Stretched non-positive Weyl connections on solvable Lie groups

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Weyl connections

A Weyl connection $\hat{\nabla}$ on a manifold M is determined by a 1-form φ satisfying the equality

$$\hat{\nabla}_X g = -2\varphi(X)g.$$

One derives the formula

$$\hat{\nabla}_X Y = \nabla_X Y + \varphi(Y)X + \varphi(X)Y - g(X,Y)E,$$

where E is dual to φ and ∇ is the Levi-Civita connection.

Gaudouchon representation

- Weyl connections locally coincide with Levi-Civita connections in *the conformal class*.
- by Gaudouchon, there is a uniqie Riemannian metric in the conformal class such that E has the property $\mathrm{div}\ E = 0$.

Gaudouchon representation

The unique pair (g, E) is the *Gaudouchon representation*.

Gaussian thermostats

A Gaussian thermostat is an ordinary differential equation on the tangent bundle to a Riemannian manifold of the form

$$\frac{dx}{dt} = v, \nabla_v v = E - \frac{g(E,v)}{g(v,v)} v, \quad (1)$$

where $x(t)$ is a parametrized curve in M, $v \in TM$.

Observation

The trajectories of the Gaussian thermostat are geodesics of a Weyl connection (M. Wojtkowski)

Gaussian thermostats are used in creating interesting models in nonequilibrium statistical mechanics. In these models, the vector field E contributes to the kinetic energy, while the second term of the thermostat equation retains the kinetic energy to be constant.

We want:

Understand Gaussian thermostats from the geometric point of view, as a geometric structure $(M, g, E, \hat{\nabla})$.

Curvature tensor of the Weyl connection

The curvature tensor of the Weyl connection:

$$\hat{R}(X,Y) = \hat{\nabla}_X \hat{\nabla}_Y - \hat{\nabla}_Y \hat{\nabla}_Y - \hat{\nabla}_{[X,Y]}.$$

Decomposition into symmetric and antisymmetric parts:

$$\hat{R}_a(X,Y)Z = R(X,Y)Z + \langle Z,E\rangle(\langle Y,E\rangle X - \langle X,E\rangle Y)$$
$$+(\langle Z,Y\rangle\langle X,E\rangle - \langle Z,X\rangle\langle Y,E\rangle)E + E^2(\langle Z,X\rangle Y - \langle Z,Y\rangle X)$$
$$+\langle Z,\nabla_X E\rangle Y - \langle Z,\nabla_Y E\rangle X + \langle Z,X\rangle \nabla_Y E - \langle Z,Y\rangle \nabla_X E.$$
$$\hat{R}_s(X,Y)Z = (\langle Y,\nabla_X E\rangle - \langle X,\nabla_Y E\rangle)Z = -d\varphi(X,Y)Z.$$

Sectional curvature

$\Pi = \langle X, Y \rangle \subset TM$, $X \perp Y, |X| = |Y| = 1$,

Definition

$$\hat{K}(\Pi) = \langle \hat{R}_a(X, Y)Y, X \rangle$$

The sectional curvature $\hat{K}(\Pi)$ depends on the choice of the Riemannian metric in the conformal class, but the sign does not.

Weyl connections, Gaussian thermostats and non-positive curvature

Problem

Describe manifolds with Weyl connections of non-positive sectional curvature $\hat{K}(\Pi)$ = describe Gaussian thermostats (M, g, E) of non-positive sectional curvature.

Some motivation from dynamical systems

Theorem (Wojtkowski)

If the all sectional curvatures of the Weyl connection are negative, then the geodesic flow has a dominated splitting with exponential growth/decay of volumes.

Corollary

For 2-dimensional manifolds one obtains the Anosov property of the geodesic flow.

Known

- Anosov property of electrical fields on surfaces of negative curvature, established by Bonetto, Gentile and Mastropietro,
- Some properties of the Gaussian thermostats with Anosov property on surfaces were established by Paternain.

Some physical motivation: Weyl geometry

Space-time version of Weyl geometry in the theories of gravity, quantum mechanics, elementary particle physics and cosmology:

E. Scholz, *The unexpected resurgence of Weyl geometry in late XX-century physics*, in: Beyond Einstein, Springer, 2018

However...

There are few examples of higher-dimensional (compact) manifolds with Weyl connections of negative sectional curvature:

$$\mathrm{Sol}_3$$

$$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle$$

$$[e_2, e_3] = e_1, [e_3, e_1] = e_2, [e_1, e_2] = e_3$$

with 2 (different) connections (T-W), one isolated, and one parametrized family.

Features of non-positivity: SNP condition

Definition

Let (M, g) be a Riemannian manifold. Let E be a vector field and $\gamma E, \gamma > 0$ be a family of Weyl connections defined by vector fields γE. A vector field E is called *streched non-positive* (SNP) if there exists $\gamma_0 \geq 0$ such that the Weyl connections defined by the fields γE are non-positive for all $\gamma \geq \gamma_0$. A Weyl connection on a compact Riemannian manifold M is called streched non-positive (SNP) if its Gauduchon representation E is SNP.

Note that the SNP condition is the property of the Gaussian thermostat itself. It says that there is a non-isolated family $(M, g, \gamma E)$ of Gaussian thermostats which yields non-positivity for the values of parameter γ exceeding some γ_0. In this sense, the classification of homogeneous spaces with invariant SNP Weyl connections answers the question: *what are Gaussian thermostats of the form $(G/H, g, \gamma E)$ which are non-positive for $\gamma \geq \gamma_0$?*

Homogeneous spaces and invariant Weyl connections

Natural problem

Describe *invariant* Weyl connections on Riemannian homogeneous spaces G/H, look for invariant Weyl connections with features of non-positivity.

Let G/H be a homogeneous Riemannian manifold with an orthogonal decomposion $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Let

$$\mathfrak{p}_0 = \{X \in \mathfrak{p} \,|\, [Y, X] = 0 \, \forall Y \in \mathfrak{h}\}.$$

The projection of an invariant vector field on G determined by $E \in \mathfrak{p}_0$ defines a Weyl connection on G/H .

Interesting: classics on homogeneous spaces with negative curvature

Theorem (Wolf)

Any hohomogeneous manifold endowed with a Riemannian metric of non-positive curvature is isometric to a solvable Lie group endowed with a left-invariant metric.

Theorem (Azencott-Wilson)

A solvable Lie group admits a left-invariant Riemannian metric if and only if its Lie algebra is an NC-algebra.

In particular, NC-algebra contains an abelian subalgebra complementary to the derived subalgebra.

Rigidity phenomena

Theorem (Alekseevsky, Heintze, A-W)

The only left-invariant Riemannian metrics on unimodular Lie groups are flat metrics.

NOTE: unimodularity is a necessary condition for a Lie group to admit a co-compact lattice, hence to get a compact manifold G/Γ with a Weyl connection of the negative curvature.

Natural problems

Are there analogues of these theorems for invariant Weyl connections with non-positive curvature?

Conditions on SNP (T-W)

Proposition

If a Weyl connection determined by a divergence-free vector field E on a compact manifold is SNP, then

- (W1) $K(\Pi) \leq 0$ for every plane Π containing $E \neq 0$,
- (W2) $\langle \nabla_Y E, Y \rangle = 0$ for every $Y \perp E$, $E \neq 0$.

SNP Weyl connections on unimodular Lie groups

Theorem (T-W)

For a unimodular Lie group G, if a left-invariant vector field $E \in \mathfrak{g}$ satisfies (W1) and (W2), then E is parallel.

Theorem (T-W)

For a unimodular Lie group G endowed with an invariant Riemannian metric g a left-invariant vector field $E \in \mathfrak{g}$ is SNP if and only if $\mathrm{ad}\ E$ is skew-symmetric and $E \perp [\mathfrak{g}, \mathfrak{g}]$.

Known

T-W looked for *isolated* examples of Gaussian thermostats (G, g, E) with non-positive Weyl curvature. The following cases were analyzed:

1. $\mathfrak{g} = \mathfrak{a} \oplus_\varphi \tilde{\mathfrak{a}}$, where $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ are abelian Lie algebras of dimesnions $n+1$ and n, respectively, and $\varphi : \mathfrak{a} \to \mathfrak{gl}(\tilde{\mathfrak{a}})$. The non-positivity of the Weyl connection determined by $E \in \mathfrak{a}$ is expressed in terms of the eigenvalues of $\varphi(E)$,
2. the case of a 3-dimensional Lie group, where non-positive Weyl connections were found on the 3-dimensional solvable Lie group Sol^3, which is semidirect product $\mathbb{R} \rtimes_\varphi \mathbb{R}^3$ determined by

$$\varphi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

SNP Weyl connections on solvable Lie groups of dimension 4

In this work we pose the following problems.

1. Find all invariant SNP Weyl connections on unimodular Lie groups of dimension 4.
2. Classify all unimodular solvable Lie groups admitting invariant Weyl connections with SNP property (thus, all solvable Lie groups which can yield SNP Gaussian thermostats $(G, g, \gamma E)$).

Classification of solvable Lie groups of dimension 4

1. Nilpotent Lie groups:

$$\mathbb{R}^4, \mathrm{Nil} \times \mathbb{R}, \mathrm{Nil}^4 .$$

2. Solvable Lie groups:

$$\mathrm{Sol}^4_{m,n}, \mathrm{Sol}^3 \times \mathbb{R}, \mathrm{Sol}^4_0, \mathrm{Sol}'^4_0, \mathrm{Sol}^4_\mu, \widetilde{\mathrm{Isom}_0(\mathbb{R}^2)} \times \mathbb{R}, \mathrm{Sol}^4_l, \widetilde{S^1 \rtimes \mathrm{Nil}^3}.$$

Groups $\mathrm{Nil} \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}^4_{m,n}, \mathrm{Sol}_3 \times \mathbb{R}, \mathrm{Sol}^4_0, \mathrm{Sol}'^4_0, \mathrm{Sol}^4_\mu$ and $\widetilde{\mathrm{Isom}_0(\mathbb{R}^2)} \times \mathbb{R}$ are all of the form

$$\mathbb{R} \rtimes_\varphi \mathbb{R}^3,$$

where φ is given as follows

Classification of solvable Lie groups of dimension 4, II

- Nil $\times \mathbb{R}$:

$$\varphi(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Nil^4 :

$$\varphi(t) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

- $\mathrm{Sol}^4_{m,n}$:

$$\varphi(t) = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-(1+\lambda)t} \end{pmatrix}$$

- $\mathrm{Sol}^3 \times \mathbb{R}$:

$$\varphi(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

Classification of solvable Lie groups of dimension 4, III

- Sol_0^4:

$$\varphi(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

- $\mathrm{Sol}_0'^4$:

$$\varphi(t) = \begin{pmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

- Sol_μ^4:

$$\varphi(t) = \begin{pmatrix} e^{\mu t}\cos t & e^{\mu t}\sin tt & 0 \\ -e^{\mu t}\sin t & e^{\mu t}\cos t & 0 \\ 0 & 0 & e^{-2\mu t} \end{pmatrix}$$

- $\widetilde{\mathrm{Isom}_0(\mathbb{R}^2)} \times \mathbb{R}$:

$$\varphi(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Classification of solvable Lie groups of dimension 4, IV

Sol_l^4 and $\widetilde{S^1 \rtimes_\varphi \mathrm{Nil}}$ are semidirect products $\mathbb{R} \rtimes_\varphi N_3$.

- Sol_1^4: $\varphi(t)$ acts on the unipotent matrix

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

  by the formula

$$\begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}$$

- $\widetilde{S^1 \rtimes_\varphi \mathrm{Nil}}$: $\varphi(t)$-action is given by the formula

$$\begin{pmatrix} 1 & x\cos t + y \sin t & z + p(x,y,t) \\ 0 & 1 & -x \sin t + y \cos t \\ 0 & 0 & 1 \end{pmatrix}$$

  where $p(x,y,t) = \frac{1}{2}(y^2 - x^2)\cos t \sin t - xy \sin^2 t$.

4-dimensional solvable Lie algebras

- algebraically: $\mathfrak{g} = \mathbb{R} \oplus_\varphi \mathbb{R}^3$, $\mathfrak{g} = \mathbb{R} \oplus_\varphi \ltimes_3$ with *standard* bases (known),
- example: for $\mathfrak{g}$ of the Lie group Sol_0^4:

$$\mathfrak{g} = \langle e_1, e_2, e_3, e_4 \rangle,$$

$$[e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_4, e_3] = -2e_3.$$

Classification of Invariant Riemannian metrics on solvable Lie groups of dimension 4, I

Method: classification up to automorphism of $\mathfrak{g}$

Determine an invariant scalar product $\langle -, - \rangle$ on $\mathfrak{g}$ by expressing the orthonormal base $X_1, ..., X_4$ ("Milnor base") through the standard base $e_1, ..., e_4$. Describe the representatives of the $\mathrm{Aut}(\mathfrak{g})$-orbits by writing down the family of Milnor bases.

Example

For Sol_0^4:

$$X_1 = e_1, X_2 = e_2, X_3 = b_{13}e_1 + e_3, X_4 = b_{44}e_4, b_{44} > 0, b_{13} \geq 0.$$

Method, cnd

The moduli space $\mathcal{M}$ of left invariant metrics on a Lie group G is $\tilde{\mathcal{M}}/\operatorname{Aut}(\mathfrak{g})$. One shows that

$$\mathcal{M} = \operatorname{Aut}(\mathfrak{g})\backslash GL(n, \mathbb{R})/O(n, \mathbb{R}).$$

Van Tuong: Ann. Global. Anal. Geom. 51(2016),109-128

Specializing $n = 4$ and quite long matrix calculations.

Main results in dimension 4

Theorem

A non-abelian solvable unimodular Lie group G admits an SNP Weyl connection if an only if it is one of the following:

$$\mathrm{Nil} \times \mathbb{R}, \widetilde{\mathrm{Isom}_0(\mathbb{R}^2)} \times \mathbb{R}, \widetilde{\mathrm{Nil} \rtimes S^1}, \mathrm{Sol}^3 \times \mathbb{R}.$$

Any of these groups admits a co-compact lattice Γ*, so any of solvmanifolds* G/Γ *is a compact 4-dimensional manifold with an SNP connection.*

Main results in dimension 4,II

Theorem

Alll possible (g, E) *determining invariant SNP Weyl connections:*

- $\mathrm{Nil} \times \mathbb{R}$, *two-parameter family*

$$\{be_1, e_2, e_3, e_4, b > 0\}, E = \alpha e_2,$$

- $\widetilde{\mathrm{Isom}_0(\mathbb{R}^2)} \times \mathbb{R}$, *three-parameter family*

$$\{e_1, be_2.e_3, e_3, ce_4, 0 < b < 1, c > 0\}, E = \alpha e_3,$$

- $\widetilde{\mathrm{Nil} \rtimes S^1}$ *four-parameter family*

$$\{ae_1, e_2, be_1 + ce_3, de_4, a, b, d > 0, 0 < c < 1\}, E = \alpha e_4,$$

- $\mathrm{Sol}^3 \times \mathbb{R}$, *three-parameter family*

$$\{e_1, e_2, be_3 + e_3, ce_4, b, c > 0\}, E = \alpha e_1.$$

Compact 4-solvmanifolds admiiting invariant SNP Weyl connections

Corollary

All 4-dimensional compact solvmanifolds with invariant SNP Weyl connections are exhaused by the list above.

How to prove

After the classification in terms of Milnor bases, use the [T-W] results on Weyl curvature which will translate into the identities:

$$\langle [E, Y], Y] \rangle,$$

$$\langle [Y, E], Z \rangle - \langle [Z, Y], E \rangle + \langle [Z, E], Y \rangle = 0.$$

Check these for each Milnor base.

Structural theorem

Theorem

Unimodular Lie group G admits an invariant SNP Weyl connection determined by a non-central $E \in \mathfrak{g}$ if and only if $\mathfrak{g}$ has the form

$$\mathfrak{g} = \langle E \rangle \oplus_{\varphi} \mathfrak{s},$$

where:

1. $\mathfrak{s}$ *is a unimodular solvable Lie algebra such that* $\mathrm{Aut}(\mathfrak{s})$ *contains a compact torus* T *of positive dimension,*
2. $\varphi : \langle E \rangle \to \mathrm{Der}(\mathfrak{s})$ *has image in* $\mathfrak{t} \subset \mathrm{Der}(\mathfrak{s})$, *that is* $\varphi(\langle E \rangle) \subset \mathfrak{t}$,
3. $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}] = [\mathfrak{g}, \mathfrak{g}]$.

CONCLUSION: We need unimodular solvable Lie algebras whose automorphism group contains a torus of positive dimension.

Vergne's types of Lie algebras and SNP Weyl connections

Definition

The Vergne type $\{d_1, ..., d_r\}$ of a nilpotent Lie algebra $\mathfrak{n}$ with descending central series $\mathfrak{n}^{(i)} = [\mathfrak{n}, \mathfrak{n}^{(i-1)}]$ is defined by

$$d_i = \dim\left(\mathfrak{g}^{(i-1)}/\mathfrak{g}^{(i)}\right).$$

Example

Nilpotent Lie algebras of type $\{n, 2\}$)-*Heisenberg* are nilpotent Lie algebras $V \oplus \langle x, y\rangle$ of dimension $n+2$ defined by a pair of alternating forms F_1 and F_2 on the n-dimensional vector space V putting for any $v, w \in V, [v, w] = F_1(v, w)x + F_2(v, w)y$.

Examples

Proposition

Any unimodular solvable Lie group which is a semidirect product $A \rtimes_\varphi N$ of an abelian Lie group A and a nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ has type $\{n, 2\}$-Heisenberg, admits an SNP Weyl connection.

Proposition

Any unimodular semidirect product $A \rtimes N$ of an abelian Lie group and a realification of a complex nilpotent Lie group of type $\{2n, 1, 1\}$ admits an SNP Weyl connection.

Metabelian Lie algebras

A finite dimensional Lie algebra $\mathfrak{g}$ is called *metabelian*, if $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. The *signature* of a metabelian Lie algebra is a pair (m, n), where $m = \dim \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, $n = \dim[\mathfrak{g}, \mathfrak{g}]$. Note that this is a particular Vergne class.

A metabelian Lie algebra structure on $\mathfrak{g}$ is completely determined by the commutator map $\Lambda^2 U \to V$, where $V = [\mathfrak{g}, \mathfrak{g}]$ and U is a complement in $\mathfrak{g}$ (different complements determine different structures). Conversely, let $\mathfrak{g} = U \oplus V$ be a direct sum of two vector spaces U and V of dimensions m and n. Then each skew symmetric bilinear surjective map $f : \Lambda^2 U \to V$ determines a metabelian Lie algebra structure on $\mathfrak{g}$ of signature (m, n).

The space of maps f is $\Lambda^2 U^* \otimes V$, and the group $GL(U) \times GL(V)$ naturally acts on it. Thus, a classification of metabelian Lie algebras can be understood in terms of the orbits of this group. Galitskii and Timashev found the canonical elements f determining the orbits.

Galitskii-Timashev classification of metabelian Lie algebras

Theorem

Let $\mathfrak{g}$ be a metabealian complex Lie algebra determined by $f : \Lambda^2 U \to V$. Then

$$\mathrm{Aut}(\mathfrak{g}) = G(f) \rtimes N$$

where N is a unipotent subgroup, and $G(f)$ denotes the $GL(U) \times GL(V)$-stabilizer.

GT-classification, tensor f

Treat $f \in \Lambda^2 U^* \otimes V$ as a tensor. Choose bases ${}_1e, \ldots, {}_me$ of U and $e_1, \ldots, e_n$ of V, and write the base of $\Lambda^2 U^* \otimes V$ in the form

$$ {}^{ij}e_k = e^i \wedge e^j \otimes e_k. $$

Note that dual elements are denoted by raising or lowering the indices. The tables in GT-paper contain the description of f in the dual form, so the following notation is used:

$$ (abc \ldots ijk) \text{ stands for } {}_{ab}e_c + \cdots +_{ij} e_k. $$

Lie groups with SNP Weyl connections in the GT-classification

Any metabelian Lie algebra over $\mathbb{C}$ of signature (m, n) such that $m, n \leq 5$ has an automorphism group which contains an algebraic torus of positive dimension with the following exceptions determined by the canonical choice of tensor f:

for $m, n \leq 5$:

132 521 415 354
125 144 153 234 243 252 342 351
125 134 153 233 243 252 342 451
125 135 144 152 234 242 251 343
125 134 143 152 233 244 342 451
125 143 154 233 242 251 341 352
125 132 144 153 234 243 252 351
125 134 141 153 243 252 342 351
121 144 153 234 243 252 342 451
125 134 143 152 233 242 251 341

Lie groups with SNP Weyl connections in the GT-classification,II

Theorem

Let $\mathfrak{n}$ be a realification of a complex metabelian Lie algebra of signature (m, n), $m, n \leq 5$ or $m \leq 6, n \leq 3$ such that its automorphism group contains an algebraic torus of positive dimension Then any solvable Lie group whose Lie algebra is a semidirect extension of the GT-algebra as above admits an SNP Weyl connection. Thus, 203 out of 223 Galitskii-Timashev classes of metabelian complex Lie algebras yield solvable Lie groups as semidirect products with SNP Weyl connections.

Solvable Lie groups with no SNP Weyl connections

Definition

W say that a nilpotent Lie algebra $\mathfrak{n}$ is *characteristically nilpotent of Dyer type*, if $\mathrm{Aut}(\mathfrak{n})$ is unipotent.

Proposition

No solvable Lie group whose Lie algebra is a semidirect product of an abelian and characteristically nilpotent Lie algebra of Dyer type admits an invariant SNP Weyl connection.

Example

$$[X_1, X_2] = X_3; \quad [X_1, X_3] = X_4; \quad [X_1, X_5] = X_7; \quad [X_1, X_8] = X_9;$$

$$[X_2, X_3] = X_5; \quad [X_2, X_4] = X_7; \quad [X_2, X_5] = X_6; \quad [X_2, X_7] = -X_8;$$

$$[X_3, X_7] = -[X_4, X_5] = X_9.$$

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