Parabolic Normalizers as Subdirect Products

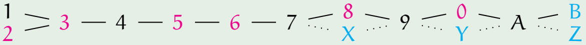
Götz Pfeiffer

School of Mathematical and Statistical Sciences
University of Galway, Ireland

Lie Theory in Prato
January 10-13, 2023

A Motivating Example.

- Let $P \leq W = W(\mathrm{D}_{12})$ be a parabolic of type A_{2211}:



- Then $\mathrm{N} = \mathrm{N}_W(\mathrm{P}) = (\mathrm{P} \times \mathrm{Q}) : \mathrm{D}$, where Q is a parabolic of type D_{211}, and D of size 16 is generated by 4 matrices:

$$
\left[\begin{array}{cccccc|cc|cccc}
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & 1 & . & & & & \\
 & & & & & & . & -1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
\quad
\left[\begin{array}{cccccc|cc|cccc}
. & . & . & 1 & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & -1 & . & & & & \\
 & & & & & & . & 1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
$$

$$
\left[\begin{array}{cccccc|cc|cccc}
1 & . & . & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
\hline
 & & & & & & 1 & . & & & & \\
 & & & & & & . & 1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
\quad
\left[\begin{array}{cccccc|cc|cccc}
. & 1 & . & . & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & . & 1 & & & & \\
 & & & & & & 1 & . & & & & \\
\hline
 & & & & & & & & . & . & . & 1 \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & 1 & . & . & .
\end{array}\right]
$$

- It turns out that $\mathrm{D} = (A \times \mathrm{B}) : \mathrm{C}$.
- Why?

Overview.

- The normalizer $\mathrm{N} := \mathrm{N}_W(\mathrm{P})$ of a parabolic subgroup P of a finite Coxeter group W has the form

$$\mathrm{N} = (\mathrm{P} \times \mathrm{Q}) : \mathrm{D}, \qquad \mathrm{D} = (\mathrm{A} \times \mathrm{B}) : \mathrm{C}.$$

P. Parabolic Subgroups and Howlett Complements.
Q. Orthogonal Complements and a Galois Connection.
D. Direct Products and Goursat Isomorphisms.
A. Orthogonal Closure.
B. Parabolic Closure.
C. Closure.
N. Concluding Remarks.

Joint w/ Gerhard Röhrle and J. Matthew Douglass

P. Parabolic Subgroups.

- Let (W, S) be a finite Coxeter system, acting as reflection group on Euclidean space $V = \mathbb{R}^n$.
- A **parabolic subgroup** of W is a subgroup of the form $W_J = \langle J \rangle$, $J \subseteq S$, or any of its W-conjugates.

- If $U \leq V$ then $Z_W(U)$ is a parabolic subgroup of W.

- We may assume that W is **irreducible**, i.e., of type A_n, B_n, D_n, or of exceptional type E_n, F_4, G_2, H_n, $I_2(m)$.

- Conjugacy classes of parabolics correspond to **shapes**:
- A_n: $2^S/\sim \longleftrightarrow \{\lambda \vdash n+1\}$.
- B_n: $\{\lambda \vdash m \mid 0 \leq m \leq n\}$.
- D_n: $\{\lambda \vdash m \mid m \neq n-1\}$, with 2 classes $\lambda_\pm$ if $\lambda \vdash n$ is even.
- Exceptional types: explicit lists (of up to 41 classes).

P. Howlett Complements.

- **Howlett's Lemma.** Let W be a reflection group on $V = \mathbb{R}^n$ with $W \trianglelefteq G \leq O(V)$. Then W has a complement H in G.

- More precisely, if $\Phi = \Phi^+ \cup \Phi^-$ is a **root system** for W then $H = \{a \in G \mid \Phi^+.a \subseteq \Phi^+\}$ is a complement of W.
- In particular, a **parabolic subgroup** P of a finite Coxeter group W has a complement H in its normalizer $N_W(P)$.
- Usually, H is itself a reflection group on $X = \mathrm{Fix}_V(P)$ …

- Howlett (1980): $H = H' : H''$, where $H' = Q : U$ is a reflection group with parabolic U, and $Q = P^\dagger$ (see below).

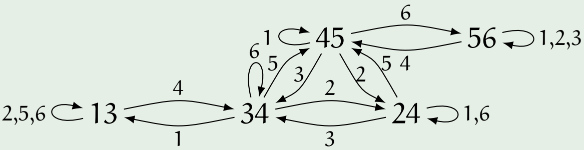
- For $P = W_J$, denote the normalizer complement H by N_J.

P. A Groupoid with Objects $J \subseteq S$.

- For $J \subseteq S$, denote by w_J the **longest element** of W_J.
- Consider the **labelled directed graph** with vertices $J \subseteq S$, and edges $J \xrightarrow{s} K$ if $s \in S \setminus J$ and $K = J^{w_J w_L}$, $L = J \cup \{s\}$.

- $J \sim K$ iff J and K are in the same connected component.

- **Example.** $J = \{1, 3\}$ in E_6 with diagram $1 - 3 - \overset{\overset{2}{|}}{4} - 5 - 6$.



- ⇝ a **presentation** of N_J as the automorphism group of J.
- In particular, N_J is generated by **involutions** $\sim w_K w_L$ . . .

Q. A Galois Connection.

- Let $\mathrm{T} := S^W$ be the set of **all reflections** in W.
- Define the **orthogonal complement** $\mathrm{P}^\dagger$ of a parabolic P as
$$\mathrm{P}^\dagger := \langle r \in \mathrm{T} \mid [r, s] = 1 \text{ for all } s \in \mathrm{P} \cap \mathrm{T} \rangle.$$

- **Proposition.** $\mathrm{P}^\dagger = Z_W(\mathrm{Fix}_V(\mathrm{P})^\perp)$ is a parabolic subgroup.

- Set $\mathrm{Q} := \mathrm{P}^\dagger$. Then $\mathrm{P} \times \mathrm{Q} \leq \mathrm{N} := N_W(\mathrm{P})$.

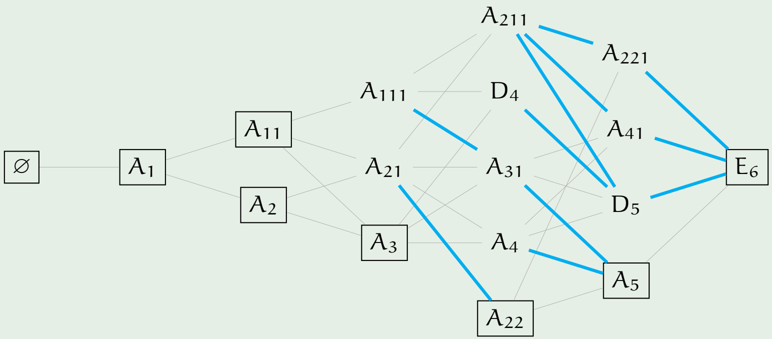
- **Theorem.** The map $\mathrm{P} \mapsto \mathrm{P}^\dagger$ is an **antitone Galois connection** on the set of parabolic subgroups of W.

- **Proof.** $\mathrm{P} \leq \mathrm{P}^{\dagger\dagger}$ and $\mathrm{P}_1 \leq \mathrm{P}_2 \implies {\mathrm{P}_2}^\dagger \leq {\mathrm{P}_1}^\dagger$. □
- Thus $\mathrm{P}^{\dagger\dagger\dagger} = \mathrm{P}^\dagger$. And $\mathrm{P}^{\dagger\dagger\dagger\dagger} = \mathrm{P}^{\dagger\dagger}$.
- Denote $\hat{\mathrm{P}} = \mathrm{P}^{\dagger\dagger}$ the **orthogonal closure** of the parabolic P.

Q. Galois Shapes.

- $(P^x)^\dagger = (P^\dagger)^x \implies \dagger$ is a Galois connection on $2^S / \sim$.
- And $P \mapsto \hat{P}$ is a closure operator on the shapes $2^S / \sim$.

- **Example:** Closed Parabolics in E_6.



Q. Orthogonal Pairs.

Theorem. Orthogonal pairs of shapes of closed parabolics.

- A_n: $\frac{A_n}{\varnothing}$ and $\frac{A_{n-m}}{A_{m-1}}$, $m \geq 2$.
- B_n: $\frac{B_m A_1^k}{B_l A_1^k}$, $n = m + l + 2k$.
- D_n: $\frac{D_m A_1^k}{D_l A_1^k}$, $n = m + l + 2k$, and A_1^k, $n = 2k + 1$,

  or $(A_1^{2k})_+$, $(A_1^{2k})_-$, $n = 4k$, or $\frac{(A_1^{2k+1})_+}{(A_1^{2k+1})_-}$, $n = 4k + 2$.
- E_6: $\frac{E_6}{\varnothing}$, $\frac{A_5}{A_1}$, $\frac{A_{22}}{A_2}$, $\frac{A_3}{A_{11}}$. E_7: $\frac{E_7}{\varnothing}$, $\frac{D_6}{A_1}$, $\frac{A_5'}{A_2}$, $\frac{D_{41}}{A_{11}}$, $\frac{A_{31}'}{A_3}$, $\frac{D_4}{A_{111}'}$, $\frac{A_{1111}}{A_{111}''}$.

  E_8: $\frac{E_8}{\varnothing}$, $\frac{E_7}{A_1}$, $\frac{E_6}{A_2}$, $\frac{D_6}{A_{11}}$, $\frac{D_5}{A_3}$, $\frac{A_5}{A_{21}}$, $\frac{D_{41}}{A_{111}}$, D_4, A_4, A_{31}, A_{22}, A_{1111}.

  F_4: $\frac{F_4}{\varnothing}$, $\frac{B_3}{\tilde{A}_1}$, $\frac{\tilde{B}_3}{A_1}$, $\frac{A_2}{\tilde{A}_2}$, A_{11}, B_2.

  H_3: $\frac{H_3}{\varnothing}$, $\frac{A_{11}}{A_1}$. H_4: $\frac{H_4}{\varnothing}$, $\frac{H_3}{A_1}$, A_{11}, A_2, $I_2(5)$.

- $\hat{P}$ is the smallest closed parabolic that contains P ...

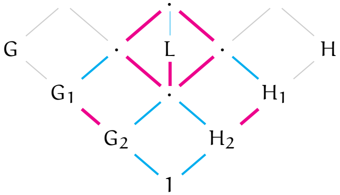
D. Goursat Isomorphisms.

- Let $G_2 \trianglelefteq G_1 \leq G$, $H_2 \trianglelefteq H_1 \leq H$ be **sections** of groups G, H.
- The **graph** of a section isomorphism $\theta \colon G_1/G_2 \xrightarrow{\sim} H_1/H_2$ is

$$\{gh \in G_1 \times H_1 \mid (G_2 g)^\theta = H_2 h\} \subseteq G \times H$$

- **Goursat's Lemma.** A subset L of $G \times H$ is a **subgroup** if and only if it is the graph of an **isomorphism** θ of sections.

- **Proof.** The graph of $\theta \colon G_1/G_2 \xrightarrow{\sim} H_1/H_2$ is a subgroup L. Conversely, $L \leq G \times H$ is the graph of such a θ, where $G_1 = \{g \mid gh \in L\}$, $H_1 = \ldots$ are **projections** with **kernels** $H_2 = L \cap H$, $G_2 = \ldots$, and $\theta \colon G_2 g \mapsto H_2 h$ for $gh \in L$.



D. Parabolic Normalizers as Subdirect Products.

- $\mathrm{X} := \mathrm{Fix}_{\mathrm{V}}(\mathrm{P})$. Then $\mathrm{V} = \mathrm{X} \oplus \mathrm{X}^{\perp}$ and $\mathrm{N} \leq \mathrm{GL}(\mathrm{X}) \times \mathrm{GL}(\mathrm{X}^{\perp})$.
- **Kernels:** $\mathrm{P} = \mathrm{N} \cap \mathrm{GL}(\mathrm{X}^{\perp})$ and $\mathrm{Q} = \mathrm{N} \cap \mathrm{GL}(\mathrm{X})$.
- **Projections:** $\mathrm{Q} : \mathrm{D}$ and $\mathrm{P} : \mathrm{D}$ are (isomorphic to) the Howlett complements of P and Q in N. (Note that $\mathrm{Q} \trianglelefteq \mathrm{N}$.)

- **Proposition.** $\mathrm{N} = (\mathrm{P} \times \mathrm{Q}) : \mathrm{D}$

- **Proof.** D is the Howlett complement of $\mathrm{P} \times \mathrm{Q}$ in N. □

- **Example.** In E_6, a parabolic P of type A_{111} has Q of type A_1 and D isomorphic to a Coxeter group of type A_2.

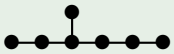
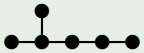
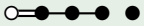
  It turns out that $\mathrm{P} : \mathrm{D}$ acts as a reflection group of type B_3 on $\mathrm{X}^{\perp}$, and $\mathrm{Q} : \mathrm{D}$ as a reflection group of type A_{21} on X.

	P	P : D	D	Q : D	Q	
●═○—○	A_{111}	B_3	A_2	A_{21}	A_1	○—○ ●

D. Special Case: Involution Centralizers [Serre 2021].

- If $u \in W$ with $u^2 = 1$ then $C_W(u)$ normalizes a parabolic.
- Richardson 1982: $N_W(W_J) = C_W(u) \iff$ "$u = -1 \in W_J$".

- **Example.** Involution Centralizers $N_W(P) = C_W(u)$ in E_7.

	P	$P:D$	D	$Q:D$	Q	
	1	1	1	E_7	E_7	
•	A_1	A_1	1	D_6	D_6	
	A_{11}	B_2	A_1	B_{41}	D_{41}	
	A'_{111}	B_3	A_2	F_4	D_4	
	A''_{111}	B_3	A_2	B_{31}	A_{1111}	
	A_{1111}	B_{31}	$\dots$			

- Nice. But: Is D always a reflection group? On X? On $X^\perp$?
- And is PQ always a maximal rank reflection subgroup?

A. Orthogonal Closure.

- Set $Y := \mathrm{Fix}_V(Q)$. Then $V = Y \oplus Y^{\perp}$, where $Y^{\perp} \leq X$.
- Thus $X = (X \cap Y) \oplus Y^{\perp}$ and $D \leq \mathrm{GL}(X \cap Y) \times \mathrm{GL}(Y^{\perp})$.
- **Goursat:** $D = (A \times B).C$, where $A = Z_D(Y^{\perp}) \leq \mathrm{GL}(X \cap Y)$ and $B = Z_D(X \cap Y) \leq \mathrm{GL}(Y^{\perp})$.

- **Proposition.** A is the Howlett complement of P in $\hat{P} \cap N$.

- Hence $A = 1$ if $P = \hat{P}$.

- **Example.** In A_n, P of type $A_1^{l_2} A_2^{l_3} \cdots A_n^{l_n}$ has $N = P : A \times Q$ with $A \cong \mathfrak{S}_{l_2} \times \cdots \times \mathfrak{S}_{l_n}$ and $Q \cong \mathfrak{S}_{l_1}$. Moreover, A acts as a reflection group on $X \cap Y$.

- In general, A acts faithfully on $X^{\perp}$ (permuting the simple roots of P) and on $X \cap Y$, not always as a reflection group.

B. Parabolic Closure.

- For $U \leq W$ let $\overline{U} := Z_W(\mathrm{Fix}_V(U))$ be its **parabolic closure**.

- **Proposition.** $B = Z_D(X \cap Y) \leq GL(Y^{\perp})$ is the Howlett complement of PQ in $N \cap \overline{PQ}$.

- Hence $B = 1$ if the reflection subgroup PQ is parabolic.

- **Example.** In B_n, P with label $[1^{l_1} 2^{l_2} \dots m^{l_m}] \vdash m \leq n$ has $N = (P \times Q) : (A \times B)$, where A has type $B_{l_3} \cdots B_{l_m}$, $B \cong \mathfrak{S}_{l_2}$ and Q has type $B_{l_1} A_1^{l_2}$. Note that $B : Q \cong B_{l_1} B_{l_2}$.

- In general, B acts faithfully on $X^{\perp}$ (permuting the simple roots of P), and on $Y^{\perp}$ (permuting the simple roots of Q).
- Usually, B is a reflection group on $X^{\perp}$, $Y^{\perp}$, so that $P : B$, $Q : B$ are reflection groups with parabolic subgroup B …

C. Closure.

- $C \neq 1$ occurs only in type E_7, E_8, D_n, $n \geq 5$.

- If $C \neq 1$ then $C = \langle c \rangle \leq N$, where c comes from a **graph automorphism** $w_J w_L$ in a situation like $J = \{3, 4, 5, 6\}$ (of type A_k, k even) and $L = \{1, 2, \ldots, 6\}$ (of type D_{k+2}):

  provided that $c \notin \hat{P}$.

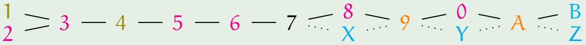
- The involution c acts as a reflection on both $X \cap Y$ and $Y^\perp$, but not on $X = (X \cap Y) \oplus Y^\perp$ (and in general not on $X^\perp$) ...

- **Example.** A_2 in D_5.

- The smallest example with all of $A, B, C \neq 1$ is A_{2211} in D_{12}.

N. Example A_{2211} in D_{12} Revisited.

- P of type A_{2211} in D_{12}:



- $Q = P^\dagger$ has type D_{211}; $\widehat{P}$ has type D_{611}; $\overline{PQ}$ has type D_{622}.
- $D = (A \times B) : C$ is generated by matrices a_1, a_2, b, c:

$$
\left[\begin{array}{cccccc|cc|cccc}
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & 1 & . & & & & \\
 & & & & & & . & -1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
\quad
\left[\begin{array}{cccccc|cc|cccc}
. & . & . & 1 & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & -1 & . & & & & \\
 & & & & & & . & 1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
\quad
\left[\begin{array}{cccccc|cc|cccc}
1 & . & . & . & . & . & & & & & & \\
. & 1 & . & . & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
\hline
 & & & & & & 1 & . & & & & \\
 & & & & & & . & 1 & & & & \\
\hline
 & & & & & & & & 1 & . & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & . & 1
\end{array}\right]
\quad
\left[\begin{array}{cccccc|cc|cccc}
. & 1 & . & . & . & . & & & & & & \\
1 & . & . & . & . & . & & & & & & \\
. & . & 1 & . & . & . & & & & & & \\
. & . & . & 1 & . & . & & & & & & \\
. & . & . & . & 1 & . & & & & & & \\
. & . & . & . & . & 1 & & & & & & \\
\hline
 & & & & & & . & 1 & & & & \\
 & & & & & & 1 & . & & & & \\
\hline
 & & & & & & & & . & . & . & 1 \\
 & & & & & & & & . & 1 & . & . \\
 & & & & & & & & . & . & 1 & . \\
 & & & & & & & & 1 & . & . & .
\end{array}\right]
$$

- $A = \langle a_1, a_2 \rangle$, both $a_1, a_2 \sim w_K w_L$ of type A_{2211} in A_{511} in $\widehat{P}$.
- $B = \langle b \rangle$, where $b \sim w_K w_L$ of type A_{2211} in A_{322} in $\overline{PQ}$.
- $C = \langle c \rangle$, where $c \sim w_K w_L$ of type A_{2211} in D_{4211} not in $\widehat{P}$.

N. More About Involution Centralizers.

- Suppose $N = N_W(P) = C_W(u)$ for $u \in W$ with $u^2 = 1$.
- Then $X = \mathrm{Fix}_V(P) = \ker(u - 1)$ and $X^\perp = \ker(u + 1)$.
- Serre [2021] observes, case by case, that

  $C_W(u)$ is generated by involutions of degree ≤ 2.

- **Corollary.** $C_W(u)$ acts as a reflection group on X and $X^\perp$.

- **Proof.** $C_W(u) = N = (P \times Q) : D$, where D is generated by involutions of degree 2. Thus
- $D = A \times B$,
- $P : (A \times B)$ is a reflection group on $X^\perp$,
- and $Q : B$ is a reflection group on X. □

N. Example E_6.

	P	Q	$\lvert D\rvert$	$\overline{PQ}$	A	B	C	$X^{\perp}$	$X\cap Y$	$Y^{\perp}$
7	A_3	3	2	16		A_1				
∗3	A_{11}	7	2	16		A_1				
6	A_{21}	4	1	(PQ)						
9	A_{22}	4	2	(W)		A_1		2 ()		
4	A_2	9	2	(W)		A_1				2 ()
∗5	A_{111}	2	6	12	A_2					●
10	A_{31}	2	1	16						●
11	A_4	2	1	(PQ)						●
15	A_5	2	1	(W)						●
∗2	A_1	15	1	(W)				●		
8	A_{211}	∅	2	(PQ)	A_1			2 ()	○	
∗12	D_4	∅	6	(PQ)	A_2					
13	A_{221}	∅	2	(PQ)	A_1			2 ()	○	
14	A_{41}	∅	1	(PQ)						
16	D_5	∅	1	(PQ)						
17	E_6	∅	1	(W)						
∗1	∅	17	1	(W)						

N. More on Orthogonally Closed Parabolics.

- If $P = \hat{P}$ then $N = N_W(P) = N_W(Q) = (P \times Q) : B$.

- **Proof.** $\overline{PQ} = W$; or $\frac{A_3}{A_{11}}$ in E_6; A_1^k in D_{2k+1}; $\frac{A_{n-m}}{A_{m-1}}$ in A_n. □
- In fact, if $w_0 = -1$ then $P = \hat{P} \implies \overline{PQ} = W$. Moreover: If $w_0 = -1$ then $N_W(P) = C_W(u)$, $u^2 = 1$, implies $P = \hat{P}$.
- In type B_n, the converse also holds: normalizers of orthogonally closed parabolics are involution centralizers.

- **Theorem.** Apart from some known exceptions,
$$N = (P : A \times Q) : B,$$
where

  A is the Howlett complement of P in $\hat{P}$, and

  B is the Howlett complement of $\hat{P}Q$ (in W).