Constructive and algorithmic aspects of Hesselink-type stratifications

Valdemar Tsanov (Bulgarian Academy of Sciences)

Conference
Lie theory: frontiers, algorithms and applications
Monash Prato Center, 11. January 2023

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \ .$$

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \ .$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \ .$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,
semistable locus $X_U^{ss} = \{x \in X : \overline{U^c x} \cap \mathcal{M} \neq 0\}$.

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \ .$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,
semistable locus $X_U^{ss} = \{x \in X : \overline{U^c x} \cap \mathcal{M} \neq 0\}$.

$$\mathcal{M}/U \cong X_U^{ss}//U^c \cong Proj(\mathbb{C}[X]^U).$$

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \ .$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,
semistable locus $X_U^{ss} = \{x \in X : \overline{U^c x} \cap \mathcal{M} \neq 0\}$.

$$\mathcal{M}/U \cong X_U^{ss}//U^c \cong Proj(\mathbb{C}[X]^U).$$

Nullcone $\mathcal{N} = \mathcal{N}_U(X) = X \setminus X_U^{ss} = Z(\mathbb{C}[X]_+^U)$.

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v \rangle}{\langle v, v \rangle} \,.$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,
semistable locus $X_U^{ss} = \{x \in X : \overline{U^c x} \cap \mathcal{M} \neq 0\}$.

$$\mathcal{M}/U \cong X_U^{ss}//U^c \cong Proj(\mathbb{C}[X]^U).$$

Nullcone $\mathcal{N} = \mathcal{N}_U(X) = X \setminus X_U^{ss} = Z(\mathbb{C}[X]_+^U)$.

$$\bigcap_{\xi \in i\mathfrak{u}} \mathcal{M}_\xi = \mathcal{M} \quad \subset \quad X \quad \supset \quad \mathcal{N} = \bigcup_{\xi \in i\mathfrak{u}} \mathcal{N}_\xi$$

where $\mathcal{M}_\xi = \mu_\xi^{-1}(0)$ and $\mathcal{N}_\xi := \{x \in X : \mu_\xi(\lim_{t \to -\infty} e^{t\xi} x) > 0\}$.

$U \subset U(V)$, compact connected Lie group, acts on

$X \subset \mathbb{P}(V)$, projective U^c-variety, with momentum

$$\mu : X \to i\mathfrak{u}^* \quad , \quad \mu_\xi[v] = \frac{\langle \xi v, v\rangle}{\langle v, v\rangle} \ .$$

Kempf-Ness set $\mathcal{M} = \mathcal{M}_U(X) := \mu^{-1}(0)$,
semistable locus $X_U^{ss} = \{x \in X : \overline{U^c x} \cap \mathcal{M} \neq 0\}$.

$$\mathcal{M}/U \cong X_U^{ss}//U^c \cong Proj(\mathbb{C}[X]^U).$$

Nullcone $\mathcal{N} = \mathcal{N}_U(X) = X \setminus X_U^{ss} = Z(\mathbb{C}[X]_+^U)$.

$$\bigcap_{\xi \in i\mathfrak{u}} \mathcal{M}_\xi = \mathcal{M} \quad \subset \quad X \quad \supset \quad \mathcal{N} = \bigcup_{\xi \in i\mathfrak{u}} \mathcal{N}_\xi$$

where $\mathcal{M}_\xi = \mu_\xi^{-1}(0)$ and $\mathcal{N}_\xi := \{x \in X : \mu_\xi(\lim_{t \to -\infty} e^{t\xi} x) > 0\}$.

$$\mathcal{M} = \emptyset \iff \mathbb{C}[X]^U = \mathbb{C} \iff \mathcal{N} = X \ .$$

$\iota : G \subseteq U^c$ real reductive, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{q} = \mathfrak{g} \cap i\mathfrak{u}$.

$\iota : G \subseteq U^c$ real reductive, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{q} = \mathfrak{g} \cap i\mathfrak{u}$.

Gradient map, K-equivariant,

$$\mu_{\mathfrak{q}} = \iota^* \circ \mu_U : X \to i\mathfrak{u}^* \to \mathfrak{q}^* \ .$$

$\iota : G \subseteq U^c$ real reductive, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{q} = \mathfrak{g} \cap i\mathfrak{u}$.

Gradient map, K-equivariant,

$$\mu_{\mathfrak{q}} = \iota^* \circ \mu_U : X \to i\mathfrak{u}^* \to \mathfrak{q}^* \ .$$

$\mathcal{M}_{\mathfrak{q}} = \mu_{\mathfrak{q}}^{-1}(0)$, $X_G^{ss} = \{x \in X : \overline{Gx} \cap \mathcal{M}_{\mathfrak{q}} \neq \emptyset\}$.
$\mathcal{N}_{\mathfrak{q}} = X \setminus X_G^{ss} = \bigcup_{\xi \in \mathfrak{q}} \mathcal{N}_{\xi}$.

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m \in \mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m \in \mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi > 0})$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m \in \mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi>0})$$

$$\mathbb{P}(V)_{\xi,m} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi}[v] \in \mathbb{P}(V^{\xi,m})\}$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m\in\mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m\in\mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t\to-\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi>0})$$

$$\mathbb{P}(V)_{\xi,m} = \{[v] \in \mathbb{P}(V) : \lim_{t\to-\infty} e^{t\xi}[v] \in \mathbb{P}(V^{\xi,m})\}$$
$$= \mathbb{P}(V^{\xi\geq m}) \setminus \mathbb{P}(V^{\xi>m}).$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m\in\mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m\in\mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t\to-\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi>0})$$

$$\mathbb{P}(V)_{\xi,m} = \{[v] \in \mathbb{P}(V) : \lim_{t\to-\infty} e^{t\xi}[v] \in \mathbb{P}(V^{\xi,m})\}$$
$$= \mathbb{P}(V^{\xi\geq m}) \setminus \mathbb{P}(V^{\xi>m}).$$

$$\mathbb{P}(V)_{\xi,m} \to \mathbb{P}(V^{\xi,m}) \, , \; x \to \lim_{t\to-\infty} e^{t\xi}x.$$

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m \in \mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi > 0})$$

$$\mathbb{P}(V)_{\xi,m} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi}[v] \in \mathbb{P}(V^{\xi,m})\}$$
$$= \mathbb{P}(V^{\xi \geq m}) \setminus \mathbb{P}(V^{\xi > m}).$$

$$\mathbb{P}(V)_{\xi,m} \to \mathbb{P}(V^{\xi,m}) \, , \; x \to \lim_{t \to -\infty} e^{t\xi} x.$$

For a G-stable $X \subset \mathbb{P}(V)$, set $X^{\xi,m} = X \cap \mathbb{P}(V^{\xi,m})$ and

$X_{\xi,m} = X \cap \mathbb{P}(V)_{\xi,m} =: S_{\xi}(X^{\xi,m})$.

Example: $X = \mathbb{P}(V)$, $\mathfrak{g} = \mathbb{R}\xi \subset i\mathfrak{u}(V)$.

$$V = \bigoplus_{m \in \mathbb{R}} V^{\xi,m} \quad , \quad V^{\xi,m} = \{v \in V : \xi v = mv\}.$$

$$\mathbb{P}(V)^{\xi} = \bigcup_{m \in \mathbb{R}} \mathbb{P}(V^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi} v = 0\} = \mathbb{P}(V^{\xi > 0})$$

$$\mathbb{P}(V)_{\xi,m} = \{[v] \in \mathbb{P}(V) : \lim_{t \to -\infty} e^{t\xi}[v] \in \mathbb{P}(V^{\xi,m})\}$$
$$= \mathbb{P}(V^{\xi \geq m}) \setminus \mathbb{P}(V^{\xi > m}).$$

$$\mathbb{P}(V)_{\xi,m} \to \mathbb{P}(V^{\xi,m}) \, , \; x \to \lim_{t \to -\infty} e^{t\xi} x.$$

For a G-stable $X \subset \mathbb{P}(V)$, set $X^{\xi,m} = X \cap \mathbb{P}(V^{\xi,m})$ and

$$X_{\xi,m} = X \cap \mathbb{P}(V)_{\xi,m} =: S_{\xi}(X^{\xi,m}).$$

$$\mathcal{N}_{\xi} = \bigsqcup_{m > 0} S_{\xi}(X^{\xi,m}) \, .$$

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{q}} &= \bigcup_{\xi\in\mathfrak{q}} \mathcal{N}_\xi \\
&= G \bigcup_{\xi\in\mathfrak{a}} \mathcal{N}_\xi = G\mathcal{N}_{\mathfrak{a}} \\
&= \bigcup_{\xi\in\mathfrak{a}^+} G\mathcal{N}_\xi && , \mathcal{Z}_X = \{Z \subset X : \textit{conn.comp.of some } X^\xi \\
&= \bigcup_{\beta\in\mathcal{B}_X} G\mathcal{N}_\beta && , \mathcal{B}_X = \{\beta_Z \in \mathfrak{a} : Z \in \mathcal{Z}_X, \textit{addapted}\} \\
&= \bigcup_{(\beta,Z)\in\mathcal{BZ}_X} G\mathcal{S}_\beta(Z) && , \mathcal{BZ}_X = \{(\beta_Z, Z) : Z \in \mathcal{Z}_X, \beta_Z \in \mathfrak{a}^+\} \\
&= \bigsqcup_{(\beta,Z)\in\mathcal{S}_X} G\mathcal{S}_\beta(Z^{ss}_{G_\beta/\beta}) && , \mathcal{S}_X = \{(\beta_Z, Z) \in \mathcal{BZ}_X : Z^{ss}_{G_\beta/\beta} \neq \emptyset\} \; .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{q}} &= \bigcup_{\xi\in\mathfrak{q}} \mathcal{N}_\xi \\
&= G \bigcup_{\xi\in\mathfrak{a}} \mathcal{N}_\xi = G\mathcal{N}_{\mathfrak{a}} \\
&= \bigcup_{\xi\in\mathfrak{a}^+} G\mathcal{N}_\xi && , \mathcal{Z}_X = \{Z \subset X : \textit{conn.comp.of some } X^\xi \\
&= \bigcup_{\beta\in\mathcal{B}_X} G\mathcal{N}_\beta && , \mathcal{B}_X = \{\beta_Z \in \mathfrak{a} : Z \in \mathcal{Z}_X, \textit{addapted}\} \\
&= \bigcup_{(\beta,Z)\in\mathcal{B}\mathcal{Z}_X} G\mathcal{S}_\beta(Z) && , \mathcal{B}\mathcal{Z}_X = \{(\beta_Z, Z) : Z \in \mathcal{Z}_X, \beta_Z \in \mathfrak{a}^+\} \\
&= \bigsqcup_{(\beta,Z)\in\mathcal{S}_X} G\mathcal{S}_\beta(Z^{ss}_{G_\beta/\beta}) && , \mathcal{S}_X = \{(\beta_Z, Z) \in \mathcal{B}\mathcal{Z}_X : Z^{ss}_{G_\beta/\beta} \neq \emptyset\} .
\end{aligned}
$$

$$
G \times_{P_\beta} \mathcal{S}_\beta(Z) \to G\mathcal{S}_\beta(Z)
$$

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{q}} &= \bigcup_{\xi\in\mathfrak{q}} \mathcal{N}_{\xi} \\
&= G \bigcup_{\xi\in\mathfrak{a}} \mathcal{N}_{\xi} = G\mathcal{N}_{\mathfrak{a}} \\
&= \bigcup_{\xi\in\mathfrak{a}^+} G\mathcal{N}_{\xi} && , \ \mathcal{Z}_X = \{Z \subset X : \textit{conn.comp.of some } X^{\xi} \\
&= \bigcup_{\beta\in\mathcal{B}_X} G\mathcal{N}_{\beta} && , \ \mathcal{B}_X = \{\beta_Z \in \mathfrak{a} : Z \in \mathcal{Z}_X, \textit{addapted}\} \\
&= \bigcup_{(\beta,Z)\in\mathcal{B}\mathcal{Z}_X} G\mathcal{S}_{\beta}(Z) && , \ \mathcal{B}\mathcal{Z}_X = \{(\beta_Z, Z) : Z \in \mathcal{Z}_X, \beta_Z \in \mathfrak{a}^+\} \\
&= \bigsqcup_{(\beta,Z)\in\mathcal{S}_X} G\mathcal{S}_{\beta}(Z^{ss}_{G_\beta/\beta}) && , \ \mathcal{S}_X = \{(\beta_Z, Z) \in \mathcal{B}\mathcal{Z}_X : Z^{ss}_{G_\beta/\beta} \neq \emptyset\} \ .
\end{aligned}
$$

$$G \times_{P_\beta} \mathcal{S}_{\beta}(Z) \to G\mathcal{S}_{\beta}(Z)$$

$$\dim G\mathcal{S}_{\beta}(Z) \leq \dim G/P_{\beta} + \dim \mathcal{S}_{\beta}(Z).$$

$$
\begin{aligned}
\mathcal{N}_{\mathfrak{q}} &= \bigcup_{\xi\in\mathfrak{q}} \mathcal{N}_\xi \\
&= G \bigcup_{\xi\in\mathfrak{a}} \mathcal{N}_\xi = G\mathcal{N}_{\mathfrak{a}} \\
&= \bigcup_{\xi\in\mathfrak{a}^+} G\mathcal{N}_\xi & , \mathcal{Z}_X = \{Z \subset X : \textit{conn.comp.of some } X^\xi \\
&= \bigcup_{\beta\in\mathcal{B}_X} G\mathcal{N}_\beta & , \mathcal{B}_X = \{\beta_Z \in \mathfrak{a} : Z \in \mathcal{Z}_X, \textit{addapted}\} \\
&= \bigcup_{(\beta,Z)\in\mathcal{BZ}_X} G\mathcal{S}_\beta(Z) & , \mathcal{BZ}_X = \{(\beta_Z, Z) : Z \in \mathcal{Z}_X, \beta_Z \in \mathfrak{a}^+\} \\
&= \bigsqcup_{(\beta,Z)\in\mathcal{S}_X} G\mathcal{S}_\beta(Z^{ss}_{G_\beta/\beta}) & , \mathcal{S}_X = \{(\beta_Z, Z) \in \mathcal{BZ}_X : Z^{ss}_{G_\beta/\beta} \neq \emptyset\} .
\end{aligned}
$$

$$G \times_{P_\beta} \mathcal{S}_\beta(Z) \to G\mathcal{S}_\beta(Z)$$

$$\dim G\mathcal{S}_\beta(Z) \leq \dim G/P_\beta + \dim \mathcal{S}_\beta(Z).$$

$$Z^{ss}_{G_\beta/\beta} \neq \emptyset \implies \dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z).$$

A pair $(\beta, Z) \in \mathcal{B}\mathcal{Z}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

A pair $(\beta, Z) \in \mathcal{B}\mathcal{Z}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

- Root: $(0, X)$.

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

- Root: $(0, X)$.
- Arrows from the root to the candidates.

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

- Root: $(0, X)$.
- Arrows from the root to the candidates.
- Iterate.

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_{\mathfrak{q}} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

- Root: $(0, X)$.
- Arrows from the root to the candidates.
- Iterate.

Signature: $sign(leaf) = +$ and for a vertex a

$$sign(a) = - \overset{def}{\iff} \exists a \to b \quad with \quad sign(b) = +$$

A pair $(\beta, Z) \in \mathcal{BZ}_X$ is called

- fit, if $\dim G\mathcal{S}_\beta(Z) = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.
- a candidate, if $\dim X = \dim G/P_\beta + \dim \mathcal{S}_\beta(Z)$.

Rem: If $\mathcal{N}_\mathfrak{q} = X$, then there is a stratifying fit candidate.

Popov's tree $\mathcal{T}_X$ with signature: Vertices in $\mathcal{BZ}_X$, with

- Root: $(0, X)$.
- Arrows from the root to the candidates.
- Iterate.

Signature: $sign(leaf) = +$ and for a vertex a

$$sign(a) = - \overset{def}{\iff} \exists a \to b \quad with \quad sign(b) = +$$

Theorem (Popov, Ts.)

$$sign(root) = + \iff \mathcal{N}_\mathfrak{q} \neq X \ .$$

Computability:

Computability:

dim G/P_β are known numbers.

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

Computability:

$\dim G/P_\beta$ are known numbers.

$\dim \mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Complex flag variety: $X = \tilde{U}^c/\tilde{P}_{\tilde{\lambda}} \subset \mathbb{P}(V_{\tilde{\lambda}})$, $G = U^c$.

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Complex flag variety: $X = \tilde{U}^c/\tilde{P}_{\tilde{\lambda}} \subset \mathbb{P}(V_{\tilde{\lambda}})$, $G = U^c$.

$\mathcal{S}_\beta(Z)$ is a parabolic orbit: $\mathcal{S}_\beta(Z) = \tilde{P}_\beta x_{\tilde{w}}$ with $\tilde{w} \in \tilde{W}$.

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Complex flag variety: $X = \tilde{U}^c/\tilde{P}_{\tilde{\lambda}} \subset \mathbb{P}(V_{\tilde{\lambda}})$, $G = U^c$.

$\mathcal{S}_\beta(Z)$ is a parabolic orbit: $\mathcal{S}_\beta(Z) = \tilde{P}_\beta x_{\tilde{w}}$ with $\tilde{w} \in \tilde{W}$.

The conn. comp. of X^β are exactly the closed $\tilde{G}_\beta$-orbits.

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Complex flag variety: $X = \tilde{U}^c/\tilde{P}_{\tilde{\lambda}} \subset \mathbb{P}(V_{\tilde{\lambda}})$, $G = U^c$.

$\mathcal{S}_\beta(Z)$ is a parabolic orbit: $\mathcal{S}_\beta(Z) = \tilde{P}_\beta x_{\tilde{w}}$ with $\tilde{w} \in \tilde{W}$.

The conn. comp. of X^β are exactly the closed $\tilde{G}_\beta$-orbits.

$Z^{ss}_{G_\beta/\beta} \neq \emptyset \iff Z^{ss}_{G'_\beta} \neq \emptyset$

Computability:

dim G/P_β are known numbers.

dim $\mathcal{S}_\beta(Z)$ (or $\mathrm{codim}_X \mathcal{S}_\beta(Z)$) is a challenge, in general.

$Z^{ss}_{G_\beta/\beta}$ is an obstacle (back to square 1).

Complex flag variety: $X = \tilde{U}^c/\tilde{P}_{\tilde{\lambda}} \subset \mathbb{P}(V_{\tilde{\lambda}})$, $G = U^c$.

$\mathcal{S}_\beta(Z)$ is a parabolic orbit: $\mathcal{S}_\beta(Z) = \tilde{P}_\beta x_{\tilde{w}}$ with $\tilde{w} \in \tilde{W}$.

The conn. comp. of X^β are exactly the closed $\tilde{G}_\beta$-orbits.

$Z^{ss}_{G_\beta/\beta} \neq \emptyset \iff Z^{ss}_{G'_\beta} \neq \emptyset$

$$\mathcal{N} = \bigcup_{j=1}^{n} \bigcup_{\tilde{w}:\langle \tilde{w}\tilde{\lambda}|\beta_j\rangle > 0} G\tilde{P}_{\beta_j} x_{\tilde{w}} \qquad \text{for some special } \beta_1, \ldots, \beta_n$$

depending only on $G \subset \tilde{G}$ and not on $\tilde{\lambda}$.

Let $X = \tilde{U}^c / \tilde{B}$, with ample cone $\mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}}$, and $G = U^c \subset \tilde{U}^c$.

Let $X = \tilde{U}^c/\tilde{B}$, with ample cone $\mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}}$, and $G = U^c \subset \tilde{U}^c$.

G-ample cone $\mathcal{A}^G(X) = Cone\{\tilde{\lambda} \in \tilde{\Lambda}^+ : \exists n \in \mathbb{N}, V^G_{n\tilde{\lambda}} \neq 0\}$

Let $X = \tilde{U}^c / \tilde{B}$, with ample cone $\mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}}$, and $G = U^c \subset \tilde{U}^c$.

G-ample cone $\mathcal{A}^G(X) = Cone\{\tilde{\lambda} \in \tilde{\Lambda}^+ : \exists n \in \mathbb{N}, V^G_{n\tilde{\lambda}} \neq 0\}$

For $\tilde{\lambda} \in \tilde{\Lambda}^{++}$, $X \subset \mathbb{P}(V_{\tilde{\lambda}})$, and

Let $X = \tilde{U}^c/\tilde{B}$, with ample cone $\mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}}$, and $G = U^c \subset \tilde{U}^c$.

G-ample cone $\mathcal{A}^G(X) = Cone\{\tilde{\lambda} \in \tilde{\Lambda}^+ : \exists n \in \mathbb{N}, V^G_{n\tilde{\lambda}} \neq 0\}$

For $\tilde{\lambda} \in \tilde{\Lambda}^{++}$, $X \subset \mathbb{P}(V_{\tilde{\lambda}})$, and

$\dim \mathcal{N}_\beta = \max\{l(\tilde{w}) : \langle \tilde{w}\tilde{\lambda} | \beta \rangle > 0\}$

Let $X = \tilde{U}^c/\tilde{B}$, with ample cone $\mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}}$, and $G = U^c \subset \tilde{U}^c$.

G-ample cone $\mathcal{A}^G(X) = Cone\{\tilde{\lambda} \in \tilde{\Lambda}^+ : \exists n \in \mathbb{N}, V^G_{n\tilde{\lambda}} \neq 0\}$

For $\tilde{\lambda} \in \tilde{\Lambda}^{++}$, $X \subset \mathbb{P}(V_{\tilde{\lambda}})$, and

$\dim \mathcal{N}_\beta = \max\{l(\tilde{w}) : \langle \tilde{w}\tilde{\lambda}|\beta\rangle > 0\}$

Put $\ell^{\tilde{\lambda}} = \mathrm{codim}_X \mathcal{N}_{\tilde{\mathfrak{a}}} = \min\{l(\tilde{w}) : \tilde{w}\tilde{\lambda} \notin Cone\tilde{\Delta}^+\}$ and

$$\begin{aligned} \ell_{\tilde{U}} = \ell_{\tilde{\Delta}} &= \min\{\ell^{\tilde{\lambda}} : \tilde{\lambda} \in \tilde{\Lambda}^+ \setminus \{0\}\} \\ &= \min\{l(\tilde{w}) : \tilde{w}\tilde{\Lambda}^+ \not\subseteq Cone\tilde{\Delta}^+\} \end{aligned}$$

Theorem (Staneva, Ts.)

$$\begin{aligned} \#\Delta^+ < \ell_{\tilde{U}} &\implies \mathcal{M}_U(X, \tilde{\lambda}) \neq \emptyset \ , \ \forall \tilde{\Lambda}^+ \\ &\implies \mathcal{A}^G(X) = \mathcal{A}(X) = \tilde{\Lambda}^+_{\mathbb{R}} \end{aligned}$$

Theorem (Staneva, Ts.)

The values of ℓ_Δ *for simple root systems are*

Type of Δ	ℓ_Δ
$\mathbf{A}_n$	1
$\mathbf{B}_n$	n
$\mathbf{C}_n$	n
$\mathbf{D}_n,\ n \geq 4$	$n-1$, *for* $n \neq 5$ 3, *for* $n = 5$
$\mathbf{E}_6$	5
$\mathbf{E}_7$	10
$\mathbf{E}_8$	$7 \leq \ell_{\mathbf{E}_8} \leq 29$
$\mathbf{F}_4$	8
$\mathbf{G}_2$	3

Theorem (Staneva, Ts.)

The values of ℓ_Δ for simple root systems are

Type of Δ	ℓ_Δ
$\mathbf{A}_n$	1
$\mathbf{B}_n$	n
$\mathbf{C}_n$	n
$\mathbf{D}_n,\ n \geq 4$	$n-1$, for $n \neq 5$ 3, for $n = 5$
$\mathbf{E}_6$	5
$\mathbf{E}_7$	10
$\mathbf{E}_8$	$7 \leq \ell_{\mathbf{E}_8} \leq 29$
$\mathbf{F}_4$	8
$\mathbf{G}_2$	3

$\ell_\Delta^{sd} = \min\{\ell^\lambda : \lambda \in \Lambda_{sd}^+ \setminus \{0\}\}$, where $\Lambda_{sd}^+ = \{\lambda \in \Lambda^+ : -w_0\lambda = \lambda\}$.

Theorem (Staneva, Ts.)

The values of ℓ_Δ *for simple root systems are*

Type of Δ	ℓ_Δ
$\mathbf{A}_n$	1
$\mathbf{B}_n$	n
$\mathbf{C}_n$	n
$\mathbf{D}_n,\ n \geq 4$	$n-1$, for $n \neq 5$ 3, for $n = 5$
$\mathbf{E}_6$	5
$\mathbf{E}_7$	10
$\mathbf{E}_8$	$7 \leq \ell_{\mathbf{E}_8} \leq 29$
$\mathbf{F}_4$	8
$\mathbf{G}_2$	3

$\ell_\Delta^{sd} = \min\{\ell^\lambda : \lambda \in \Lambda_{sd}^+ \setminus \{0\}\}$, where $\Lambda_{sd}^+ = \{\lambda \in \Lambda^+ : -w_0\lambda = \lambda\}$.

$$w_0 = -1,\ \#\Delta^+ < \ell_{\tilde{U}}^{sd} \implies \mathcal{M}_U(X, \tilde{\lambda}) \neq \emptyset\ ,\ \forall \tilde{\Lambda}^+$$
$$\implies \mathcal{A}^G(X) = \mathcal{A}(X) = \tilde{\Lambda}_{\mathbb{R}}^+$$

Theorem (Staneva, Ts.)

The values of ℓ_Δ and ℓ_Δ^{sd} for simple root systems are

Type of Δ	ℓ_Δ	$\ell_\Delta^{\mathrm{sd}}$
$\mathbf{A}_n$	1	$\lfloor \frac{n+2}{2} \rfloor$
$\mathbf{B}_n$	n	
$\mathbf{C}_n$	n	
$\mathbf{D}_n,\ n \geq 4$	$n-1$, for $n \neq 5$ 3, for $n = 5$	$n-1$
$\mathbf{E}_6$	5	9
$\mathbf{E}_7$	10	
$\mathbf{E}_8$	$7 \leq \ell_{\mathbf{E}_8} \leq 29$	
$\mathbf{F}_4$	8	
$\mathbf{G}_2$	3	

Corollary

If $G \cong SL_2$ or E_8 and $\tilde{G}$ has no simple factors isomorphic to G, then

$$\mathcal{A}^G(\tilde{G}/\tilde{B}) \cong \mathcal{A}(\tilde{G}/\tilde{B}) .$$