algebras

Definition

*Let $\mathfrak{g}^E$ be a Lie algebra over the field E containing the subfield $F \subset E$. A Lie algebra $\mathfrak{h}^F$ over F is called an F-**form** of $\mathfrak{g}^E$ if $\mathfrak{g}^E \cong \mathfrak{g}^F \otimes_F E$.*

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1. Every free Lie algebra over E contains a unique F-form up to isomorphism. The same holds for free nilpotent Lie algebras.
2. The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ has both $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{u}(n)$ as real forms, which are non-isomorphic in general.

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3. Consider the Lie algebra $\mathfrak{g}_\lambda$ for $\lambda \in \mathbb{C}$ with basis X_1, X_2, X_3 and relations

$$[X_1, X_2] = X_2 \qquad\qquad [X_1, X_3] = \lambda X_3.$$

It is easy to see that this Lie algebra has a real form if and only if there exists $\mu \in \mathbb{C}$ such that $\mathrm{ad}_{\mu X_1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu\lambda \end{pmatrix}$ is conjugate to a real matrix. This is the case if and only if

$$\overline{\mu} = \lambda\mu$$

or hence if and only if $|\lambda| = 1$.

1 Second method: underlying Lie algebras

Definition

Let $\mathfrak{g}^E$ be a Lie algebra over the field E containing the subfield $F \subset E$. By restricting the scalar multiplication on $\mathfrak{g}^E$ to F we find a new Lie algebra $\mathfrak{g}_F$, which is called the ***underlying Lie algebra****.*

Note that $\dim_F(\mathfrak{g}_F) = \dim_E(\mathfrak{g}^E)[E : F]$, which we assume to be finite.

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Consider the Heisenberg Lie algebra $\mathfrak{h}_3(F)$ over any field F, so with basis X, Y, Z and $[X, Y] = Z$. For example, from $\mathfrak{h}_3(\mathbb{C})$ we find the real underlying Lie algebra $(\mathfrak{h}_3(\mathbb{C}))_\mathbb{R}$ with basis

$$X_1 = X \qquad X_2 = iX$$
$$X_3 = Y \qquad X_4 = iY$$
$$X_5 = Z \qquad X_6 = iZ$$

and relations

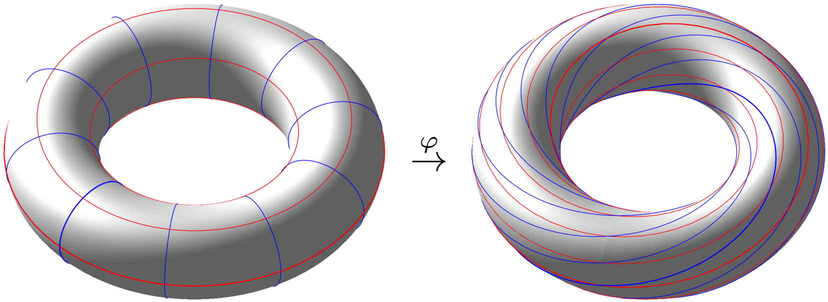
$$[X_1, X_3] = X_5 \qquad [X_2, X_3] = X_6$$
$$[X_1, X_4] = X_6 \qquad [X_2, X_4] = -X_6.$$

2 Overview

Why do we care about $F = \mathbb{Q}$?

2 Arnold's cat map

Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then A induces a diffeomorphism φ on $\mathbb{T}^2 = {}^{\mathbb{R}^2}\!/_{\mathbb{Z}^2}$:



The map φ is called **Arnold's cat map**.

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Observation

This map has interesting dynamical properties. You can experiment ‣ here .

2 What is an Anosov diffeomorphism?

Definition

Let M be a closed Riemannian manifold with diffeomorphism $f : M \to M$, then we call f an ***Anosov diffeomorphism*** *if:*

(i) *there exists a continuous splitting of the tangent bundle*

$$TM = E^s \oplus E^u;$$

(ii) *the subbundles E^s and E^u are preserved under the map $Df : TM \to TM$, i.e.*

$$Df(E^s) = E^s \text{ and } Df(E^u) = E^u;$$

(iii) *there exist real constants $0 < \lambda < 1$ and $c > 0$ such that*

$$\forall v \in E^s,\ \forall k > 0 : \|Df^k(v)\| \leq c\lambda^k \|v\|,$$
$$\forall v \in E^u,\ \forall k > 0 : \|Df^k(v)\| \geq \frac{1}{c\lambda^k} \|v\|.$$

2 Lattices in nilpotent Lie groups

Note that $\mathbb{Z}^2$ is a lattice in the Lie group $\mathbb{R}^2$. In a similar fashion, you can build examples on other (nilpotent) Lie groups.

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Although $\log(N)$ is in general not closed under addition or the Lie bracket, the rational span $\mathfrak{n}^{\mathbb{Q}} = \mathbb{Q}\log(N)$ always is and thus forms a rational Lie algebra. The corresponding group $N^{\mathbb{Q}} = \exp(\mathfrak{n}^{\mathbb{Q}})$ is a subgroup of G called the rational Mal'cev completion of N.

2 Examples

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2 Denote by $H_3(F)$ the group corresponding to the Lie algebra $\mathfrak{h}_3(F)$ under the exponential map. For every $k \in \mathbb{Z}$, $k \neq 0$, the groups

$$N_k = \left\{ \begin{pmatrix} 1 & x & \frac{z}{k} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

are lattices in $H_3(\mathbb{R})$. It holds that

$$(N_k)^{\mathbb{Q}} = H_3(\mathbb{Q}).$$

Indeed, the diffeomorphism $\log$ is given by

$$\log \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & z - \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

2 Properties of $N^{\mathbb{Q}}$

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Every rational nilpotent Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ *is induced by a lattice* N. *The group* N *is unique up to commensurability.*

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The existence of an Anosov diffeomorphism on G/N only depends on the rational Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ and is equivalent to the existence of an integer-like, hyperbolic automorphism.

Recall that a matrix is called **hyperbolic** if its eigenvalues have absolute value $\neq 1$ and is integer-like if its characteristic polynomial has integer coefficients and constant term ± 1. In particular, the eigenvalues will be algebraic units.

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Corollary

In order to study the existence of Anosov diffeomorphisms, we have to understand rational nilpotent Lie algebras and their automorphisms.

2 Examples

Although the Lie algebra $\mathfrak{h}_3(\mathbb{Q})$ does not have hyperbolic integer-like automorphisms, by extending the field to $\mathfrak{h}_3(\mathbb{Q}(\sqrt{2}))$ we find an interesting automorphism φ defined as

$$\begin{aligned}\varphi(X) &= \lambda X\\ \varphi(Y) &= \lambda Y\\ \varphi(Z) &= \lambda^2 Z,\end{aligned}$$

where $\lambda = 1 + \sqrt{2}$ is an algebraic unit, with conjugate $1 - \sqrt{2} = -\lambda^{-1}$.

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In fact, both examples are the same, but realised in two distinct ways. This works for any $\sqrt{d}$ with $d > 0$ and d not a square.

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1. Every compact quasi-Kähler Chern-flat manifolds is constructed from $\mathfrak{n}_{\mathbb{R}}$ where $\mathfrak{n}$ is a complex 2-step nilpotent Lie algebra by the work of Di Scala, Lauret and Vezzoni.

2. The study of nilmanifolds which are isospectral but not isometric is related to the study of almost inner derivations on nilpotent Lie algebras. It is possible to construct such derivations on underlying Lie algebras by the work of Burde, Dekimpe and Verbeke.

3 Overview

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3 Nilpotent Lie algebras associated to graphs

Let $\mathcal{G} = (V, E)$ be a simple, undirected graph with vertices V and edges E. Associated to $\mathcal{G}$ is a 2-step nilpotent Lie algebra $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ over the real numbers, defined as follows.

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- A basis for $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ is given by the vertices $v_1, \ldots, v_n \in V$ and the edges $e_{ij} = \{v_i, v_j\} \in E$.
- The subspace $\langle e_{ij} \mid e_{ij} \in E \rangle$ spanned by the edges is central.
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2. If $\mathcal{G}$ is the full graph on n vertices, then $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ is the free 2-step nilpotent Lie algebra on n generators.

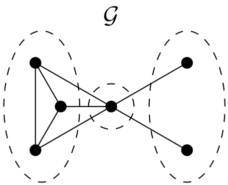
3 Some notation

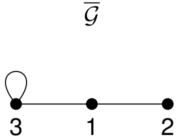
Note that for every map $d : V \to \mathbb{R}_0$ we have an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}})$ such that $\varphi(v) = d(v)v$. This uniquely characterizes the 2-step nilpotent Lie algebras associated to graphs and allows us to describe the automorphism group.

Let $\mathcal{G} = (S, E)$ be a simple undirected graph.

- For any vertex $\alpha \in V$, we define the *open and closed neighbourhoods of* α by
$$\Omega'(\alpha) = \{\beta \in V \mid \{\alpha, \beta\} \in E\} \quad \text{and} \quad \Omega(\alpha) = \Omega'(\alpha) \cup \{\alpha\}, \tag{1}$$
respectively.
- Define a relation $\prec$ on the vertices V by $\alpha \prec \beta \Leftrightarrow \Omega'(\alpha) \subset \Omega(\beta)$.
- Define an equivalence relation $\sim$ on V by $\alpha \sim \beta \Leftrightarrow \alpha \prec \beta \wedge \alpha \succ \beta$. Note that $\alpha \sim \beta$ if and only if the transposition of α with β defines an automorphism of $\mathcal{G}$.
- The equivalence classes of $\sim$ are called the *coherent components of* $\mathcal{G}$ and are denoted by $\lambda \in \Lambda := V/ \sim$.
- Write $\overline{\mathcal{G}}$ for the induced graph on Λ, where we remember the number of elements in each $\lambda \in \Lambda$.

3 Examples





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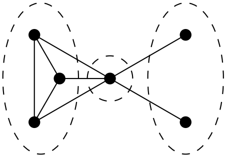

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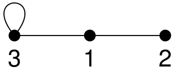

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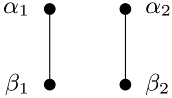

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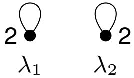

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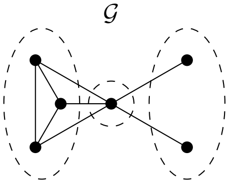

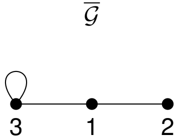

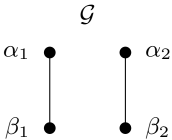

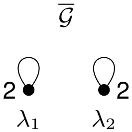

The Lie algebra $\mathfrak{h}_3(\mathbb{R}) \oplus \mathfrak{h}_3(\mathbb{R})$ is a Lie algebra associated to a graph, of which we already know several rational forms. Can we describe them all?

3 Automorphisms of $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$

It is not hard to show that every coherent component has either no or all edges, so they generate either an abelian or a free nilpotent Lie algebra. For every automorphism $\pi \in \mathrm{Aut}(\mathcal{G})$, one can define an automorphism $P(\pi)$ by permuting the vertices V accordingly.

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Theorem (D.- Mainkar, 2020)

For every Lie algebra $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ associated to a graph $\mathcal{G}$, we have that

$$\mathrm{Aut}(\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}) = U \left(\prod_{\lambda \in \Lambda} \mathrm{GL}(V_\lambda) \right) P(\mathrm{Aut}(\mathcal{G}))$$

where U is equal to the unipotent radical of G.

Note that there is a way to embed $\mathrm{Aut}(\overline{\mathcal{G}})$ into $\mathrm{Aut}(\mathcal{G})$, and thus we can replace $P(\mathrm{Aut}(\mathcal{G}))$ by $P(\mathrm{Aut}(\overline{\mathcal{G}}))$ in the theorem as well.

3 Rational forms

Let $\rho : \mathrm{Gal}(E, \mathbb{Q}) \to \mathrm{Aut}(\mathfrak{n}_{\mathcal{G}}^{\mathbb{Q}})$ be a morphism, then the subalgebra

$$\mathfrak{n}_{\rho}^{\mathbb{Q}} = \{v \in \mathfrak{n}_{\mathcal{G}}^{E} \mid \forall \sigma \in \mathrm{Gal}(E, \mathbb{Q}) : \ \rho_{\sigma}({}^{\sigma}v) = v\}$$

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Theorem (D. - Witdouck, 2022)

Let $\mathcal{G}$ be a simple undirected graph and $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ the associated 2-step nilpotent Lie algebra.

- *Any rational form of $\mathfrak{n}_{\mathcal{G}}^{\mathbb{R}}$ is $\mathbb{Q}$-isomorphic to $\mathfrak{n}_{\rho}^{\mathbb{Q}}$ for some finite degree real Galois extension $L/\mathbb{Q}$ and an injective group morphism $\rho : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$.*
- *If $K/\mathbb{Q}$ is another finite degree Galois extension with an injective group morphism $\eta : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(\overline{\mathcal{G}})$, then $\mathfrak{n}_{\rho}^{\mathbb{Q}}$ and $\mathfrak{n}_{\eta}^{\mathbb{Q}}$ are $\mathbb{Q}$-isomorphic if and only if $L = K$ and there exists a $\varphi \in \mathrm{Aut}(\overline{\mathcal{G}})$ such that $\varphi\rho(\sigma)\varphi^{-1} = \eta(\sigma)$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$.*

3 How to compute the rational forms?

Corollary

Every Lie algebra associated to a graph has either 1 *or infinitely many rational forms, depending on whether* $\mathrm{Aut}(\overline{\mathcal{G}})$ *is trivial or not.*

For $\mathbb{R} \subset \mathbb{C}$ on the other hand, there are at most finitely many real forms in each complex Lie algebra.

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For $\mathfrak{h}_3(\mathbb{R}) \oplus \mathfrak{h}_3(\mathbb{R})$, we have found all rational forms. However, it is not so easy to give an explicit description in general.

3 How to compute the rational forms?

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Question (Inverse Galois problem)

Given a finite group H*, does there exist a field extension* $\mathbb{Q} \subset E$ *with* $\mathrm{Gal}(E, \mathbb{Q}) \approx H$*?*

For $\mathbb{Z}_2$, this problem is easy, and all extensions are of the form $\mathbb{Q}(\sqrt{d})$.

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Theorem (D., Witdouck - 2023)

We can characterize the actions ρ leading to a rational form that is Anosov.

4 Isomorphic underlying Lie algebras

This last part is motivated by a question from Di Scala, Lauret and Vezzoni:

Question

Do there exist two non-isomorphic complex 2-step nilpotent Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that $\mathfrak{g}_{\mathbb{R}} \approx \mathfrak{h}_{\mathbb{R}}$?

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Example (D., 2019)

Consider the Lie algebra $\mathfrak{g}_\lambda$ of dimension 10 with basis $X_1, \ldots, X_8, Z_1, Z_2$:

$$\begin{array}{lll} [X_1, X_5] = Z_1 & [X_2, X_6] = Z_1 & [X_3, X_7] = Z_1 \\ [X_4, X_8] = Z_1 & [X_2, X_5] = Z_2 & [X_3, X_6] = Z_2 \\ [X_4, X_7] = Z_2 & [X_1, X_8] = -Z_2 & [X_2, X_7] = -\lambda Z_2. \end{array}$$

Why does this example work?

4 Conjugate Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over the field E with basis $X_1, \ldots, X_n$. The Lie bracket is completely determined by the structural constants $c_{ij}^k \in E$:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

For any $\sigma \in \mathrm{Aut}(E, F)$ an automorphism that fixes F, one can define the σ-conjugate Lie algebra $\mathfrak{g}^\sigma$ with structural constants $\sigma(c_{ij}^k)$.

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Observation

1. *This does not depend on the choice of basis* $X_1, \ldots, X_n$.
2. *The underlying Lie algebra* $(\mathfrak{g}^\sigma)_F$ *is isomorphic to* $\mathfrak{g}_F$.
3. *If* $\mathfrak{g}$ *has an* F*-form, then* $\mathfrak{g}^\sigma \approx \mathfrak{g}$.

Corollary

For $\mathfrak{g}_\lambda$*, it holds that* $\overline{\mathfrak{g}_\lambda} = \mathfrak{g}_{\overline{\lambda}}$. *By showing that* $\mathfrak{g}_\lambda \not\approx \mathfrak{g}_{\overline{\lambda}}$*, the conclusion follows.*

4 When are underlying Lie algebras isomorphic?

Theorem (D., 2019)

Let $F \subseteq E$ be a Galois extension and $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over E such that $\mathfrak{g}_F \approx \mathfrak{h}_F$. If we decompose

$$\mathfrak{g} \approx \bigoplus_{i=1}^{k} \mathfrak{g}_i$$

into indecomposable ideals, then there exists $\sigma_i \in \mathrm{Gal}(E, F)$ such that

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Theorem (D., 2019)

If both $\mathfrak{g}$ and $\mathfrak{h}$ have an F-form, then $\mathfrak{g}_F \approx \mathfrak{h}_F$ implies $\mathfrak{g} \approx \mathfrak{h}$.

4 Sketch of the proof

Theorem (D., 2019)

Let $F \subseteq E$ be a Galois extension and let $\mathfrak{g}$ be a Lie algebra over the field E. Let $\mathfrak{g}_F$ be the underlying Lie algebra over F, then there is an isomorphism

$$\mathfrak{g}_F \otimes_F E \approx \bigoplus_{\sigma \in \mathrm{Gal}(E,F)} \mathfrak{g}^\sigma$$

where $\mathfrak{g}^\sigma$ is the σ-conjugate of the Lie algebra $\mathfrak{g}$.

The isomorphism is induced by the natural maps $\mathfrak{g}_F \to \mathfrak{g}^\sigma$.

4 Open question

So far, the easiest strategy to show that $\mathfrak{g}$ has no F-form is to show that $\mathfrak{g}^\sigma$ is not isomorphic to $\mathfrak{g}$ for some $\sigma \in \mathrm{Aut}(E, F)$.

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I don't know any example where $\mathfrak{g} \approx \mathfrak{g}^\sigma$ for all $\sigma \in \mathrm{Gal}(E, F)$ but such that $\mathfrak{g}$ does not have an F-form. (Even not for $\mathbb{R} \subset \mathbb{C}$.)

Question

Does there exist a complex Lie algebra with no real forms, but still $\mathfrak{g} \approx \overline{\mathfrak{g}}$?

Questions?