A finite volume scheme for noncoercive Dirichlet problems with right-hand sides in H^{-1}

Jérôme Droniou* — **Thierry Gallouët****

**UMPA, ENS Lyon,*
46 allée d'Italie,
69364 Lyon cedex 07, France.
jdroniou@umpa.ens-lyon.fr

***CMI, Université de Provence,*
Technopôle de Château Gombert,
39 rue F.Joliot Curie,
13453 Marseille Cedex 13, France.
gallouet@cmi.univ-mrs.fr

ABSTRACT. *We prove the convergence of a finite volume scheme for convection-diffusion equations with right-hand sides in H^{-1}; the convection terms we consider are non-regular and can entail the loss of coercivity of the operator associated to the equation.*

KEYWORDS: *finite volume methods, noncoercive elliptic equations, little regular data.*

1. Introduction

Let Ω be a polygonal open subset of $\mathbb{R}^d$ ($d = 2$ or 3). The problem under study is

$$\begin{cases} -\Delta u + \operatorname{div}(\mathbf{v}u) = L & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad [1]$$

with $\mathbf{v} \in (L^p(\Omega))^d$ for some $p > d$ and $L \in H^{-1}(\Omega)$. We consider solutions to [1] in the classical weak sense (for which existence and uniqueness have been proved in [DRO 01]).

There are numerous works about the discretization of convection-diffusion problems with finite volumes methods, either on structured or unstructured meshes (see e.g. [GAL 00], [EYM 00]). We intend here to define (and of course prove the convergence of) a finite volume discretization of [1], using the same grids as in [EYM 00] (see next section).

Our work contains two main originalities. First, we consider right-hand sides with little regularity; previous papers take in general this right-hand side in $L^2(\Omega)$, but $H^{-1}(\Omega)$ is, both mathematically and physically speaking, a natural space for L. The second originality is in the convection term of [1]: $\mathbf{v}$ has little regularity too, but, above all, we impose no hypothesis on this datum (such as "$\mathrm{div}(\mathbf{v}) \geq 0$") to ensure that the problem is coercitive; to handle this last point, we adapt to the discrete setting the techniques of [DRO 01].

2. Definition of the scheme and main result

The idea of finite volumes methods is to integrate [1] on the elements of a discretization mesh of Ω and to find suitable approximations of the quantities appearing in this integration. Let us first give the geometrical properties we impose on the discretization mesh.

Definition 2.1 *An admissible mesh $\mathcal{T}$ of Ω is a finite family of polygonal open convex subsets of Ω (the "control volumes"), together with a finite family $\mathcal{E}$ of disjoint subsets of $\overline{\Omega}$ contained in affine hyperplanes (the "edges") and a family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ of points in Ω such that*

i) $\overline{\Omega} = \bigcup_{K\in\mathcal{T}} \overline{K}$,

ii) each $\sigma \in \mathcal{E}$ is a non-empty open subset of ∂K for some $K \in \mathcal{T}$,

iii) denoting $\mathcal{E}_K = \{\sigma \in \mathcal{E} \mid \sigma \subset \partial K\}$, we have $\partial K = \cup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$ for all $K \in \mathcal{T}$,

iv) for all $K \neq L$ in $\mathcal{T}$, either the $(d-1)$-dimensional measure of $\overline{K} \cap \overline{L}$ is null, or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, that we denote then $\sigma = K|L$,

v) for all $K \in \mathcal{T}$, $x_K \in K$,

vi) for all $\sigma = K|L \in \mathcal{E}$, the line (x_K, x_L) intersects and is orthogonal to σ,

vii) for all $\sigma \in \mathcal{E}$, $\sigma \subset \partial\Omega \cap \partial K$, the line which is orthogonal to σ and going through x_K intersects σ.

We define the size of the mesh by $\mathrm{size}(\mathcal{T}) = \sup_{K\in\mathcal{T}} \mathrm{diam}(K)$. $\mathbf{n}_{K,\sigma}$ *is the unit normal to $\sigma \in \mathcal{E}_K$ outward to K. We let $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E} \mid \sigma \not\subset \partial\Omega\}$ and $\mathcal{E}_{\mathrm{ext}} = \mathcal{E}\backslash\mathcal{E}_{\mathrm{int}}$. If $\sigma \in \mathcal{E}$, $m(\sigma)$ is the $(d-1)$-dimensional measure of σ; if $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, d_σ is the distance between the points (x_K, x_L) and $d_{K,\sigma}$ denotes the distance between x_K and σ; if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, $d_\sigma = d_{K,\sigma}$ is the distance between x_K and σ. The transmissibility through an edge σ is $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$. The following quantity measures the "regularity" of the mesh:*

$$\mathrm{reg}(\mathcal{T}) = \inf_{K\in\mathcal{T}} \left(\inf_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{d_\sigma} \right).$$

Since $L \in H^{-1}(\Omega)$, we can write $L = f + \mathrm{div}(G)$ with $f \in L^2(\Omega)$ and $G \in (L^2(\Omega))^d$ (in models of physical problems, L naturally appears in this form, see e.g. [FIA 94] — this is why we have kept f which, theoritically, can be taken equal to 0). Formally integrating L on a control volume K and using Stokes' formula, we find $\int_K f(x)\,dx - \sum_{\sigma\in\mathcal{E}_K}\int_\sigma G(x)\cdot\mathbf{n}_{K,\sigma}\,d\gamma(x)$ (γ is the $(d-1)$-dimensional measure on ∂K). The first term is not a problem to define since $f \in L^2(\Omega)$, but G is not regular enough for the second term to make sense; so we must introduce a suitable approximation of G on σ.

Let T be an admissible mesh; if $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, the "half-diamond" $\triangle_{K,\sigma}$ is defined by $\triangle_{K,\sigma} = \{tx_K + (1-t)x\,,\ t\in[0,1]\,,\ x\in\sigma\}$. Denoting

$$v_{K,\sigma} = \left(\frac{1}{\mathrm{meas}(\triangle_{K,\sigma})}\int_{\triangle_{K,\sigma}}\mathbf{v}(x)\,dx\right)\cdot\mathbf{n}_{K,\sigma}\,,\quad f_K = \frac{1}{\mathrm{meas}(K)}\int_K f(x)\,dx$$
$$\text{and}\quad G_{K,\sigma} = \left(\frac{1}{\mathrm{meas}(\triangle_{K,\sigma})}\int_{\triangle_{K,\sigma}}G(x)\,dx\right)\cdot\mathbf{n}_{K,\sigma}\,,$$

a finite volume dicretization of [1] is written

$$\begin{aligned}&\forall K\in\mathcal{T}\,,\\ &\sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma} + m(\sigma)v_{K,\sigma}u_{K,\sigma,+} = \mathrm{meas}(K)f_K + \sum_{\sigma\in\mathcal{E}_K}m(\sigma)G_{K,\sigma}\,,\end{aligned}\qquad [2]$$

$$\forall K\in\mathcal{T}\,,\forall\sigma\in\mathcal{E}_K\,,\quad F_{K,\sigma} = -\tfrac{m(\sigma)}{d_{K,\sigma}}(u_\sigma - u_K)\,,\qquad [3]$$

$$\begin{aligned}\forall\sigma = K|L\in\mathcal{E}_{\mathrm{int}}\,,\quad & F_{K,\sigma} + m(\sigma)v_{K,\sigma}u_{K,\sigma,+} - m(\sigma)G_{K,\sigma}\\ &= -(F_{L,\sigma} + m(\sigma)v_{L,\sigma}u_{L,\sigma,+} - m(\sigma)G_{L,\sigma})\,,\\ \forall\sigma\in\mathcal{E}_{\mathrm{ext}}\,,\quad & u_\sigma = 0\,,\end{aligned}\qquad [4]$$

$$\begin{aligned}\forall\sigma = K|L\in\mathcal{E}_{\mathrm{int}}\,,\quad & u_{K,\sigma,+} = u_K \text{ if } v_{K,\sigma}\ge 0\,,\ u_{K,\sigma,+} = u_L \text{ otherwise,}\\ \forall\sigma\in\mathcal{E}_{\mathrm{ext}}\cap\mathcal{E}_K\,,\quad & u_{K,\sigma,+} = u_K \text{ if } v_{K,\sigma}\ge 0\,,\ u_{K,\sigma,+} = 0 \text{ otherwise.}\end{aligned}\qquad [5]$$

Using [4] (conservativity of the fluxes) to eliminate the unknowns $(u_\sigma)_{\sigma\in\mathcal{E}}$, we see that [2]—[5] is a linear square system in $(u_K)_{K\in\mathcal{T}} \in \mathbb{R}^{\mathrm{Card}(\mathcal{T})}$ (we identify the set $\mathbb{R}^{\mathrm{Card}(\mathcal{T})}$ to the set $X(\mathcal{T})$ of functions defined a.e. on Ω and constant on each control volume $K\in\mathcal{T}$).

Our main result is the following.

Theorem 2.1 *If $\mathcal{T}$ is an admissible mesh, then there exists a unique solution to [2]—[5]. Moreover, let $\alpha > 0$; denoting by $u_\mathcal{T} \in X(\mathcal{T})$ the solution to [2]—[5], $u_\mathcal{T}$ converges, as* $\mathrm{size}(\mathcal{T})\to 0$ *with* $\mathrm{reg}(\mathcal{T})\ge\alpha$*, and in $L^q(\Omega)$ for all $q < \frac{2d}{d-2}$, to the unique weak solution of [1].*

Due to lack of room, we only give, in the following proofs, the main arguments; for more details, we refer the reader to [DRO 02].

3. A Priori Estimates

Let us first prove some *a priori* estimates on the solutions to [2]—[5], which will entail existence and uniqueness of a solution to this system as well as the convergence result, thanks to some compactness arguments.

These estimates are obtained in the discrete H^1-norm on $X(\mathcal{T})$, defined for $v_{\mathcal{T}} \in X(\mathcal{T})$ by $||v_{\mathcal{T}}||_{1,\mathcal{T}} = \left(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma v)^2\right)^{1/2}$, where $D_\sigma v_{\mathcal{T}} = |v_K - v_L|$ if $\sigma = K|L \in \mathcal{E}_{\rm int}$ and $D_\sigma v_{\mathcal{T}} = |v_K|$ if $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$. Let us notice two important properties of this norm (see [EYM 00]):

– Poincaré's inequality: on $X(\mathcal{T})$, we have $||\cdot||_{L^2(\Omega)} \le \mathrm{diam}(\Omega)||\cdot||_{1,\mathcal{T}}$.

– Sobolev's inequality: if $0 < \zeta \le \mathrm{reg}(\mathcal{T})$, there exists C only depending on ζ such that, on $X(\mathcal{T})$ and for all $q < \frac{2d}{d-2}$, we have $||\cdot||_{L^q(\Omega)} \le Cq||\cdot||_{1,\mathcal{T}}$.

Proposition 3.1 (Estimate on $\ln(1+|u_{\mathcal{T}}|)$) *There exists $C > 0$ such that, if $\mathcal{T}$ is an admissible mesh and $u_{\mathcal{T}} = (u_K)_{K\in\mathcal{T}}$ is a solution to [2]—[5], then*

$$||\ln(1+|u_{\mathcal{T}}|)||_{1,\mathcal{T}} \le C\left(||f||_{L^1(\Omega)} + ||G||_{(L^2(\Omega))^d} + ||\mathbf{v}||_{(L^2(\Omega))^d}\right).$$

PROOF. Let $\varphi(s) = \int_0^s \frac{dt}{(1+|t|)^2}$. We multiply [2] by $\varphi(u_K)$ and sum on the meshes $K \in \mathcal{T}$. Gathering by edges, using the Cauchy-Schwarz inequality and since φ is bounded by 1, we find

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma (u_K - u_L)(\varphi(u_K) - \varphi(u_L)) \qquad [6]$$

$$\le \quad ||f||_{L^1(\Omega)} + C||G||_{(L^2(\Omega))^d} \left(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (\varphi(u_K) - \varphi(u_L))^2\right)^{1/2} \qquad [7]$$

$$+\sum_{\sigma\in\mathcal{E}} m(\sigma)\left(\frac{d_{L,\sigma}}{d_\sigma} v_{L,\sigma} u_{L,\sigma,+} - \frac{d_{K,\sigma}}{d_\sigma} v_{K,\sigma} u_{K,\sigma,+}\right)(\varphi(u_K) - \varphi(u_L)) \; [8]$$

(with the notations — which we also use in the sequel of this paper — $\sigma = K|L$ if $\sigma \in \mathcal{E}_{\rm int}$ and $u_L = u_{L,\sigma,+} = v_{L,\sigma} = d_{L,\sigma} = G_{L,\sigma} = 0$ if $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$).

φ being nondecreasing and Lipschitz-continuous with Lipschitz constant 1, we have $(\varphi(u_K) - \varphi(u_L))^2 \le (u_K - u_L)(\varphi(u_K) - \varphi(u_L))$ and, thanks to Young's inequality, the second term of [7] is bounded by $C^2||G||^2_{(L^2(\Omega))^d}/2$ plus one half of [6].

To estimate [8], we first notice that all $\sigma \in \mathcal{E}_{\rm ext}$ in this sum give non-positive terms. Studying then, for $\sigma = K|L \in \mathcal{E}_{\rm int}$, each case (according to the signs of $v_{K,\sigma}$ and $v_{L,\sigma}$), we notice that the contribution of σ to this sum is bounded from above by 0 if $u_K u_L \le 0$ and by $m(\sigma)(|\frac{d_{L,\sigma}}{d_\sigma} v_{L,\sigma}| +$

$|\frac{d_{K,\sigma}}{d_\sigma}v_{K,\sigma}|)\inf(|u_K|,|u_L|)|\varphi(u_K)-\varphi(u_L)|$ otherwise. Thus, by denoting $\mathcal{A}=\{\sigma=K|L\in\mathcal{E}_{\rm int}\mid u_Ku_L>0\}$, [8] is bounded from above by

$$C||\mathbf{v}||_{(L^2(\Omega))^d}\left(\sum_{\sigma\in\mathcal{A}}\tau_\sigma\inf(|u_K|,|u_L|)^2(\varphi(u_K)-\varphi(u_L))^2\right)^{1/2}.$$

But it is easy to see that, if u_K and u_L have the same sign, then

$$\inf(|u_K|,|u_L|)^2(\varphi(u_K)-\varphi(u_L))^2\le(u_K-u_L)(\varphi(u_K)-\varphi(u_L)).$$

Thus, thanks to Young's inequality, [6] is bounded by $C'(||f||_1+||G||_2+||\mathbf{v}||_2)^2$. By construction of φ we have $(\ln(1+|u_K|)-\ln(1+|u_L|))^2\le(u_K-u_L)(\varphi(u_K)-\varphi(u_L))$ and this concludes the proof.

Proposition 3.2 (Estimate on $||u_\mathcal{T}||_{1,\mathcal{T}}$). *Let $\mathcal{T}$ be an admissible mesh and $0<\zeta\le\mathrm{reg}(\mathcal{T})$. There exists $C>0$ only depending on $(\Omega,\mathbf{v},\zeta)$ such that, if $u_\mathcal{T}$ is a solution to [2]—[5], then $||u_\mathcal{T}||_{1,\mathcal{T}}\le C(||f||_{L^2(\Omega)}+||G||_{(L^2(\Omega))^d})$.*

Proof. [2]—[5] being a linear system, it is sufficient to bound $u_\mathcal{T}$ in the case $||f||_{L^2(\Omega)}+||\,|G|\,||_{L^2(\Omega)}\le 1$. We denote, for $k>0$, $T_k(s)=\max(-k,\min(s,k))$ and $S_k(s)=s-T_k(s)$.

Let us first estimate $S_k(u_\mathcal{T})$ for k large enough. We have $(S_k(u_K)-S_k(u_L))^2\le(u_K-u_L)(S_k(u_K)-S_k(u_L))$; thus, multiplying [2] by $S_k(u_\mathcal{T})$, gathering by edges and using the Cauchy-Schwarz inequality, we find

$$||S_k(u_\mathcal{T})||^2_{1,\mathcal{T}}\le||f||_{L^2(\Omega)}||S_k(u_\mathcal{T})||_{L^2(\Omega)}+C||G||_{(L^2(\Omega))^d}||S_k(u_\mathcal{T})||_{1,\mathcal{T}} \quad [9]$$

$$+\sum_{\sigma\in\mathcal{E}}m(\sigma)\left(\frac{d_{L,\sigma}}{d_\sigma}v_{L,\sigma}u_{L,\sigma,+}-\frac{d_{K,\sigma}}{d_\sigma}v_{K,\sigma}u_{K,\sigma,+}\right)(S_k(u_K)-S_k(u_L)). \quad [10]$$

Thanks again to the Cauchy-Schwarz inequality, [10] is bounded by

$$\left(\sum_{\sigma\in\mathcal{E}}m(\sigma)d_\sigma\left(\frac{d_{L,\sigma}}{d_\sigma}v_{L,\sigma}u_{L,\sigma,+}-\frac{d_{K,\sigma}}{d_\sigma}v_{K,\sigma}u_{K,\sigma,+}\right)^2\right)^{1/2}||S_k(u_\mathcal{T})||_{1,\mathcal{T}}. \quad [11]$$

Gathering by control volumes and using Hölder's inequality (with $p/2>1$ and $p/(p-2)$), we notice that the first factor of [11] is bounded by

$$C_0||\mathbf{v}||_{L^p(\Omega)}\left(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)d_{K,\sigma}|u_{K,\sigma,+}|^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}}. \quad [12]$$

Using the definition of ζ and the fact that $\sum_{\sigma\in\mathcal{E}_K}m(\sigma)d_{K,\sigma}=d\,\mathrm{meas}(K)$, we have

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}m(\sigma)d_{K,\sigma}|u_{K,\sigma,+}|^{\frac{2p}{p-2}}\le\frac{d}{\zeta}||u_\mathcal{T}||^{\frac{2p}{p-2}}_{L^{\frac{2p}{p-2}}(\Omega)}. \quad [13]$$

Let $q \in]\frac{2p}{p-2}, \frac{2d}{d-2}[$ (recall that $p > d$). Since $|u_{\mathcal{T}}| \le k + |S_k(u_{\mathcal{T}})|$ and $S_k(u_{\mathcal{T}}) = 0$ outside $E_k = \{|u_{\mathcal{T}}| \ge k\}$, we have, thanks to Hölder's inequality and to the discrete Sobolev's inequalities, by denoting $\theta = \frac{p-2}{2p} - \frac{1}{q} > 0$,

$$||u_{\mathcal{T}}||_{L^{\frac{2p}{p-2}}(\Omega)} \le C_1 k + C_1 ||S_k(u_{\mathcal{T}})||_{L^{\frac{2p}{p-2}}(\Omega)} \le C_1 k + C_2 \mathrm{meas}(E_k)^\theta ||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}$$

where C_1 and C_2 do not depend on k nor $\mathcal{T}$. Using this last inequality in [13] and gathering with [9]-[10], [11] and [12], we deduce

$$||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}^2 \le ||S_k(u_{\mathcal{T}})||_{L^2(\Omega)} + C_3 ||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}$$
$$+ C_3 ||\mathbf{v}||_{(L^p(\Omega))^d} \left(k ||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}} + \mathrm{meas}(E_k)^\theta ||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}^2 \right). \quad [14]$$

But thanks to proposition 3.1, to the discrete Poincaré inequality and to Tchebycheff's inequality, $\mathrm{meas}(E_k) \le \frac{C_4}{\ln(1+|k|)^2}$ (where C_4 does not depend on k nor $\mathcal{T}$). Thus, taking k large enough (not depending on $\mathcal{T}$) in [14], we can bound $||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}$.

The estimate on $T_k(u_{\mathcal{T}})$ is quite straightforward (multiply [2] by $T_k(u_K)$, sum on $K \in \mathcal{T}$, gather by edges, use the fact that $T_k(u_K)$ is bounded by k, that $|u_{\mathcal{T}}| \le k + |S_k(u_{\mathcal{T}})|$ and that we have a bound on $||S_k(u_{\mathcal{T}})||_{1,\mathcal{T}}$), and the proof is completed by writing $u_{\mathcal{T}} = T_k(u_{\mathcal{T}}) + S_k(u_{\mathcal{T}})$.

4. Proof of Theorem 2.1

The existence an uniqueness of a solution to [2]—[5] is an immediate consequence of proposition 3.2, which shows that the square matrix defining this system is injective, thus bijective.

Using the same methods as in [EYM 00], we prove that a subsequence of the solutions to [2]—[5], corresponding to meshes $(\mathcal{T}_n)_{n\ge 1}$ such that $\mathrm{size}(\mathcal{T}_n) \to 0$ and $\inf_n(\mathrm{reg}(\mathcal{T}_n)) > 0$, converges in $L^q(\Omega)$, for all $q < \frac{2d}{d-2}$, to a weak solution of [1]. Since this weak solution is unique (see [DRO 01]), this proves theorem 2.1. To handle the difficulties brought by the non-regularity of $\mathbf{v}$ and G (in [EYM 00], $\mathbf{v}$ is C^1-continuous), we approximate these functions by regular ones.

5. Another scheme

We present here a variant of the preceding scheme, but in which we discretize $\mathbf{v}$ and G in a conservative way.

Let $\mathcal{T}$ be an admissible mesh. If $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$, we define the "full-diamond" around σ by $\triangle_\sigma = \triangle_{K,\sigma} \cup \triangle_{L,\sigma}$; if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, the "full-diamond" around σ is simply $\triangle_\sigma = \triangle_{K,\sigma}$. We let, for $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$,

$$\mathbf{v}_\sigma = \frac{1}{\mathrm{meas}(\triangle_\sigma)} \int_{\triangle_\sigma} \mathbf{v}(x)\,dx \quad \text{and} \quad G_\sigma = \frac{1}{\mathrm{meas}(\triangle_\sigma)} \int_{\triangle_\sigma} G(x)\,dx.$$

$(f_K)_{K\in\mathcal{T}}$ being defined as before, the new scheme for [1] is

$$\forall K\in\mathcal{T}\,,\quad \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}+m(\sigma)\mathbf{v}_\sigma\cdot\mathbf{n}_{K,\sigma}u_{\sigma,+}=\mathrm{meas}(K)f_K+\sum_{\sigma\in\mathcal{E}_K} m(\sigma)G_\sigma\cdot\mathbf{n}_{K,\sigma}\,, \quad [15]$$

$$\begin{array}{ll}\forall K\in\mathcal{T}\,,\forall\sigma=K|L\in\mathcal{E}_K\cap\mathcal{E}_{\rm int}\,, & F_{K,\sigma}=\frac{m(\sigma)}{d_\sigma}(u_K-u_L)\,,\\ \forall K\in\mathcal{T}\,,\forall\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}\,, & F_{K,\sigma}=\frac{m(\sigma)}{d_\sigma}u_K\,,\end{array} \quad [16]$$

$$\begin{array}{l}\forall\sigma=K|L\in\mathcal{E}_{\rm int}\,,\ u_{\sigma,+}=u_K\ \text{ if }\ \mathbf{v}_\sigma\cdot\mathbf{n}_{K,\sigma}\geq 0\,,\ u_{\sigma,+}=u_L\ \text{ otherwise,}\\ \forall\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}_K\,,\ u_{\sigma,+}=u_K\ \text{ if }\ \mathbf{v}_\sigma\cdot\mathbf{n}_{K,\sigma}\geq 0\,,\ u_{\sigma,+}=0\ \text{ otherwise.}\end{array} \quad [17]$$

Notice that [15]—[17] is exactly [2]—[5], provided that we define $v_{K,\sigma}=\mathbf{v}_\sigma\cdot\mathbf{n}_{K,\sigma}$, $G_{K,\sigma}=G_\sigma\cdot\mathbf{n}_{K,\sigma}$ and $u_{K,\sigma,+}=u_{\sigma,+}$; thus, the techniques used before prove the existence and uniqueness of the solution to [15]—[17] as well as the convergence of this approximation to the weak solution of [1].

6. Numerical results

All the results we present here concern the scheme of section 5, and the open set is $\Omega=]-1,1[^2$.

We consider first the equation $-\Delta u=\mathrm{div}(G)$, with $u(x,y)=(1-|x|)(1-|y|)$, and we use an unstructured discretization of Ω. The L^2-norm of the error converges in $\sqrt{h}$, but the discrete H^1-norm does not seem to converge.

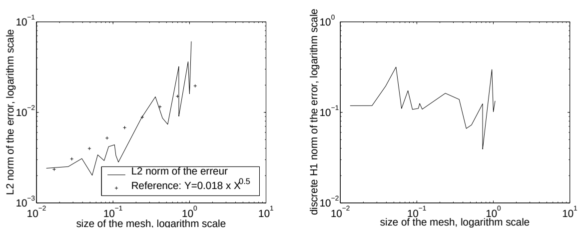

Figure 1: *Convergence results, unstructured mesh*

We then use structured (cartesian) meshes. The second numerical experiment still concerns the equation $-\Delta u=\mathrm{div}(G)$, but with $u(x,y)=A(x)A(y)$, where $A(t)=(1+w-(t-w)^-+\frac{(1+w)(t-w)^+}{1-w})$ and $w=1/\sqrt{2}$ (the preceding function, corresponding to $w=0$, gives, because of symetries between the function and the grid, too good convergence results); notice that $u\in H^{\frac{3}{2}-\epsilon}(\Omega)$ for all $\epsilon>0$ but that $u\notin H^{\frac{3}{2}}(\Omega)$. The convergence is still a bit chaotic (certainly because some meshes have more symetries with the function than others), but we notice a rate of convergence of order 1 in L^2-norm and 1/2 in discrete H^1-norm (this also shows a super-convergence result in the L^2-norm).

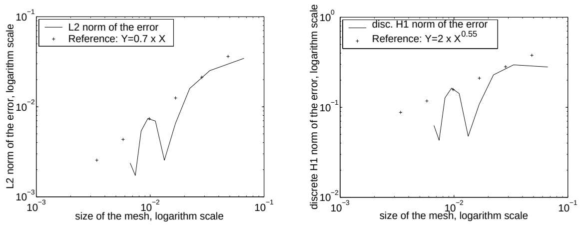

Figure 2: *Convergence results, structured mesh*

Considering the same function and discretization grid, we finally add a convection term $\mathrm{div}(\mathbf{v}u)$ with $\mathbf{v} = -6(x, y)$ (the problem is thus not coercive). The convergence is harder to obtain (we must discretize on quite thin meshes, comparing to the preceding cases), but a rate of convergence is still noticeable.

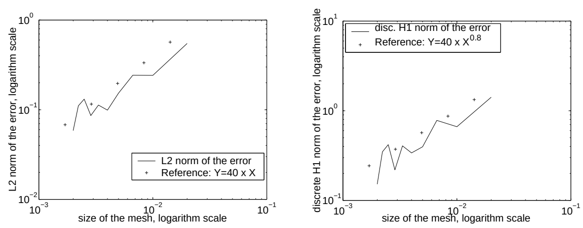

Figure 3: *Convergence results, structured mesh, non-coercive problem*

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