Semisimple elements and the little Weyl group of real semisimple $\mathbf{Z}_m$-graded Lie algebras

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OUTLINE

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1. Algebraic θ-group and classification of homogeneous semisimple elements.

$$\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}_m} \mathfrak{g}_i$$

is a real $\mathbf{Z}_m$-graded semisimple Lie algebra,

$$\mathfrak{g}^{\mathbf{C}} := \bigoplus_{i \in \mathbf{Z}_m} \mathfrak{g}_i^{\mathbf{C}} \quad \text{where} \quad \mathfrak{g}_i^{\mathbf{C}} = \mathfrak{g}_i \otimes \mathbf{C}.$$

• $\widetilde{\mathbf{G}}$ denotes the simply connected algebraic group with Lie algebra $\mathfrak{g}^{\mathbf{C}}$. The automorphism θ of $\mathfrak{g}^{\mathbf{C}}$ that defines the $\mathbf{Z}_m$-grading on $\mathfrak{g}^{\mathbf{C}}$ can be lifted uniquely to an automorphism Θ of $\widetilde{\mathbf{G}}$.

$$\mathbf{G}_0 := \{g \in \widetilde{\mathbf{G}} | \, \Theta(g) = g\}$$

is a connected reductive group (by Steinberg's Theorem). $\mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{g}_0$ are reductive Lie algebras.

• $\mathbf{G}_0$ is defined over $\mathbf{R}$. Let $\mathbf{G}_0(\mathbf{R})$ denote the group of $\mathbf{R}$-points of $\mathbf{G}_0$.

Definition. The group $\mathrm{Ad}_{\mathbf{G}_0(\mathbf{R})}$ will be called the algebraic θ-group of a real semisimple $\mathbf{Z}_m$-graded Lie algebra $(\mathfrak{g}, \theta)$.

• Aim: investigate the equivalence classes of homogeneous semisimple elements of $\mathfrak{g}^{\mathbf{C}}$ and $\mathfrak{g}$ under the action of $\mathbf{G}_0 \sim \mathrm{Ad}_{\mathbf{G}_0}$ and $\mathbf{G}_0(\mathbf{R}) \sim \mathrm{Ad}_{\mathbf{G}_0(\mathbf{R})}$, respectively.

• $\mathfrak{h}^{\mathbf{C}} \subset \mathfrak{g}_1^{\mathbf{C}}$- a Cartan subspace (a maximal subspace of commuting semisimple elements)

• (Vinberg 1976) Two elements in $\mathfrak{h}^{\mathbf{C}}$ are conjugate if and only if and they are in the same orbit of the little Weyl group of $(\mathfrak{g}^{\mathbf{C}}, \theta)$ defined as follows

$$W := W(\mathfrak{g}^{\mathbf{C}}, \theta) := \frac{\mathcal{N}_{\mathbf{G}_0}(\mathfrak{h}^{\mathbf{C}})}{\mathcal{Z}_{\mathbf{G}_0}(\mathfrak{h}^{\mathbf{C}})}.$$

• Vinberg's theorem generalizes results by Kostant and Kostant-Rally for $m = 1$ and $m = 2$. His theorem gives us infinitely many orbits of homogeneous semisimple elements in $\mathfrak{g}_1^{\mathbf{C}}$.

• For $m = 1$ the "classification" of homogeneous elements in real reductive Lie algebras using the decomposition of a semisimple element into commuting elliptic and hyperbolic parts has been proposed by Rothschild in 1972. Note that Rothschild considered the action/orbit of the identity component of $\mathbf{G}_0(\mathbf{R})$.

• For $m = 2$ a scheme of classification homogeneous elements has been proposed by Lê in 2011, generalizing Rothschild's method.

• Rothschild and Lê' methods give infinite number of $\mathbf{G}_0(\mathbf{R})$-conjugacy classes of p.

• In 1978 Vinberg-Elashvili proposed a method to enumerate conjugacy classes of $\mathcal{Z}_{\mathfrak{g}^{\mathbf{C}}}(p)$ as the type of p. There is only a finite number of types. They showed that for $\mathfrak{g}^{\mathbf{C}} = e_8 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ there is a 1-1 correspondence between the $\mathbf{G}_0$-conjugacy classes of $\mathcal{Z}_{\mathfrak{g}^{\mathbf{C}}}(p)$ and the W-conjugacy classes of $W_p = \mathcal{Z}_W(p)$.

• We wish to classify homogeneous elements in $\mathfrak{g}$ using classification of homogeneous semisimple elements in $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \otimes \mathbf{C}$ and Galois theory, which asserts that the set of $\mathbf{G}_0(\mathbf{R})$-conjugacy classes in the $\mathbf{G}_0$-orbit of element $p \in \mathfrak{g}_1$ is in a canonical bijection with

$$\ker[\mathsf{H}^1 \mathcal{Z}_{\mathbf{G}_0}(p) \to \mathsf{H}^1(\mathbf{G}_0)].$$

We need to determine the $\mathbf{G}_0$-conjugacy class of $\mathcal{Z}_{\mathbf{G}_0}(p)$.

• The geometry of $\mathbf{G}_0(p)$ is defined by $\mathbf{G}_0$-conjugacy class of $\mathcal{Z}_{\mathbf{G}_0}(p)$.

2. The Vinberg-Elashvili proposal

$$\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}_m} \mathfrak{g}_i$$

- $\mathfrak{h}^{\mathrm{C}}$- a Cartan subspace in $\mathfrak{g}_1^{\mathrm{C}}$.

In order to apply Vinberg-Elashvili's proposal to classify homogeneous semisimple elements in $\mathfrak{g}_1^{\mathrm{C}}$ we need the validity of the following.

- Statement 1: Let $\mathfrak{g}$ be a $\mathbf{Z}_m$-graded semisimple complex Lie algebra and $p \in \mathfrak{h}^{\mathrm{C}}$. The $\mathbf{G}_0$-conjugacy class of $\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p)$ is determined uniquely by the W-conjugacy class of W_p.

• Statement 1 is true if $m = 1$, since in this case W_p is generated by (complex) reflection r_α, where α is a root of $\mathcal{Z}_{\mathfrak{g}^{\mathbb{C}}}(p)$.

• Statement 1 has been proved by Vinberg-Elashivili (1978) for $\mathbf{Z}_3$-graded Lie algebra e_8 by explicit computation of W_p and $\mathcal{Z}_{\mathfrak{g}^{\mathbb{C}}}(p)$. Later, using similar methods, Statement 1 has been verified for several cases of $\mathbf{Z}_2$-graded semisimple complex Lie algebras (Antonyan 1981, Antonyan-Elashvili 1982, Dietrich-de Graaf-Marrani-Origlia 2022).

3. Our main results

A complex semisimple $\mathbf{Z}_m$-graded Lie algebra $(\mathfrak{g}^{\mathbf{C}}, \theta)$ is called of maximal rank, if the centralizer $\mathcal{Z}_{\mathfrak{g}^{\mathbf{C}}}(\mathfrak{h}^{\mathbf{C}})$ of a Cartan subspace $\mathfrak{h}^{\mathbf{C}} \subset \mathfrak{g}_1^{\mathbf{C}}$ is a Cartan subalgebra in $\mathfrak{g}^{\mathbf{C}}$.

• **Theorem 1**(De Graaf-L. 2022) Statement 1 is true for complex semisimple $\mathbf{Z}_m$-graded Lie algebra $\mathfrak{g}$ if m is simple or $\mathfrak{g}$ is of maximal rank.

• Statement 1 has many important consequences, which we discovered during the proof of Theorem 1.

Outline of the proof and consequences

In the first step we shall show that if p, q are in the same $\mathfrak{g}^{\mathrm{C}}$-family, equivalently, if $\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p)$ and $\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(q)$ are $\mathbf{G}_0$-conjugate, then they are in the same W-family, equivalently, if W_p and W_q are W-conjugate (easy part). Then **in the second step** we show that under the condition of Theorem 1, if p, q are in the same W-family then they are in the same $\mathfrak{g}^{\mathrm{C}}$-family.

Proposition 1 Assume that two elements $p, q \in \mathfrak{h}^{\mathbf{C}}$ are in the same $\mathfrak{g}^{\mathbf{C}}$-family. Then their centralizers $\mathcal{Z}_{\mathbf{G}_0}(p)$, $\mathcal{Z}_{\mathbf{G}_0}(q)$ are $\mathbf{G}_0$-conjugate and p, q are in the same W-family.

Corollary 1. The type of $\mathbf{G}_0$-orbit of p, in particular the $\mathbf{G}_0$-conjugacy class of $\mathcal{Z}_{\mathbf{G}_0}(p)$, which we need for classification of real orbits using Galois cohomology, is defined by the $\mathfrak{g}^{\mathbf{C}}$-family of p.

Idea of the proof of Proposition 1. Using the Steinberg theorem, we show that if $p, q \in \mathfrak{h}^{\mathbf{C}}$ are in the same $\mathfrak{g}^{\mathbf{C}}$-family their centralizers $\mathcal{Z}_{\mathbf{G}_0}(p)$, $\mathcal{Z}_{\mathbf{G}_0}(q)$ are conjugate. Next use the Vinberg theorem, we infer from the first assertion of Proposition 1 the second assertion.

The second step of the proof of Theorem 1:

It suffices to show that if $W_p = W_q$ then $\mathcal{Z}_{\mathfrak{g}^{\mathbf{C}}}(p) = \mathcal{Z}_{\mathfrak{g}^{\mathbf{C}}}(q)$.

Lemma 1. Assume that a $\mathbf{Z}_m$-graded reductive complex Lie algebra $(\mathfrak{g}^{\mathrm{C}}, \theta)$ satisfies one of the condition of Theorem 1. Then for any $p \in \mathfrak{h}^{\mathrm{C}}$ we have $W_p = W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p), \theta)$. Hence if $p, q \in \mathfrak{h}^{\mathrm{C}}$ such that $W_p = W_q$ then $W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p), \theta) = W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(q), \theta)$.

<u>Outline of the proof</u>. The condition of Theorem 1 implies that every $\mathbf{G}_0$-orbit of p is invariant under the action of

$$\widetilde{\mathbf{G}}^{\theta}_{\mathcal{Z}} = \{g \in \widetilde{\mathbf{G}} | \, \Theta(g) = g \mod (\mathcal{Z}(\widetilde{\mathbf{G}}))\}.$$

Using this we show that $W_p = W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p), \theta)$.

• For the completion of the proof of Theorem 1 we need new notation. For $p \in \mathfrak{h}^{\mathbf{C}}$ we let

$$\mathfrak{h}_p^{\mathbf{C}} := \{q \in \mathfrak{h}^{\mathbf{C}} \mid W_p \subset W_q\},$$

$$\mathfrak{h}_p^{\mathbf{C},\circ} := \{q \in \mathfrak{h}_p^{\mathbf{C}} \mid W_q = W_p\}.$$

• $\Sigma(\mathfrak{h}_p^{\mathbf{C}})$ - the weight system of the adjoint representation of $\mathfrak{h}_p^{\mathbf{C}}$.

$$\mathfrak{h}_p^{\mathbf{C},\mathrm{reg}} := \{q \in \mathfrak{h}_p^{\mathbf{C}} | \, \sigma(q) \neq 0 \text{ for all } \sigma \in \Sigma(\mathfrak{h}_p^{\mathbf{C}}) \setminus \{0\}\}.$$

Elements in $\mathfrak{h}_p^{\mathbf{C},\mathrm{reg}}$ will be called $\Sigma(\mathfrak{h}_p^{\mathbf{C}})$-regular.

• Now assume the opposite, i.e. $\exists\, p, q$ s.t. (i) $W_p = W_q$ and (ii) $\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p) \neq \mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(q)$. W.l.o.g. we replace (ii) by (ii') $q \notin \mathfrak{h}^{\mathrm{C},\text{reg}}$

$$(\text{i}) \Longrightarrow (\mathfrak{h}_p^{\mathrm{C}} = \mathfrak{h}_q^{\mathrm{C}}) \Longrightarrow \exists p^{\text{reg}} \in (\mathfrak{h}_p^{\mathrm{C},\circ} \cap \mathfrak{h}_p^{\mathrm{C},\text{reg}})$$

$$W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(p^{\text{reg}}, \theta)) \overset{\text{Lemma 1}}{=} W_{p^{\text{reg}}} \overset{*}{=} W_q$$

$$\overset{\text{Lemma 1}}{=} W(\mathcal{Z}_{\mathfrak{g}^{\mathrm{C}}}(q), \theta),$$

which contradicts (ii').

This completes the proof of Theorem 1.

Consequences of Statement 1

Theorem 2. Assume that Statement 1 holds for a complex $\mathbf{Z}_m$-graded semisimple Lie algebra $(\mathfrak{g}, \theta)$. Then

(1) $q \in \mathfrak{h}_p^{\mathbf{C},\circ}$ then $\mathcal{Z}_{\mathbf{G}_0}(p) = \mathcal{Z}_{\mathbf{G}_0}(\mathfrak{h}_p^{\mathbf{C}})$.

(2) $\Gamma_p (:= \mathcal{N}_W(W_p)/W_p) = \dfrac{\mathcal{N}_{\mathbf{G}_0}(\mathfrak{h}_p^{\mathbf{C}})}{\mathcal{Z}_{\mathbf{G}_0}(\mathfrak{h}_p^{\mathbf{C}})}$

(3) Assume that $\mathsf{H}^1\mathbf{G}_0 = 1$. Let $\mathcal{O} = \mathbf{G}_0 \cdot p$. Write $\mathcal{N} = \mathcal{N}_{\mathbf{G}_0}(\mathfrak{h}_p^{\mathbf{C}})$, $\mathcal{Z} = \mathcal{Z}_{\mathbf{G}_0}(\mathfrak{h}_p^{\mathbf{C}})$. Write $\mathsf{H}^1\Gamma_p = \{[\gamma_1], \ldots, [\gamma_s]\}$, and assume that for $1 \leq i \leq s$ there is a cocycle $n_i \in \mathsf{Z}^1(\mathcal{Z})$ projecting to γ_i. Let $g_i \in \mathbf{G}_0$ be such that $g_i^{-1}\bar{g}_i = n_i$ for $1 \leq i \leq s$. Then $\mathcal{O}$ has an $\mathbf{R}$-point if and only if $\bar{p} = \gamma_i^{-1} \cdot p$ for some i. In that case $g_i \cdot p$ is an $\mathbf{R}$-point of $\mathcal{O}$.

• Let $R(W)$ be the subset of all complex reflections of W.

Theorem 2 (De Graaf-L. 2023). Let $(\mathfrak{g}^{\mathbf{C}}, \theta)$ be complex semisimple $\mathbf{Z}_m$-graded Lie algebra and $W = W(\mathfrak{g}, \theta, \mathfrak{h}^{\mathbf{C}})$. The following assertions are equivalent.

(1) Statement 1 is valid for $(\mathfrak{g}^{\mathbf{C}}, \theta)$.

(2) $\forall\ w \in R(W)\ \exists\ \sigma \in \Sigma(\mathfrak{h}^{\mathbf{C}})$ s.t. w preserves the hyperplane $\sigma = 0$.

(3) #(reflecting hyperplanes in W)= #(proportional roots in $\Sigma(\mathfrak{h}^{\mathbf{C}})$).

(4) $\mathfrak{h}_r^{\mathbf{C},\circ} = \mathfrak{h}_r^{\mathbf{C},\mathrm{reg}}$ for all $r \in \mathfrak{h}^{\mathbf{C}}$.

Idea of proof. For $A \subset R(W)$ we set

$$\mathfrak{h}_A^{\mathbf{C}} : \{p \in \mathfrak{h}^{\mathbf{C}} | \, w \in W_p \text{ for all } w \in A\}. \qquad (1)$$

$W(A)$- the subgroup of W generated by A. Then

$$\mathfrak{h}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C},\circ} \bigcup_{A \subset R(W)} \mathfrak{h}_A^{\mathbf{C},\circ} \cup \{0\}, \qquad (2)$$

where $\mathfrak{h}_A^{\mathbf{C},\circ} := \{p \in \mathfrak{h}^{\mathbf{C}} | \, W_p = W(A)\}$ if $W(A) \neq W$ and $h_A^{\mathbf{C},\circ} = \emptyset$ otherwise.

Proposition (1) The action of W on $\mathfrak{h}^{\mathbf{C}}$ preserves the stratification (2).

(2) p, q belong to the same W-family, iff their projections on the quotient space $\mathfrak{h}^{\mathbf{C}}/W$ belong to the same cell in the stratification of $\mathfrak{h}^{\mathbf{C}}/W$ induced by the stratification (2).

• For $\mathcal{A} \subset \Sigma(\mathfrak{h}^{\mathbf{C}})$ (the weight of $\mathfrak{h}^{\mathbf{C}}$ w.r.t. the adjoint representation) we let

$$\mathfrak{h}^{\mathbf{C}}_{\mathcal{A}} := \{p \in \mathfrak{h}^{\mathbf{C}} | \, \sigma(p) = 0 \text{ for all } \sigma \in \mathcal{A}\} \tag{3}$$

$$\mathfrak{h}^{\mathbf{C},\mathrm{reg}}_{\mathcal{A}} := \{p \in \mathfrak{h}^{\mathbf{C}}_{\mathcal{A}} | \, \sigma(p) \neq 0 \text{ for all } \sigma \in \Sigma(\mathfrak{h}^{\mathbf{C}}) \setminus \mathcal{A}\}. \tag{4}$$

Now we consider a stratification of $\mathfrak{h}^{\mathbf{C}}$

$$\mathfrak{h}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C},\mathrm{reg}} \bigcup_{A \subset \Sigma(\mathfrak{h}^{\mathbf{C}})} \mathfrak{h}^{\mathbf{C},\mathrm{reg}}_{A} \cup \{0\}, \tag{5}$$

where $\mathfrak{h}^{\mathbf{C},\mathrm{reg}}$ consists of all $\Sigma(\mathfrak{h}^{\mathbf{C}})$-regular points of $\mathfrak{h}^{\mathbf{C}}$.

Proposition (1) The action of the little Weyl group W on $\mathfrak{h}^{\mathbf{C}}$ preserves the stratification (5).

(2) Two points p, q belong to the same $\mathfrak{g}^{\mathbf{C}}$-family, if and only if their projection on $\mathfrak{h}^{\mathbf{C}}/W$ belong to the same cell of the stratification induced by (5).

(3) For every $\mathcal{A} \subset \Sigma(\mathfrak{h}^{\mathbf{C}})$ there exists $A \subset R(W)$ such that $\mathfrak{h}_{\mathcal{A}}^{\mathbf{C},\mathsf{reg}} \subset \mathfrak{h}_{A}^{\mathbf{C},\circ}$. In particular, for any $\sigma \in \Sigma(\mathfrak{h}^{\mathbf{C}})$ there exists $w \in R(W)$ such that w preserves the hyperplane $\sigma = 0$.

Remark: Theorem 2 suggests a way to verify Statement 1 for all complex $\mathbf{Z}_m$-graded semisimple Lie algebra.

4. Conjugacy classes of Cartan subspaces in $\mathbf{Z}_m$-graded semisimple Lie algebras over $\mathbf{R}$

Theorem 3 Let $\mathfrak{h}^{\mathbf{C}}$ be a Cartan subspace in $\mathfrak{g}^{\mathbf{C}}$ and $\mathcal{N}_0 := \mathcal{N}_{\mathbf{G}_0}(\mathfrak{h}^{\mathbf{C}})$. There is a canonical bijection between the conjugacy classes of Cartan subspaces in $\mathfrak{g}_1$ and the kernel $\ker[\mathrm{H}^1\mathcal{N}_0 \to \mathrm{H}^1\mathbf{G}_0]$.

Proposition All Cartan subspaces in $\mathfrak{g}_1$ are $\mathbf{G}_0(\mathbf{R})$-conjugate, if $\mathfrak{g}$ is a compact Lie algebra.

Proposition (1) If $p \in \mathfrak{h}$ is regular, then $G_0(p) \cap \mathfrak{g}_1$ consists of L $\mathbf{G}_0(\mathbf{R})$-orbits, where L is the number of conjugacy classes of Cartan subspaces in $\mathfrak{g}_1$.

(2) Assume that all Cartan subspaces in $\mathfrak{g}_1$ are conjugate. Let $p \in \mathfrak{h}^{\mathbf{C}}$. Then the orbit $\mathbf{G}_0(p)$ contains a real point in $\mathfrak{g}_1$ if and only if the orbit $W(p)$ contains a real point in $\mathfrak{h}$. For any $q \in \mathfrak{h}$ the set of $\mathbf{G}_0(\mathbf{R})$-orbits in $\mathbf{G}_0(q) \cap \mathfrak{g}_1$ is in a canonical bjection with the set

$$\ker[H^1 W_q \to H^1 W].$$

THANK YOU FOR YOUR ATTENTION!