Representations of cyclotomic Hecke algebras

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Complex reflection groups

A complex reflection $s \in \mathrm{GL}(\mathbb{C}^r)$ *is an element of finite order such that* $\mathrm{Ker}(s - \mathbf{1})$ *is an hyperplane.*

A finite complex reflection group *is a finite subgroup of* $\mathrm{GL}(\mathbb{C}^r)$ *generated by complex reflections.*

The irreducible finite complex reflection groups are:

- $G(de, e, r)$: the monomial matrices with coefficients in μ_{de} and product of non-zero coefficients in μ_d (where μ_i is the group of i-th roots of unity in $\mathbb{C}$).

A_r	B_r	D_r	$I_2(e)$
$G(1,1,r+1)$	$G(2,1,r)$	$G(2,2,r)$	$G(e,e,2)$

- exceptional groups denoted $G_4, \ldots, G_{37}$.

H_3	F_4	H_4	E_6	E_7	E_8
G_{23}	G_{28}	G_{30}	G_{35}	G_{36}	G_{37}

They can be generated by r (*well-generated*) or $r+1$ reflections.

Braid groups

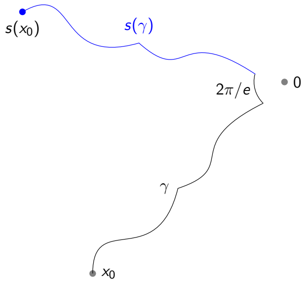
Given $W \subset \mathrm{GL}(V)$ with $V = \mathbb{C}^r$, irreducible, with reflection hyperplanes $\mathcal{H}$, let $\mathrm{V}^{\mathrm{reg}} = V - \bigcup_{H\in\mathcal{H}} H$.

- The *braid group* of W is $B := \Pi_1(\mathrm{V}^{\mathrm{reg}}/W, x)$.
- We have an exact sequence $1 \to \Pi_1(\mathrm{V}^{\mathrm{reg}}) \to B \to W \to 1$.
- B can be generated by the same number (r or $r+1$) of *braid reflections* as W needs reflections.
- The image $s \in W$ of a braid reflection $\mathbf{s} \in B$ is a *distinguished* reflection, i.e. its non-trivial eigenvalue is $\exp(2i\pi/e)$.
- B is presented by *braid relations* of the form $w_1 = w_2$ where w_1, w_2 are positive word of the same length in the braid reflections (Bessis 2001). When W is real (that is, a Coxeter group) these relations are $\mathbf{sts}\ldots = \mathbf{tst}\ldots$, but in general they may involve 3 generators.
- We get a presentation of W by adding the relations $s^e = 1$.

Broué-Malle-Rouquier 1998, Bessis-M. 2004 gave Coxeter-Like diagrams presenting the braid groups.

Braid reflection

for a reflection s of order e . . .



Hecke algebras

Let $s_1, \ldots s_c$ be representatives of conjugacy classes of distinguished reflections in W, and let e_j be the order of s_j.

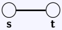
The generic Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ of W is the quotient of the group algebra RB where $R = \mathbb{Z}[u_{j,k}^{\pm 1}]$ by the ideal generated by $(\mathbf{s}_j - u_{j,0}) \ldots (\mathbf{s}_j - u_{j,e_j - 1})$, where $\mathbf{s}_j$ is a braid reflection above s_j and where $u_{j,k}$ are algebraically independent variables.

- By the specialization $u_{j,k} \mapsto \exp 2i\pi k/e_j$, the Hecke algebra over $R \otimes \mathbb{C}$ specializes to the group algebra $\mathbb{C}W$, since the generators of the ideal specializes to $\mathbf{s}_j^{e_j} - 1 = 0$.

(Ariki, Koike, Marin, Chavli, Pfeiffer, Tsuchioka 2018)

$\mathcal{H}$ is a free R-module of dimension $|W|$.

Example

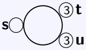
type A_2 : 

$$B = \langle \mathbf{s}, \mathbf{t} \mid \mathbf{sts} = \mathbf{tst} \rangle$$
$$W = \langle s, t \mid sts = tst, s^2 = t^2 = 1 \rangle$$
$$\mathcal{H} = \langle T_{\mathbf{s}}, T_{\mathbf{t}} \mid T_{\mathbf{s}} T_{\mathbf{t}} T_{\mathbf{s}} = T_{\mathbf{t}} T_{\mathbf{s}} T_{\mathbf{t}}, (T_{\mathbf{s}} - u_{s,0})(T_{\mathbf{s}} - u_{s,1}) = 0 \rangle$$

We can *normalize* the generators so that $u_{s,0} = q$, $u_{s,1} = -1$ thus $(T_{\mathbf{s}} - q)(T_{\mathbf{s}} + 1) = 0$.

Example

type G_7 :



$$
\begin{aligned}
B &= \langle \mathbf{s}, \mathbf{t}, \mathbf{u} \mid \mathbf{stu} = \mathbf{tus} = \mathbf{ust} \rangle \\
W &= \langle s, t, u \mid stu = tus = ust, s^2 = t^3 = u^3 = 1 \rangle \\
\mathcal{H} &= \langle T_\mathbf{s}, T_\mathbf{t}, T_\mathbf{u} \mid T_\mathbf{s} T_\mathbf{t} T_\mathbf{u} = T_\mathbf{t} T_\mathbf{u} T_\mathbf{s} = T_\mathbf{u} T_\mathbf{s} T_\mathbf{t}, \\
&\quad (T_\mathbf{s} - u_{s,0})(T_\mathbf{s} - u_{s,1}) = 0, (T_\mathbf{t} - u_{t,0})(T_\mathbf{t} - u_{t,1})(T_\mathbf{t} - u_{t,2}) = 0, \\
&\quad (T_\mathbf{u} - u_{u,0})(T_\mathbf{u} - u_{u,1})(T_\mathbf{u} - u_{u,2}) = 0 \rangle
\end{aligned}
$$

we can normalize so the Hecke algebra relations become
$(T_\mathbf{s} - q)(T_\mathbf{s} + 1) = (T_\mathbf{t} - 1)(T_\mathbf{t} - u)(T_\mathbf{t} - v) = (T_\mathbf{u} - 1)(T_\mathbf{u} - u')(T_\mathbf{u} - v') = 0$
but there are still 5 indeterminates.

Tits' deformation thorem

It follows from the freeness conjecture, the trace conjecture and Tits' deformation theorem that representations of $\mathcal{H}$ on the integral closure $\overline{R}$ of R specialize for $u_{j,k} \mapsto \exp 2i\pi k/e_j$ to the irreducible representations of W. By [Malle 1999], for each $\rho \in \mathrm{Irr}(\mathcal{H})$ we know the extension of R needed, obtained by taking some roots of indeterminates and adding some roots of unity.

(Trace conjecture)

There exists a section $\mathbf{W}$ *of* W *in* B *such that*

- $T_\mathbf{1} = 1$, *and* $\{T_\mathbf{w} \mid \mathbf{w} \in \mathbf{W}\}$ *is an* R*-basis of* $\mathcal{H}$.
- *Let* $\mathrm{Tr} : \mathcal{H} \to R$ *be the linear form "coefficient on* $T_\mathbf{1}$*". Then the matrix* $\{\mathrm{Tr}(T_\mathbf{w} T_{\mathbf{w}'})\}_{\mathbf{w},\mathbf{w}'}$ *is symmetric and invertible over* R.
- $\mathrm{Tr}(T_\mathbf{w}) = \delta_{w,1}$ *for* $\mathbf{w} \in \mathbf{W}$.
- $\mathrm{Tr}(T_{\mathbf{w}^{-1}\boldsymbol{\pi}}) = 0$ *for* $\mathbf{w} \in \mathbf{W} - \{1\}$.

where $\boldsymbol{\pi}$ is a generator of the center of the pure braid group $\Pi_1(V^{\mathrm{reg}})$, which is cyclic.

Simple models?

The third item above says that $\mathcal{H}$ is a symmetric R-algebra. Conversely knowing the irreducible representations of $\mathcal{H}$ have models over $\overline{R}$ helps show the trace conjecture.

(Conjecture)

All irreducible representations of $\mathcal{H}$ have very simple models over $\overline{R}$.

For $G(de, e, r)$ Ariki (1995) gave models over $\mathrm{Frac}(\overline{R})$, not over $\overline{R}$.
Models over $\overline{R}$ are already not known for $D_r = G(2, 2, r)$.
For Coxeter groups the conjecture would result from knowing explicit W-graphs for irreducible representations.
By Kazhdan-Lusztig theory left cells are given by W-graphs but these are irreducible only in types A_n.

W-graphs

For a Coxeter system (W, S) with one conjugacy class of reflections, in Kazhdan-Lusztig theory the Hecke algebra is normalized such that $(T_\mathbf{s} - v)(T_\mathbf{s} + v^{-1}) = 0$ where $v^2 = q$.
A W-graph is a finite graph with vertices Γ and integral weights $\mu(\gamma, \delta)$ for each edge $\gamma, \delta \in \Gamma$. In addition for each $s \in S$ there is a function $\gamma \mapsto s(\gamma)$ which associates one of the possible eigenvalues of $T_\mathbf{s}$, so that $s(\gamma) \in \{v, -v^{-1}\}$. This defines on the vector space of basis Γ the representation:

$$T_\mathbf{s}(\gamma) = \begin{cases} s(\gamma)\gamma & \text{if } s(\gamma) = -v^{-1} \\ s(\gamma)\gamma + \sum_{\delta \mid s(\delta) \neq v} \mu(\delta, \gamma)\delta & \text{if } s(\gamma) = v \end{cases}$$

These matrices satisfy automatically the relations $(T_\mathbf{s} - v)(T_\mathbf{s} + v^{-1}) = 0$.
The graph is a W-graph if is they satisfy the braid relations.

Exceptional groups

W-graphs have been constructed for ireducible representations of the Hecke algebras of H_3, H_4, E_6, E_7 and E_8 (Howlett-Yin 2010). Also for a suitable definition for 2-parameter algebras for F_4.
A tentative definition has been given in (Malle-Michel 2010) for complex reflection groups:

- Define functions $s(\gamma)$ associating to γ an eigenvalue $u_{s,i}$.
- Choose a total order on the $u_{s,i}$.
- Put 0 in some entries of a matrix for $T_{\mathbf{s}}$ depending on this order, unknowns $\mu(\delta, \gamma)$ in the other entries.

A representation has a "pre-W-graph" if there is such a graph compatible with restrictions to parabolic subgroups and W-graphs already constructed on these parabolic subgroups. Make it a W-graph by solving the equations given by the braid relations for the unknowns $\mu(\delta, \gamma)$.
Almost all representations of G_{25} and G_{26} have a pre-W-graph and the equations can be solved. For G_{32}, 12 representations do not have a pre-W-graph. For the 90 which have, the equations have been solved only for 30 of them.

Rank 2 exceptional groups

These 3 groups have the same braid group

For the other groups in $G_4, \ldots, G_{22}$, the Hecke algebra is a subalgebra of the Hecke algebra of one of these groups where some generators have been specialized to the group algebra, and the representations are restrictions of specializations of the representations of one the algebras $\mathcal{H}(G_7)$, $\mathcal{H}(G_{11})$ or $\mathcal{H}(G_{19})$.
The representations are in dimension 1,2,3 for $\mathcal{H}(G_7)$, dimension $1, \ldots, 4$ for $\mathcal{H}(G_{11})$ and dimension $1, \ldots, 6$ for $\mathcal{H}(G_{19})$. The representations of $\mathcal{H}(G_{11})$ and $\mathcal{H}(G_7)$ are special cases of those of $\mathcal{H}(G_{19})$ which don't use all eigenvalues. Up to Galois action (permutations of the $u_{s,i}$) there is one representation of $\mathcal{H}(G_{19})$ in each dimension, so at the end it is sufficient to find one representation of $\mathcal{H}(G_{19})$ in each dimension 1 to 6.

A example: dimension 3

$$\mathbf{s}_1 \mapsto \begin{pmatrix} x_1 & 0 & ((z_2z_3+z_1z_3+z_1z_2)x_2x_1r^{-1}-\frac{(y_3+y_1+y_2)r}{y_1y_2y_3})z_1^{-1} \\ 0 & x_1 & -r(y_1y_2y_3z_1)^{-1} \\ 0 & 0 & x_2 \end{pmatrix},$$

$$\mathbf{s}_2 \mapsto \begin{pmatrix} y_1+y_2+y_3-r(x_1z_1)^{-1} & az_1{}^{-1} & -1 \\ 1 & r(x_1z_1)^{-1} & 0 \\ y_1y_2y_3x_1z_1r^{-1} & 0 & 0 \end{pmatrix},$$

$$\mathbf{s}_3 \mapsto \begin{pmatrix} 0 & 0 & z_2z_3x_1r^{-1} \\ 0 & z_1 & 0 \\ -rx_1{}^{-1} & a & z_3+z_2 \end{pmatrix},$$

where
$a=(y_3+y_1+y_2)rx_1^{-1}-(y_1y_3+y_1y_2+y_3y_2)z_1+y_1y_2y_3(x_1z_1^2-x_2z_2z_3)r^{-1}$
and where $r=\sqrt[3]{x_1^2x_2y_1y_2y_3z_1z_2z_3}$.

Marin-Pfeiffer

The remaining non-Coxeter exceptional groups of rank ≥ 3 are $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$. Their reflections are of order 2 and are all conjugate, so their Hecke algebra have one parameter q.
We managed in Malle-Michel to construct the representations of G_{24} and G_{27} (rank 3) by ad hoc methods, including W-graphs and Hensel lifting.
Since then I could handle G_{29}, G_{33}, most representations of G_{31}, and "small" (dimension ≤ 100) representations of $\mathcal{H}(G_{34})$.
The main tool is the freeness proof of Marin and Pfeiffer which constructed explicitely $\mathcal{H}(W)$ as a free $\mathcal{H}(W')$-module where W' is some maximal parabolic subgroup of W.
We have $|G_{29}/B_3|=160$, $|G_{31}/G(4,2,3)|=240$, $|G_{33}/D_4|=270$ and $|G_{34}/G_{33}|=756$.

Cutting representations

To get, for instance, the representations of $\mathcal{H}(G_{29})$ we get by Marin-Pfeiffer, if ρ is a representation of $H(B_3)$ of dimension d, a model (of dimension $160d$) of the induced representation from $H(B_3)$ to $H(G_{29})$ of ρ. Assume a representation σ of $H(G_{29})$ occurs with multiplicity 1 in this induced; how to get a model of σ?
The element $\boldsymbol{\pi}$ acts by a scalar on irreducible representations. If we are lucky σ is an eigenspace of $\boldsymbol{\pi}$ in the induced representation.
Even this computation with polynomials is too costly. We specialize q to a prime number and do the specialized computation. Then, if the representation simplifies enough by spinning, we can lift again. This fails for 10 of the 60 irreducibles of $\mathcal{H}(G_{31})$.
For some representations we can only find a model of the sum of 2 copies. We compute the commuting algebra and try to find an idempotent in it. In the specialized version this means find an integral solution to a bunch of quadratic equations. There is a Pari-GP package of Denis Simon for that.

Spinning (hint of Richard Parker)

Let $m_1, \ldots, m_n$ be a list of matrices over $\mathrm{Frac}(R)$ defining an irreducible representation.
How to find (heuristically) a basis where the matrices are simple and over R?

- Chose a “simple” vector v (for example a “simple” row of one of the m_i).
- Compute the iterated images of v under the m_i until one gets a basis.
- scale the vectors, clearing out the denominators, at each step.

Try different vectors and/or repeat until one gets “simpler“ matrices.